DENSE $\mathbb{P}GL$-ORBITS IN PRODUCTS OF GRASSMANNIANS

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Abstract. In this paper, we find some necessary and sufficient conditions on the dimension vector $\underline{\mathbf{d}} = (d_1, \ldots, d_k; n)$ so that the diagonal action of $\mathbb{P}GL(n)$ on $\prod_{i=1}^k Gr(d_i; n)$ has a dense orbit. Consequently, we obtain some algorithms for finding dense and sparse dimension vectors and classify dense dimension vectors with small length or size. We also characterize the dense dimension vectors of the form $(d, d, \ldots, d; n)$.

Contents

1. Introduction

Let $\{p_1, p_2, p_3\}$ and $\{q_1, q_2, q_3\}$ be two ordered sets of distinct points on $\mathbb{P}^1$. Then there exists a unique Möbius transformation $M \in \mathbb{P}GL(2)$ such that $M(p_i) = q_i$ for $1 \leq i \leq 3$. Hence, the diagonal action of $\mathbb{P}GL(2)$ on $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ has a dense orbit consisting of distinct triples. More generally, given two ordered sets $\{p_1, \ldots, p_{n+2}\}$ and $\{q_1, \ldots, q_{n+2}\}$ of $n+2$ points in general linear position in $\mathbb{P}^n$, there exists a unique element $M \in \mathbb{P}GL(n+1)$ such that $M(p_i) = q_i$ for $1 \leq i \leq n+2$ (see [4, Section 1.6]). In this paper, we consider a natural generalization of this classical fact.

Let V be an n-dimensional vector space. Then any ordered set $S = \{U_1, \ldots, U_k\}$ of linear subspaces of V corresponds to a point in $\prod_{i=1}^k Gr(d_i; n)$, where d_i denotes the dimension of U_i for $1 \leq i \leq k$. Therefore, the $GL(n)$-action on V induces a $\mathbb{P}GL(n)$ action on $\prod_{i=1}^k Gr(d_i; n)$. In this paper, we address the following question posed to us by János Kollár.

Question 1.1. *For which dimension vectors $(d_1, \ldots, d_k; n)$ does the diagonal action of $\mathbb{P}GL(n)$ have a dense orbit in $\prod_{i=1}^k Gr(d_i; n)$?*

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In this paper, we characterize the dimension vectors of length $k \leq 4$ that have a dense $\mathbb{P}GL(n)$ orbit (Theorem 5.1), the dimension vectors of size $d_i \leq 4$ (for all i) that have a dense $\mathbb{P}GL(n)$ orbit (Section 6) and equidimensional vectors $d_i = d$ (for all i) that have a dense $\mathbb{P}GL(n)$ orbit (Theorem 7.1). More importantly, we prove reduction lemmas that allow to reduce the density of a dimension vector to one with smaller ambient dimension under suitable assumptions. For many dimension vectors, our results give an efficient algorithm for checking the density of the $\mathbb{P}GL(n)$ action (see Section 8).

If $\mathbb{P}GL(n)$ acts with a dense orbit on $\prod_{i=1}^k Gr(d_i; n)$, then the dimension of $\mathbb{P}GL(n)$ has to be greater than or equal to the dimension of $\prod_{i=1}^k Gr(d_i; n)$. We thus obtain a necessary inequality

$$\sum_{i=1}^{k} d_i(n - d_i) \leq n^2 - 1. \tag{1}$$

However, as the following example shows, this inequality is not sufficient.

Example 1.2. Consider the dimension vector $(1, 1, 2, 2; 3)$. Geometrically, this dimension vector represents a configuration (p_1, p_2, l_1, l_2) consisting of a pair of points (p_1, p_2) and a pair of lines (l_1, l_2) in $\mathbb{P}^2$. Note that we have

$$\dim(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^{2*} \times \mathbb{P}^{2*}) = 8 = \dim(\mathbb{P}GL(3)).$$

However, $\mathbb{P}GL(3)$ does not act with a dense orbit. Briefly, the points p_1 and p_2 span a line l in $\mathbb{P}^2$ and the lines l_1 and l_2 intersect l in two points q_1 and q_2. The cross-ratio of the four points p_1, p_2, q_1, and q_2 on l is an invariant of the $\mathbb{P}GL(3)$ action. Furthermore, by fixing p_1, p_2, and l_1 and varying l_2, we can get every cross-ratio. Hence, all orbits of the $\mathbb{P}GL(3)$-action in this case have codimension at least 1.

There are several things to notice about Example 1.2. First, it can be generalized to the dimension vector $(1, 1, n-1, n-1; n)$. Geometrically, this dimension vector represents a configuration (p_1, p_2, H_1, H_2) consisting of a pair of points (p_1, p_2) and a pair of hyperplanes (H_1, H_2) in $\mathbb{P}^{n-1}$. Again, the hyperplanes H_1 and H_2 intersect the line l spanned by the points p_1 and p_2 in two points q_1 and q_2 and the cross-ratio of the four points p_1, p_2, q_1, and q_2 on l is an invariant of the $\mathbb{P}GL(n)$ action. Therefore, $\mathbb{P}GL(n)$ does not act with a dense orbit in this case. On the other hand, $\dim(\mathbb{P}^{n-1} \times \mathbb{P}^{n-1} \times \mathbb{P}^{n-1*} \times \mathbb{P}^{n-1*}) = 4(n-1)$ can be arbitrarily smaller than $\dim(\mathbb{P}GL(n)) = n^2 - 1$.

More importantly, in all the above examples there is a smaller configuration of linear spaces (the points p_1, p_2, q_1, and q_2 on l) obtained by taking spans and intersections of the original linear spaces, that trivially cannot be dense, as it fails the inequality (1). This smaller configuration of linear spaces is the obstruction for the density of the original dimension vector. In this paper, we will produce many classes of examples which show that this phenomenon is typical. In fact, we expect that whenever $\mathbb{P}GL(n)$ fails to act with a dense orbit on $\prod_{i=1}^k Gr(d_i; n)$, there is a configuration of vector spaces, obtained by repeatedly taking spans and intersections of the original ones, which does not satisfy the inequality (1) and accounts for this failure.

Knowing the density of a dimension vector has many applications. We close the introduction by briefly mentioning simple examples of two general applications. First, it allows one to choose convenient coordinates. Many geometric problems, such as enumerative problems and interpolation problems, become simpler to solve if the constraints have special coordinates. For example, it is easy to see that the unique quadric surface in $\mathbb{P}^3$ that contains the three lines $x = y = 0, z = w = 0, x - z = y - w = 0$ is $xw - yz = 0$. Since $(2, 2, 2; 4)$ is dense and these three lines belong to the dense orbit (see proof of Theorem 5.1), we conclude that three general lines in $\mathbb{P}^3$ impose independent conditions on quadrics in $\mathbb{P}^3$ and there is a unique quadric surface containing three general lines (see [4]). More generally, $(k, k, k; 2k)$ is dense and $x_1 = \cdots = x_k = 0$, $x_{k+1} = \cdots = x_{2k} = 0$ and $x_1 - x_{k+1} = \cdots = x_k - x_{2k} = 0$ is in the dense orbit of $\mathbb{P}GL(2k)$. It is then easy to see that there is a unique Segre image of $\mathbb{P}^1 \times \mathbb{P}^{k-1}$ containing these three $\mathbb{P}^{k-1}$'s in $\mathbb{P}^{2k-1}$ (see [4]). This simple classical calculation is the basis for the study of the genus zero Gromov-Witten invariants of Grassmannians (see [2]).

Second, knowing the density of dimension vectors allows one to determine automorphism groups of blowups of Grassmannians. For example, the dimension vector $(2, 2, 2, 2; 5)$ is dense (see Theorem 5.1). Hence, any 4 general points in $G(2, 5)$ can be taken to any other 4 general points by an action of $\mathbb{P}GL(5)$. By Lemma 2.2, if a 4-tuple of points is in the dense orbit of $\mathbb{P}GL(5)$, then any ordering of the 4 points is also in the dense orbit. Hence, there is an $\mathfrak{S}_4$-symmetry of the blowup of $G(2, 5)$ at 4 general points. More interestingly, the blowup X of $G(2, 5)$ at a general $\mathbb{P}^3$ section under the Plücker embedding has an $\mathfrak{S}_5$-symmetry (arising from the $\mathfrak{S}_4$-symmetry) that plays an important role in the Kawamata-Morrison cone conjecture for the log Calabi-Yau variety X (see [3]).

Organization of the paper: In §2, we will prove a useful criterion for checking density in terms of dimensions of stabilizer groups. In §3, we will collect some numerical lemmas. In §4, we will prove several lemmas that allow us to reduce checking the density of a dimension vector to simpler dimension vectors. In §5, we will characterize dense dimension vectors of length at most four. In §6, we will characterize dense dimension vectors of size at most four. In §7, we characterize the dense equidimensional vectors. Finally, in §8, we discuss several additional examples and further questions.

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2. Setup and the main lemma

In this section, we prove a basic criterion for the density of the $\mathbb{P}GL(n)$ action in terms of stabilizers.

Let $(U_1, \ldots, U_k)$ be a configuration of linear subspaces of an n-dimensional vector space V. The corresponding dimension vector will be denoted by $\underline{\mathbf{d}} = (d_1, \ldots, d_k; n)$, where d_i is the dimension of U_i for all i. The number of subspaces k is called the *length* of the dimension vector $\underline{\mathbf{d}}$ and will be denoted by $l(\underline{\mathbf{d}})$. The value $\max_i d_i$ is called the *size* of $\underline{\mathbf{d}}$ and will be denoted by $|\underline{\mathbf{d}}|$. The number n is the *ambient dimension*, while the sum $\sum_{i=1}^k d_i$ is the *total dimension* and the difference $\sum_{i=1}^k d_i - n$ is the *excess dimension*. When convenient, we will express a dimension vector in exponential notation $\underline{\mathbf{d}} = (1^{e_1}, \ldots, (n-1)^{e_{n-1}}; n)$. Finally, the stabilizer of the point $(U_1, \ldots, U_k) \in \prod_{i=1}^k Gr(d_i; n)$ will be denoted by $\mathrm{Stab}(U_1, \ldots, U_k)$.

Definition 2.1. We say that a dimension vector $\underline{\mathbf{d}} = (d_1, \ldots, d_k; n)$ is *dense* if the diagonal action of $\mathbb{P}GL(n)$ on $\prod_{i=1}^k Gr(d_i; n)$ has a Zariski dense orbit. Otherwise, $\underline{\mathbf{d}}$ is *sparse*. Furthermore, we say $\underline{\mathbf{d}}$ is *trivially sparse* if it does not satisfy the dimension inequality (1).

The following basic lemma is going to be the main tool in this article.

Lemma 2.2. *Let $(U_1, \ldots, U_k) \in \prod_{i=1}^k Gr(d_i; n)$ be a tuple of vector spaces with dimension vector $\underline{\mathbf{d}} = (d_1, \ldots, d_k; n)$. The $\mathbb{P}GL(n)$ orbit of this point is dense in $\prod_{i=1}^k Gr(d_i; n)$ if and* *only if*

$$\dim(\mathrm{Stab}(U_1,\dots,U_k)) = (n^2-1) - \sum_{i=1}^{k} d_i(n-d_i).$$

Proof. If an algebraic group G acts on an irreducible projective variety X, the orbit $G.x$ of any point $x \in X$ under G is open in its Zariski closure $\overline{G.x}$ [1, I.1.8]. On the other hand, the orbit $G.x$ is isomorphic to G/H, where H is the stabilizer of x. Consequently, the dimension of the Zariski closure of the orbit of x is

$$\dim(\overline{G.x}) = \dim(G) - \dim(H).$$

Hence, the orbit $G.x$ is dense in X if and only if $\dim(X) = \dim(G) - \dim(H)$. Specializing to the case $G = \mathbb{P}GL(n)$ and $X = \prod_{i=1}^{k} Gr(d_i;n)$, we obtain the lemma since $\dim(\mathbb{P}GL(n)) = n^2-1$ and

$$\dim\left(\prod_{i=1}^{k} Gr(d_i;n)\right) = \sum_{i=1}^{k} d_i(n-d_i).$$

□

Remark 2.3. Observe that $\dim(\mathrm{Stab}(U_1,\dots,U_k)) \geq (n^2-1) - \sum_{i=1}^{k} d_i(n-d_i)$. A dimension vector $\underline{\mathbf{d}}$ is dense if there is a k-tuple $(U_1,\dots,U_k) \in \prod_{i=1}^{k} G(d_i;n)$ where equality is achieved.

Let us conclude this section with the following two evident but useful observations.

Definition 2.4. Let $\underline{\mathbf{d}} = (d_1,\dots,d_k;n)$ be a dimension vector. Then the *complement* of $\underline{\mathbf{d}}$ is the dimension vector $\underline{\mathbf{d}}^c = (n-d_1,\dots,n-d_k;n)$.

Lemma 2.5. *A dimension vector $\underline{\mathbf{d}}$ is dense if and only if the complement dimension vector $\underline{\mathbf{d}}^c$ is dense.*

Proof. Consider the ambient vector space V and its dual V^* with the dual $\mathbb{P}GL(n)$ action. Taking quotient spaces and passing to the dual defines an isomorphism $\prod_{i=1}^{k} Gr(d_i;n)$ and $\prod_{i=1}^{k} Gr(n-d_i;n)$, which respects the $\mathbb{P}GL(n)$ action. Therefore, $\mathbb{P}GL(n)$ has a dense orbit on one if and only if it does on the other. □

Definition 2.6. We say that a dimension vector $\underline{\mathbf{d}} = (d_1,\dots,d_k;n)$ *dominates* a dimension vector $\underline{\mathbf{d}}' = (d'_1,\dots,d'_{k'};n)$ if for every $1 \leq d \leq n-1$ the number of times that d appears in $\underline{\mathbf{d}}$ is greater than or equal to the number of times it appears in $\underline{\mathbf{d}}'$.

Lemma 2.7. *Let $\underline{\mathbf{d}}$ and $\underline{\mathbf{d}}'$ be two dimension vectors such that $\underline{\mathbf{d}}$ dominates $\underline{\mathbf{d}}'$. Then if $\underline{\mathbf{d}}'$ is sparse, so is $\underline{\mathbf{d}}$. Equivalently, if $\underline{\mathbf{d}}$ is dense, so is $\underline{\mathbf{d}}'$.*

Proof. The lemma immediately follows by comparing dimensions of stabilizers. □

3. A numerical lemma

In this section, we prove a numerical lemma which will be very useful in the next section in reducing the density problem of a dimension vector to a smaller one.

Lemma 3.1. *Let $\underline{\mathbf{d}} = (d_1,\dots,d_k,d_{k+1},d_{k+2};n)$ be a dimension vector listed in increasing order $d_1 \leq d_2 \leq \dots \leq d_{k+2}$ and with $|\underline{\mathbf{d}}| \leq \frac{n}{2}$. Then either $\underline{\mathbf{d}}$ is trivially sparse or $\sum_{i=1}^{k} d_i < n$.*

Proof. First, if all d_i are between 1 and $\frac{n}{2}$, then $d_i(n-d_i)$ is a strictly increasing function of d_i. Hence, by decreasing the d_i's if necessary, it suffices to show that if $\sum_{i=1}^{k} d_i = n$, then $\sum_{i=1}^{k+2} d_i(n-d_i) \geq n^2$. Second, by decreasing d_{k+1} and d_{k+2} if necessary, we can assume that $d_k = d_{k+1} = d_{k+2} = |\underline{\mathbf{d}}|$. Finally, if there are at least two dimension values $a \leq b$ strictly between 1 and $|\underline{\mathbf{d}}|$, then replacing a and b by $a-1$ and $b+1$, respectively, changes the sum $\sum_{i=1}^{k+2} d_i(n-d_i)$ by

$$(a-1)(n-a+1) + (b+1)(n-b-1) - a(n-a) - b(n-b) = 2(a-b) - 2,$$

which is a negative number. Thus, we may assume that $\underline{\mathbf{d}}$ consists of r-many 1's, $(s+2)$-many $|\underline{\mathbf{d}}|$'s (with $s \geq 1$), and at most one number a between 1 and $|\underline{\mathbf{d}}|$. Let $\epsilon \in \{0,1\}$ be the number of times a appears in $\underline{\mathbf{d}}$.

Now, assuming $\sum_{i=1}^{k} d_i = r + \epsilon a + s|\underline{\mathbf{d}}| = n$, we have $\sum_{i=1}^{k+2} d_i = n + 2|\underline{\mathbf{d}}|$, and thus

$$\sum_{i=1}^{k+2} d_i(n-d_i) = n^2 + 2n|\underline{\mathbf{d}}| - \sum_{i=1}^{k+2} d_i^2 = n^2 + (s-2)|\underline{\mathbf{d}}|^2 + \epsilon a(2|\underline{\mathbf{d}}| - a) + r(2|\underline{\mathbf{d}}| - 1).$$

Since $|\underline{\mathbf{d}}| > a > 1$, if $s \geq 2$, then the right hand side is evidently at least n^2. On the other hand, if $s = 1$, $r + \epsilon a = n - |\underline{\mathbf{d}}| \geq |\underline{\mathbf{d}}|$ and

$$-|\underline{\mathbf{d}}|^2 + (\epsilon a + r)|\underline{\mathbf{d}}| + \epsilon a(|\underline{\mathbf{d}}| - a) + r(|\underline{\mathbf{d}}| - 1) \geq 0.$$

We thus conclude that $\sum_{i=1}^{k+2} d_i(n-d_i) \geq n^2$ as desired. □

4. Reduction techniques

In this section, we prove a series of lemmas that reduce the density/sparsity problem for certain dimension vectors to the same problem for smaller ones.

Lemma 4.1. *The dimension vector* $\underline{\mathbf{d}} = (d_1, \ldots, d_k; n)$ *is dense if* $\sum_{i=1}^{k} d_i \leq n$.

Proof. Let X be an n-dimensional vector space and $\mathcal{B} = \{e_1, \ldots, e_n\}$ be a basis for X. For any $1 \leq i \leq k$, let $\alpha_i := \sum_{j=1}^{i-1} d_i$ and consider the subspace $V_i = \langle e_{\alpha_i+1}, \ldots, e_{\alpha_{i+1}} \rangle$ of X. Then, in the basis $\mathcal{B}$, the stabilizer of the configuration $(V_1, \ldots, V_k; X)$ has the form

$$\begin{pmatrix} M_1 & 0 & \ldots & 0 & A_1 \\ 0 & M_2 & \ldots & 0 & A_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & M_k & A_k \\ 0 & 0 & \ldots & 0 & B \end{pmatrix},$$

where B is an $(n - \sum_{i=1}^{k} d_i) \times (n - \sum_{i=1}^{k} d_i)$ matrix and, for each $1 \leq i \leq k$, M_i is a $d_i \times d_i$ matrix and A_i is a $d_i \times (n - \sum_{i=1}^{k} d_i)$ matrix. Therefore, we have

$$\dim(\mathrm{Stab}(V_1, \ldots, V_k; X)) = \sum_{i=1}^{k} d_i^2 + n(n - \sum_{i=1}^{k} d_i) - 1 = n^2 - 1 - \sum_{i=1}^{k} d_i(n-d_i).$$

Hence, by Lemma 2.2, we conclude that $\underline{\mathbf{d}}$ is dense. □

Lemma 4.2. *Let* $\underline{\mathbf{d}} = (a_1, \ldots, a_r, b_1, \ldots, b_s; n)$ *be a dimension vector such that* $\sum_{i=1}^{r} a_i = n-k < n$ *and* $\sum_{j=1}^{s}(n-b_j) \leq n-k$. *Then* $\underline{\mathbf{d}}$ *is dense if* $\underline{\mathbf{d}}' = (a_1, \ldots, a_r, b_1-k, \ldots, b_s-k; n-k)$ *is dense.*

Proof. Let $(V_1, \dots, V_r, U_1, \dots, U_s; X)$ be a generic configuration of vector spaces corresponding to the dimension vector $\underline{\mathbf{d}}$. Set $W := V_1 + \cdots + V_r$ and $T_j := W \cap U_j$, $1 \le j \le s$. By our numerical assumptions, we have $\dim(W) = n - k$ and $\dim(T_j) = b_j - k$, for all j. Observe that this construction yields a generic configuration with dimension vector $\underline{\mathbf{d}}'$. Every element M of $\mathbb{P}GL(n)$ that stabilizes the configuration $(V_1, \dots, V_r, U_1, \dots, U_s; X)$, preserves the subspace W and the restriction of M to W stabilizes the configuration $(V_1, \dots, V_r, T_1, \dots, T_s; W)$. Hence, we get a group homomorphism

$$f : \mathrm{Stab}(V_1, \dots, V_r, U_1, \dots, U_s; X) \to \mathrm{Stab}(V_1, \dots, V_r, T_1, \dots, T_s; W).$$

We are going to bound the dimension of $\mathrm{Stab}(V_1, \dots, V_r, U_1, \dots, U_s; X)$ by the sum of the dimensions of the kernel and the image of f.

Set $Y = \cap_{j=1}^s U_j$ and note that $\dim(Y) = n - \sum_{j=1}^s (n - b_j) \ge k$. We may choose a basis $\mathcal{B} = \{e_1, \dots, e_n\}$ for X such that

$$W = \langle e_1, \dots, e_{n-k} \rangle, \ Y = \langle e_{n-\dim(Y)+1}, \dots, e_n \rangle.$$

Then, with respect to the basis $\mathcal{B}$, the kernel of f consists of matrices of the form

$$\begin{pmatrix} I_{n-\dim(Y)} & 0 & 0 \\ 0 & I_{\dim(Y)-k} & A \\ 0 & 0 & B \end{pmatrix}$$

where A is a $(\dim(Y)-k) \times k$ matrix and B is a $k \times k$ matrix. Hence, $\dim(\mathrm{Ker}(f)) = \dim(Y) \times k$. On the other hand, the dimension vector $\underline{\mathbf{d}}'$ is dense by our assumption. Therefore,

$$\dim(\mathrm{Stab}(V_1, \dots, V_r, T_1, \dots, T_s; W)) = (n-k)^2 - 1 - \sum_{i=1}^r a_i(n-k-a_i) - \sum_{j=1}^s (b_j - k)(n - b_j).$$

We conclude that

$$\begin{aligned} \dim(\mathrm{Stab}(U_1, \dots, U_r, V_1, \dots, V_s; X)) &\le \dim(\mathrm{Stab}(V_1, \dots, V_r, T_1, \dots, T_s; W)) + \dim(\mathrm{Ker}(f)) \\ &= n^2 - 1 - \sum_{i=1}^r a_i(n-a_i) - \sum_{j=1}^s b_j(n-b_j). \end{aligned}$$

The lemma now follows from Lemma 2.2. $\square$

Lemma 4.3. *Let $\underline{\mathbf{d}} = (a_{11}, \dots, a_{1s_1}, a_{21}, \dots, a_{2s_2}, \dots, a_{r1}, \dots, a_{rs_r}; n)$ be a dimension vector such that $\sum_{j=1}^{s_i} a_{ij} \le n$ for all $1 \le i \le r$. Then $\underline{\mathbf{d}}$ is sparse if the dimension vector $(\sum_{j=1}^{s_1} a_{1j}, \dots, \sum_{j=1}^{s_r} a_{rj}; n)$ is sparse.*

Proof. By induction on r and on s_r, it suffices to prove that the dimension vector $\underline{\mathbf{d}} = (a_1, \dots, a_t, b, c; n)$ is sparse if $\underline{\mathbf{d}}' = (a_1, \dots, a_t, b+c; n)$ is sparse. By Lemma 2.2, this would follow from the inequality

$$\dim(\mathrm{Stab}(\underline{\mathbf{d}}')) \le \dim(\mathrm{Stab}(\underline{\mathbf{d}})) + 2bc. \tag{2}$$

Let $(V_1, \dots, V_t, U, W; X)$ be a generic configuration of vector spaces corresponding to $\underline{\mathbf{d}}$. Then, the configuration $(V_1, \dots, V_t, U + W; X)$ corresponds to $\underline{\mathbf{d}}'$. Every element M of $\mathbb{P}GL(n)$ that stabilizes $(V_1, \dots, V_t, U, W; X)$, also stabilizes $(V_1, \dots, V_t, U+W; X)$. Therefore, $\mathrm{Stab}(V_1, \dots, V_t, U, W; X)$ is a subgroup of $\mathrm{Stab}(V_1, \dots, V_t, U + W; X)$. On the other hand, the map

$$f : \mathrm{Stab}(V_1, \dots, V_t, U+W; X)/\mathrm{Stab}(V_1, \dots, V_t, U, W; X) \hookrightarrow Gr(b, U+W) \times Gr(c, U+W),$$

which sends an element M to $(M_{|U+W}U, M_{|U+W}W)$ is injective. This proves the desired inequality (2) since $\dim(Gr(b, U+W)) = \dim(Gr(c, U+W)) = bc$. $\square$

This result allows us to improve slightly on Lemma 4.1:

Lemma 4.4. *The dimension vector* $\underline{\mathbf{d}} = (d_1, \dots, d_k; n)$ *is dense if* $\sum_{i=1}^k d_i \leq n+1$.

Proof. Indeed, we know that the dimension vector $(1^r; n)$ is dense for $r \leq n+1$ (see §1 and [4, Section 1.6]), so the dimension vector $\underline{\mathbf{d}} = (d_1, \dots, d_k; n)$ is dense when $\sum_{i=1}^k d_i \leq n+1$ by Lemma 4.3. □

Lemma 4.5. *Suppose that an element M of $\mathbb{P}GL(n)$ stabilizes a generic configuration*

$$(V_1, \dots, V_r; X)$$

corresponding to a dimension vector $\underline{\mathbf{d}} = (a_1, \dots, a_r; n)$ *with* $\sum_{i=1}^r a_i = n - k \leq n$. *Suppose moreover that M preserves a generic subspace U of X with* $\dim(U) \geq |\underline{\mathbf{d}}| + k$ *and acts on it as the identity. Then M is the identity element.*

Proof. For every $1 \leq i \leq r$, set $\alpha_i := \sum_{j=1}^i a_i$, and choose a basis $\mathcal{B} = \{e_1, \dots, e_n\}$ for X such that

$$V_i = \langle e_{\alpha_{i-1}+1}, \dots, e_{\alpha_i} \rangle, \quad \forall 1 \leq i \leq r,$$

and that $U = \langle f_1, \dots, f_{\dim(U)-k}, e_{n-k+1}, \dots, e_n \rangle$, with $f_j \in \langle e_1, \dots, e_{n-k} \rangle$. We can express the vectors f_j, $1 \leq j \leq \dim(U) - k$, in terms of the vectors e_i, $1 \leq i \leq n-k$, in the form

$$(f_1, \dots, f_{\dim(U)-k}) = (E_1, \dots, E_r) \begin{pmatrix} A_1 \\ \vdots \\ A_r \end{pmatrix},$$

where $E_i = (e_{\alpha_{i-1}+1}, \dots, e_{\alpha_i})$ and A_i is an $a_i \times (\dim(U) - k)$ matrix. Since U is assumed to be a generic subspace, each A_i has an invertible full minor. Thus, after suitable change of basis for each V_i, we may assume that for all $1 \leq i \leq r$, the matrix A_i has the form $A_i = (I_{a_i} \ *)$. Then, for an element $M \in \mathrm{Stab}(V_1, \dots, V_r; X)$, $M(f_j) = f_j$ for all $1 \leq j \leq \dim(U) - k$, implies that $M(e_i) = e_i$ for all $1 \leq i \leq n-k$. Therefore, if M stabilizes the configuration $(V_1, \dots, V_r; X)$ and acts as identity on U, M has to be the identity element. □

Lemma 4.6. *Consider the dimension vector* $\underline{\mathbf{d}} = (a_1, \dots, a_r, b, b; n)$ *with* $b + \sum_{i=1}^r a_i = n$ *and* $b \leq \frac{n}{2}$. *Then* $\underline{\mathbf{d}}$ *is dense if the dimension vector* $\underline{\mathbf{d}}' = (a_1, \dots, a_r, b; n-b)$ *is dense.*

Proof. Let $(V_1, \dots, V_r, U_1, U_2; X)$ be a generic configuration of vector spaces corresponding to $\underline{\mathbf{d}}$. Set $W := V_1 + \cdots + V_r$ and $U' := W \cap (U_1 + U_2)$. Then $\dim(W) = n - b$ and $\dim(U') = b$. Observe that this construction yields a generic configuration with dimension vector $\underline{\mathbf{d}}'$. Consider the map induced by restriction

$$f : \mathrm{Stab}(V_1, \dots, V_r, U_1, U_2; X) \to \mathrm{Stab}(V_1, \dots, V_r, U'; W).$$

If the dimension vector $\underline{\mathbf{d}}'$ is dense, then by Lemma 2.2,

$$\dim(\mathrm{Stab}(V_1, \dots, V_r, U'; W)) = (n-b)^2 - 1 - \sum_{i=1}^r a_i(n - b - a_i) - b(n - 2b)$$

$$= n^2 - 1 - \sum_{i=1}^r a_i(n - a_i) - 2b(n-b).$$

By another application of Lemma 2.2, it suffices to prove that the map f is injective. An element $M \in \mathrm{Ker}(f)$ preserves U_1 and U_2 and acts as the identity on W, and thus has to be the identity element by Lemma 4.5. □

Lemma 4.7. *Let* $\underline{\mathbf{d}} = (a_1, \dots, a_r, b; n)$ *be a dimension vector with* $\sum_{i=1}^r a_i = n$, $|\underline{\mathbf{d}}| = b$, *and* $a_i + b \le n$ *for all* $1 \le i \le r$. *Then* $\underline{\mathbf{d}}$ *is dense if the dimension vector* $\underline{\mathbf{d}}' = (a_1, \dots, a_r; b)$ *is dense.*

Proof. Let $(V_1, \dots, V_r, U; X)$ be a generic configuration of vector spaces corresponding to $\underline{\mathbf{d}}$. For any $1 \le i \le r$, set $W_i := V_1 + \dots + \widehat{V_i} + \dots + V_r$ and $U_i := W_i \cap U$. Then $\dim(W_i) = n - a_i$, $\dim(U_i) = b - a_i$, and we have a homomorphism induced by restriction

$$f : \mathrm{Stab}(V_1, \dots, V_r, U; X) \to \mathrm{Stab}(U_1, \dots, U_r; U).$$

By Lemma 4.5, f has trivial kernel. On the other hand, if $\underline{\mathbf{d}}'$ is dense, so is its complement $\underline{\mathbf{d}}'^c = (b - a_1, \dots, b - a_r; b)$, and thus we have:

$$\begin{aligned}\dim(\mathrm{Stab}(U_1, \dots, U_r; U)) &= b^2 - 1 - \sum_{i=1}^r a_i(b - a_i) \\ &= n^2 - 1 - \sum_{i=1}^r a_i(n - a_i) - b(n - b).\end{aligned}$$

Therefore, by Lemma 2.2, $\underline{\mathbf{d}}$ is dense as well. □

Lemma 4.8. *Let* $\underline{\mathbf{d}} = (a_1, \dots, a_r, b_1, b_2; n)$ *be a dimension vector with* $\sum_{i=1}^r a_i = n - k < n$, $k \le b_1, b_2$, *and* $b_1 + b_2 = n$. *Then* $\underline{\mathbf{d}}$ *is dense if the dimension vector* $\underline{\mathbf{d}}' = (a_1, \dots, a_r, b_1 - k, b_2 - k; n - k)$ *is dense.*

Proof. Let $(V_1, \dots, V_r, U_1, U_2; X)$ be a generic configuration of vector spaces corresponding to $\underline{\mathbf{d}}$ and set $W := V_1 + \dots + V_r$ and $U_i' := W \cap U_i$ for $i = 1, 2$. Then $\dim(W) = n - k$, $\dim(U_i') = b_i - k$ for $i = 1, 2$. Observe that this construction yields a generic configuration with dimension vector $\underline{\mathbf{d}}'$. We have the homomorphism induced by restriction

$$f : \mathrm{Stab}(V_1, \dots, V_r, U_1, U_2; X) \to \mathrm{Stab}(V_1, \dots, V_r, U_1', U_2'; W).$$

Since any element $M \in \mathrm{Ker}(f)$ preserves U_1 and U_2 and acts as identity on W, by Lemma 4.5, f is injective. Therefore, if $\underline{\mathbf{d}}'$ is dense, we have

$$\begin{aligned}\dim(\mathrm{Stab}(V_1, \dots, V_r, U_1, U_2; X)) &\le \dim(\mathrm{Stab}(V_1, \dots, V_r, U_1', U_2'; W)) \\ &= (n-k)^2 - 1 - \sum_{i=1}^r a_i(n - k - a_i) - \sum_{j=1}^2 (b_j - k)(n - b_j) \\ &= n^2 - 1 - \sum_{i=1}^r a_i(n - a_i) - \sum_{j=1}^2 b_j(n - b_j).\end{aligned}$$

This, together with Lemma 2.2, implies that the dimension vector $\underline{\mathbf{d}}$ is dense. □

Lemma 4.9. *Let* $\underline{\mathbf{d}} = (a_1, \dots, a_r, b_1, b_2; n)$ *be a dimension vector with* $\sum_{i=1}^r a_i = n - k < n$, $k \le b_1, b_2$, *and* $b_1 + b_2 < n$. *Set* $m = b_1 + b_2 - k$. *Then* $\underline{\mathbf{d}}$ *is dense if the dimension vector* $\underline{\mathbf{d}}' = (a_1, \dots, a_r, b_1, b_2; m)$ *is dense.*

Proof. We prove the assertion in two steps. Let $(V_1, \dots, V_r, U_1, U_2; X)$ be a generic configuration of vector spaces corresponding to $\underline{\mathbf{d}}$ and set $W := V_1 + \dots + V_r$ and $U := U_1 + U_2$. Also, let $T = U \cap W$ and $U_i' = U_i \cap W$ for $i = 1, 2$. Note that $\dim(W) = n - k$ and $\dim(T) = b_1 + b_2 - k = m$. Any element in the stabilizer of $(V_1, \dots, V_r, U_1, U_2; X)$ stabilizes W, T, U_1', and U_2', and thus we get a homomorphism:

$$f : \mathrm{Stab}(V_1, \dots, V_r, U_1, U_2; X) \to \mathrm{Stab}(V_1, \dots, V_r, U_1', U_2', T; W).$$

The new twist in this argument is that we are considering a configuration of vector spaces which are not in general position, as T contains U_1' and U_2'.

Lemma 4.5 implies that f has trivial kernel. Now assume that $\mathrm{Stab}(V_1,\dots,V_r,U_1',U_2',T;W)$ has the expected dimension. Since T contains U_1' and U_2', this expected dimension is

$$(n-k)^2-1-\sum_{i=1}^{r}a_i(n-k-a_i)-m(n-k-m)-\sum_{j=1}^{2}(b_j-k)(m+k-b_j). \tag{3}$$

Using the equalities $m=b_1+b_2-k$ and $\sum_{i=1}^r a_i=n-k$, the expression (3) equals

$$n^2-1-\sum_{i=1}^{r}a_i(n-a_i)-\sum_{j=1}^{2}b_j(n-b_j).$$

Then Lemma 2.2 would imply that $\underline{\mathbf{d}}$ is dense.

Hence, it suffices to prove that $\mathrm{Stab}(V_1,\dots,V_r,U_1',U_2',T;W)$ has the expected dimension. For every $1\le i\le r$, let $W_i:=V_1+\cdots+\widehat{V_i}+\cdots+V_r$ and $V_i':=T\cap W_i$. Then $(V_1',\dots,V_r',U_1',U_2';T)$ is a generic configuration of vector spaces corresponding to the complement of the dimension vector $\underline{\mathbf{d}}'$. Restricting from W to T, induces a homomorphism

$$g:\mathrm{Stab}(V_1,\dots,V_r,U_1',U_2',T;W)\to\mathrm{Stab}(V_1',\dots,V_r',U_1',U_2';T),$$

whose kernel is trivial by Lemma 4.5. If $\underline{\mathbf{d}}'$ is dense, then so is its complement, and hence the dimension of the image of g is at most

$$m^2-1-\sum_{i=1}^{r}a_i(m-a_i)-\sum_{j=1}^{2}(b_j-k)(m+k-b_j),$$

which using $m=b_1+b_2-k$ and $\sum_{i=1}^r a_i=n-k$ simplifies to

$$n^2-1-\sum_{i=1}^{r}a_i(n-a_i)-\sum_{j=1}^{2}b_j(n-b_j)$$

as desired. □

Lemma 4.10. *Consider the dimension vector $\underline{\mathbf{d}}=(a_1,\dots,a_r;n)$. Assume that there are k elements $i_1,\dots,i_k\in\{1,\dots,r\}$ such that $\sum_{j=1}^k a_{i_j}=(k-1)n$ and let $\underline{\mathbf{d}}'$ be the dimension vector obtained from $\underline{\mathbf{d}}$ after replacing a_{i_j} with $b_{i_j}:=\sum_{\substack{t=1\\t\neq j}}^k a_{i_t}-(k-2)n$. Then $\underline{\mathbf{d}}$ is dense if and only if $\underline{\mathbf{d}}'$ is dense.*

Proof. Let $(V_1,\dots,V_r;X)$ be a generic configuration of vector spaces corresponding to $\underline{\mathbf{d}}$. For every $1\le t\le r$, set $U_t:=V_t$ if $t\neq i_1,\dots,i_k$, and $U_t:=\cap_{\substack{s=1\\s\neq j}}^k V_{i_s}$ if $t=i_j$ for some $1\le j\le k$. Then, by our numerical assumptions, the configuration $(U_1,\dots,U_r;X)$ corresponds to the dimension vector $\underline{\mathbf{d}}'$ and the original vector spaces V_i can be recovered from the vector spaces U_j. Indeed, for any $1\le j\le k$, we have

$$\sum_{\substack{t=1\\t\neq j}}^{k}b_{i_t}=a_{i_j}+(k-2)\sum_{t=1}^{k}a_{i_t}-(k-1)(k-2)n=a_{i_j}$$

and thus

$$V_{i_j}=\cup_{\substack{t=1\\t\neq j}}^k U_{i_t}.$$

This implies that the density/sparsity of $\underline{\mathbf{d}}$ is equivalent to that of $\underline{\mathbf{d}}'$. □

Remark 4.11. The converses of Lemmas 4.2, 4.6, 4.7, 4.8 and 4.9 are easily seen to hold. In each case, the generic subspaces with invariants $\underline{\mathbf{d}}'$ can be obtained via the construction in the proof. Hence, if $\underline{\mathbf{d}}'$ is not dense, $\underline{\mathbf{d}}$ is certainly not dense.

5. Dimension vectors with small length

Our goal in this section is to characterize dense dimension vectors of small length. Recall that the length of a dimension vector $\underline{\mathbf{d}} = (d_1, \dots, d_k; n)$ is defined to be the number k. Every dimension vector of length one is dense since the Grassmannian $Gr(k, n)$ is a quotient of $\mathbb{P}GL(n)$. In the following result, we show that most dimension vectors with length at most four are dense.

Theorem 5.1. *Let $\underline{\mathbf{d}}$ be a dimension vector with length $k \leq 4$. Then $\underline{\mathbf{d}}$ is sparse if and only if $k = 4$ and $\underline{\mathbf{d}} = (a, b, c, d; n)$ with $a + b + c + d = 2n$.*

Proof. First, we show that all dimension vectors of length two and three are dense. Let $\underline{\mathbf{d}} = (a, b; n)$ be a dimension vector of length two. By taking the complement if necessary (see Lemma 2.5), we can assume that $a + b \leq n$. Then $\underline{\mathbf{d}}$ is dense by Lemma 4.1.

Now let $\underline{\mathbf{d}} = (a, b, c; n)$ be a dimension vector of length 3. We consider the following two cases.

- If $n = 2k$ and $a = b = c = k$, then let $(V_1, V_2, V_3; X)$ be the vector spaces
$$V_1 = \langle e_1, \dots, e_k \rangle,\ V_2 = \langle e_{k+1}, \dots, e_{2k} \rangle,\ V_3 = \langle e_1 + e_{k+1}, \dots, e_k + e_{2k} \rangle,$$
where $\mathcal{B} = \{e_1, \dots, e_{2k}\}$ is a basis for X. Then, in this basis $\mathcal{B}$, $\operatorname{Stab}(V_1, V_2, V_3; X)$ consists of matrices of the form
$$M = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}.$$
Hence,
$$\dim(\operatorname{Stab}(V_1, V_2, V_3; X)) = k^2 - 1 = (2k)^2 - 1 - 3k(2k - k).$$
Therefore, Lemma 2.2 implies that $\underline{\mathbf{d}}$ is dense.
- If a, b, and c are not all equal to $n/2$, by taking the complement and rearranging if necessary, we can assume that $a + b = n - k < n$. If $c \leq k$, $\underline{\mathbf{d}}$ is dense by Lemma 4.1. If $c > k$, then, by Lemma 4.2, the density of $\underline{\mathbf{d}}$ follows from the density of $(a, b, c - k; n - k)$. We are done since either we are in the first case and $\underline{\mathbf{d}}$ is dense, or we can inductively continue reducing the ambient dimension n.

Finally, let $\underline{\mathbf{d}} = (a, b, c, d; n)$ be a dimension vector of length four. We will show that $\underline{\mathbf{d}}$ is dense if and only if $a + b + c + d \neq 2n$. We begin by studying the case $a + b + c + d = 2n$.

- First, consider the case $n = 2k$ and $a = b = c = d = k$. The dimension vector $(k, k, k, k; 2k)$ is trivially sparse by Lemma 2.2 since
$$4k(2k - k) = 4k^2 > 4k^2 - 1 = \dim(\mathbb{P}GL(2k)).$$
- Now assume $a + b + c + d = 2n$, but a, b, c, and d are not all equal. Then, by taking the complement and rearranging if necessary, we can assume that $a + b < n$. We apply Lemma 4.2. Let $(V_1, \dots, V_4; X)$ be any configuration of vector spaces corresponding to $\underline{\mathbf{d}}$, and consider the $(a + b)$-dimensional space $W := V_1 + V_2$. Then

the configuration $(V_1, V_2, W \cap V_3, W \cap V_4; W)$ corresponds to the dimension vector $\underline{\mathbf{d}}' = (a, b, a+b+c-n, a+b+d-n; a+b)$. Since

$$a + b + (a + b + c - n) + (a + b + d - n)$$

$$= (a + b + c + d - 2n) + 2(a + b) = 2(a + b),$$

by induction on the dimension of the ambient space, $\underline{\mathbf{d}}'$ is sparse and, therefore, so is $\underline{\mathbf{d}}$.

Now we suppose that $a+b+c+d \neq 2n$ and show by induction on n that $\underline{\mathbf{d}}$ is dense. By taking the complement if necessary, we can assume that $a + b + c + d < 2n$.

- If $a + b + c + d \leq n$, then $\underline{\mathbf{d}}$ is dense by Lemma 4.1.
- If sum of the two larger dimensions is bigger than n, we take the complement and rearrange so that $a + b + c + d > 2n$ and $a + b = n - k < n$. This implies that

  $$(n - c) + (n - d) = 2n - (c + d) < a + b = n - k.$$

  Thus, by Lemma 4.2, $\underline{\mathbf{d}}$ is dense if $\underline{\mathbf{d}}' = (a, b, c - k, d - k; n - k)$ is dense. Since

  $$a + b + (c - k) + (d - k) = (a + b + c + d) - 2k > 2(n - k),$$

  by induction on n, $\underline{\mathbf{d}}'$ is dense and we are done. In the remaining cases, we may assume that $n < a + b + c + d < 2n$, $a + b = n - k < n$, and $c + d \leq n$ (suppose for simplicity that we have ordered the dimensions so that $a \leq b \leq c \leq d$).
- If $c < k$, then $a + b + c = n - t < n$. Lemma 4.2 implies that $\underline{\mathbf{d}}$ is dense if $\underline{\mathbf{d}}' = (a, b, c, d - t; n - t)$ is dense. Since

  $$a + b + c + (d - t) = (n - t) + (d - t) < 2(n - t),$$

  $\underline{\mathbf{d}}'$ is dense by induction. Therefore, in the following cases we may assume that $k \leq c, d$.
- If $c + d = n$, then by Lemma 4.8, $\underline{\mathbf{d}}$ is dense if $\underline{\mathbf{d}}' = (a, b, c - k, d - k; n - k)$ is dense. Since

  $$a + b + (c - k) + (d - k) = (a + b + c + d) - 2k < 2(n - k),$$

  $\underline{\mathbf{d}}'$ is dense by induction.
- Finally, assume that $c + d < n$. Then Lemma 4.9 implies that $\underline{\mathbf{d}}$ is dense if $\underline{\mathbf{d}}' = (m - a, m - b, c - k, d - k; m)$ is dense, where $m = c + d - k$. Since

  $$(m - a) + (m - b) + (c - k) + (d - k) = 2m - (a + b) + (c + d) - 2k$$

  $$< 2m - (n - k) + n - 2k = 2m - k < 2m,$$

  by induction, we conclude that $\underline{\mathbf{d}}'$ is dense.

□

Corollary 5.2. *Let $\underline{\mathbf{d}} = (d_1, \ldots, d_k; n)$ be a dimension vector such that there is a subsequence of $d_1, \ldots, d_k$ with total sum $2n$. Then $\underline{\mathbf{d}}$ is sparse.*

Proof. This follows from Theorem 5.1 and Lemma 4.3. □

6. Dimension vectors with small size

In the previous section we characterized all dense dimension vectors of length at most four. In this section we study dimension vectors of a given size, without any restriction on the length. We illustrate how the techniques we developed in Section 4 can be employed in classifying dense dimension vectors, and fully classify dense dimension vectors with size at most four. We begin by the following useful consequence of Lemma 3.1.

Proposition 6.1. *Let $\underline{\mathbf{d}} = (d_1, \dots, d_r; n)$ be a dimension vector of size $|\underline{\mathbf{d}}| = k$ with $2k \le n$. Then $\underline{\mathbf{d}}$ is either trivially sparse, or is dense by Lemma 4.1, or the density/sparsity problem for $\underline{\mathbf{d}}$ can be reduced to the density/sparsity problem for a dimension vector in a smaller dimensional ambient space.*

Proof. Let $\underline{\mathbf{d}}$ be as in the statement of the proposition and suppose that it is not trivially sparse. Then, assuming that the d_i's are arranged in increasing order, by Lemma 3.1, we have

$$\sum_{i=1}^{r-2} d_i < n.$$

This leads to the following cases:

(I) If $\sum_{i=1}^{r} d_i \le n$, then $\underline{\mathbf{d}}$ is dense by Lemma 4.1.

(II) If $\sum_{i=1}^{r-1} d_i = n - k < n$ and $k < d_r$, then we can apply Lemma 4.2 to reduce the density problem for $\underline{\mathbf{d}}$ to the density problem for $(d_1, \dots, d_{r-1}, d_r - k; n - k)$.

(III) If $\sum_{i=1}^{r-2} d_i = n-k < n$ and $k \le d_{r-1}, d_r$, then by Lemma 4.8 or Lemma 4.9 (depending on whether $d_{r-1} + d_r$ is equal to or less than n), we reduce the density problem of $\underline{\mathbf{d}}$ to one with smaller ambient dimension.

□

Suppose we want to determine if a dimension vector $\underline{\mathbf{d}} = (d_1, \dots, d_r; n)$ with $|\underline{\mathbf{d}}| \le t$ is dense or sparse. By Proposition 6.1, if $n \ge 2|\underline{\mathbf{d}}|$ and the problem is not trivially answered, this problem can be reduced to one with smaller ambient dimension. Lemmas 4.1, 4.2, 4.8 and 4.9 used in the proof of Proposition 6.1 preserve the size of the dimension vectors. Hence, we may assume that $|\underline{\mathbf{d}}| < n < 2|\underline{\mathbf{d}}|$. On the other hand, as the minimum value for $d_i(n - d_i)$, with $1 \le d_i \le n - 1$, is $n - 1$, if $r > n + 1$ we have:

$$\sum_{i=1}^{r} d_i(n - d_i) > (n-1)(n+1) = n^2 - 1,$$

and hence $\underline{\mathbf{d}}$ is trivially sparse. Therefore, we may assume that $r \le n + 1$. Furthermore, if $r = n + 1$, the same calculation shows that if any of d_i's are not equal to 1 or $n - 1$, then $\underline{\mathbf{d}}$ is trivially sparse. Hence, the density/sparsity problem for dimension vectors of bounded size is reduced to a finite collection of low-dimensional cases. Here we will demonstrate how to use the results of the previous sections to classify all dense vectors with $|\underline{\mathbf{d}}| \le 4$.

In the following, we will frequently use the exponential notation for dimension vectors (for example the dimension vector $(1, 1, 2, 2, 2, 4; 7)$ will be denoted as $(1^2, 2^3, 4; 7)$). We will also use the following simple observation.

Lemma 6.2. *For any $n \ge 2$, the dimension vector $(1^n, n-1; n)$ is dense.*

Proof. Let X be an n-dimensional vector space with basis $\mathcal{B} = \langle e_1, \dots, e_n \rangle$. Consider the configuration $(V_1, \dots, V_n, U; X)$ of vector spaces with $V_i = \langle e_i \rangle$ for $1 \le i \le n$, and $U = \{(x_1, \dots, x_n) : \sum_{i=1}^{n} x_i = 0\}$ (the coordinates are with respect to the basis $\mathcal{B}$). Then

$(V_1, \ldots, V_n, U; X)$ corresponds to the dimension vector $(1^n, n-1; n)$ and it is easy to check that $\mathrm{Stab}(V_1, \ldots, V_n, U; X)$ is trivial. Hence, we are done by Lemma 2.2. $\square$

We now begin the classification of dimension vectors of size $|\underline{\mathbf{d}}|$ at most four.

$|\underline{\mathbf{d}}| = 1$: This case is trivial. The dimension vector has the form $\underline{\mathbf{d}} = (1^r; n)$, which is trivially sparse if $r > n+1$ and is dense otherwise (see §1 and [4, Section 1.6]).

$|\underline{\mathbf{d}}| = 2$: We first consider the case $n = 3$, where Proposition 6.1 does not apply. In this case $r \leq 4$, and by Theorem 5.1 all vectors of the form $(1^a, 2^b; 3)$ with $a + b \leq 4$ are dense except for $(1^2, 2^2; 3)$.

Now assume that $n \geq 4$. A dimension vector $(1^a, 2^b; n)$ is dense if $a+2b \leq n+1$ by Lemma 4.4 and trivially sparse if $a+2b \geq n+4$ by Lemma 3.1. If $a+2b = n+2$, then we check that $(1^a, 2^b; n)$ is trivially sparse unless $a \leq 3$. If this holds, we use either Lemma 4.8 (if $b \geq 2$ and $n = 4$) or Lemma 4.9 (if $b = 1$ or $b \geq 2$ and $n \geq 5$) to reduce to the dimension vector $(1^a; 2)$, which is dense if $a \leq 3$. If $a+2b = n+3$, then $(1^a, 2^b; n)$ is trivially sparse unless $a \leq 5-n$, so $n \leq 5$ and we get only two vectors $(1, 2^3; 4)$ and $(2^4; 5)$, both of which are dense by Theorem 5.1.

Putting all this together, we obtain the following list of all dense vectors of size 2, listed by excess dimension:

- $(1^a, 2^b; n)$ with $a + 2b \leq n + 1$.
- $(1^a, 2^b; n)$ with $a + 2b = n + 2$ and $a \leq 3$.
- Finitely many vectors with $a + 2b \geq n + 3$: $(2^3; 3)$, $(1, 2^3; 3)$, $(2^4; 3)$, $(1, 2^3; 4)$ and $(2^4; 5)$.

$|\underline{\mathbf{d}}| = 3$: We first consider the cases $n = 4$ and $n = 5$ where Proposition 6.1 does not apply. We have the following cases:

- $n = 4$, $r \leq 4$: By Theorem 5.1, all dimension vectors are dense except for $(1^2, 3^2; 4)$ and $(1, 2^2, 3; 4)$.
- $n = 4$, $r = 5$: Since $r = n+1$, $\underline{\mathbf{d}}$ is trivially sparse unless it is of the form $(1^a, 3^b; 4)$ with $a+b = 5$. On the other hand, if $2 \leq a, b$, then $\underline{\mathbf{d}}$ dominates $(1^2, 3^2; 4)$ and hence is sparse. The remaining cases are $(1^4, 3; 4)$ (dense by Lemma 6.2), $(1, 3^4; 4)$ (dense by taking the complement), and $(3^5; 4)$ (dense by taking the complement).
- $n = 5$, $r \leq 4$: Theorem 5.1 implies that all dimension vectors are dense except for $(1, 3^3; 5)$ and $(2^2, 3^2; 5)$.
- $n = 5$, $r = 5$: All dimension vectors in this case are trivially sparse except for the following three, which can be reduced to smaller dimension vectors that are considered in the previous cases:

$$(1^4, 3; 5) \xrightarrow{\text{Lemma 4.2}} (1^4, 2; 4) \text{ (trivially sparse)},$$

$$(1^3, 2, 3; 5) \xrightarrow{\text{Lemma 4.8}} (1^4; 3) \text{ (dense)},$$

$$(1^3, 3^2; 5) \xrightarrow{\text{complement}} (2^2, 4^3; 5) \xrightarrow{\text{Lemma 4.2}} (2^2, 3^3; 4) \text{ (trivially sparse)}.$$

- $n = 5$, $r = 6$: All dimension vectors in this case are trivially sparse.

We now assume that $n \geq 6$. A dimension vector $(1^a, 2^b, 3^c; n)$ is dense if $a+2b+3c \leq n+1$ by Lemma 4.4 and trivially sparse if $a + 2b + 3c \geq n + 6$ by Lemma 3.1. Hence we need to consider the following cases:

- $a+2b+3c=n+2$. Here the dimension vector $(1^a,2^b,3^c;n)$ is not trivially sparse if $a\le 3c+3$. In this case we apply Lemma 4.2 to reduce to the dimension vector $(1^a,2^{b+1},3^{c-1};n-1)$, and hence by induction to the vector $(1^a,2^{b+c};n-c)$, which is dense if and only if $a\le 3$.
- $a+2b+3c=n+3$. In this case the dimension vector $(1^a,2^b,3^c;n)$ is not trivially sparse if $a+b\le 4$. If this holds, then we use Lemma 4.8 (when $c\ge 2$ and $n=6$) or Lemma 4.9 (when $c=1$ or $n\ge 7$) to reduce to the dimension vector $(1^a,2^b;3)$, which is dense if $a+b\le 4$ and $(a,b)\ne(2,2)$.
- $a+2b+3c=n+4$. The dimension vector $(1^a,2^b,3^c;n)$ is not trivially sparse if $3a+4b+3c\le 15$, so there are only finitely many possibilities. If the length $r=a+b+c\le 4$, then the only possibilities are $(1,3^3;6)$, $(2^2,3^2;6)$ $(2,3^3;7)$, $(3^4;8)$, which are all dense by Theorem 5.1. If $a+b+c=5$ then $b=0$, giving the three vectors $(1^2,3^3;7)$, $(1,3^4;9)$ and $(3^5;11)$. By Lemma 4.9 these vectors reduce to $(1^2,3^3;4)$, $(1,3^4;4)$ and $(3^5;4)$, respectively, so $(1^2,3^3;7)$ is sparse and $(1,3^4;9)$ and $(3^5;11)$ are dense.
- $a+2b+3c=n+5$. In this case the dimension vector $(1^a,2^b,3^c;n)$ is trivially sparse unless $a+b\le 7-n$, so the only possibilities are $(2,3^3;6)$ and $(3^4;7)$, which are dense by Theorem 5.1.

Putting all this together, we obtain a complete list of dense vectors of size 3, listed by excess dimension.

- $(1^a,2^b,3^c;n)$ with $a+2b+3c\le n+1$.
- $(1^a,2^b,3^c;n)$ with $a+2b+3c=n+2$ and $a\le 3$.
- $(1^a,2^b,3^c;n)$ with $a+2b+3c=n+3$, $a+b\le 4$ and $(a,b)\ne 2,2$.
- Finitely many vectors with $a+2b+3c\ge n+4$: $(2,3^2;4)$, $(2^3,3;4)$, $(1,2,3^2;4)$, $(3^3;4)$, $(1,3^3;4)$, $(2,3^3;4)$, $(3^4;4)$, $(1,3^4;4)$, $(2^3,3;5)$, $(1,2,3^2;5)$, $(3^3;5)$, $(1,3^3;5)$, $(2,3^3;5)$, $(3^4;5)$, $(1,3^3;6)$, $(2^2,3^2;6)$, $(2,3^3;6)$, $(2,3^3;7)$, $(3^4;8)$, $(1,3^4;9)$ and $(3^5;11)$.

Before proceeding further, we generalize the method that we used above. For a dimension vector of the form $(1^{e_1},\dots,k^{e_k};n)$ with excess dimension $\sum_{i=1}^k ie_i-n\le k$, we can reduce to a vector of smaller size.

Theorem 6.3. *Let $\underline{\mathbf{d}}=(1^{e_1},\dots,k^{e_k},n)$ be a dimension vector with total dimension $\sum_{i=1}^k ie_i=n+l+1$, where $l<k$ and $e_k>0$. Then $\underline{\mathbf{d}}$ is dense if and only if the dimension vector $(1^{e_1},\dots,l^{e_l};l+1)$ is dense.*

Proof. First assume that $k\ge l+2$. In this case we can repeatedly apply Lemma 4.2 to replace each such k with $l+1$, without changing the excess dimension. Hence we can assume that $k=l+1$, and we have reduced $\underline{\mathbf{d}}$ to the dimension vector $(1^{e_1},\dots,l^{e_l},(l+1)^f;m)$, where $f=\sum_{i=l+1}^k e_i>0$ and $\sum_{i=1}^l ie_i+(l+1)f=m+l+1$. If $f\ge 2$ and $m=2l+4$, then Lemma 4.8 reduces to the dimension vector $(1^{e_1},\dots,l^{e_l};l+1)$ (we drop extra components of dimensions 0 and $l+1$). If $f\ge 2$ and $m\ge 2l+5$, then Lemma 4.9 also reduces to the dimension vector $(1^{e_1},\dots,l^{e_l};l+1)$. Finally, if $f=1$, then either $\underline{\mathbf{d}}$ has no more than two components (in which case the theorem holds trivially), or we can use Lemma 4.9 to reduce to the dimension vector $(1^{e_1},\dots,l^{e_l};l+1)$, which completes the proof.

□

It follows that we've reduced the classification problem of dense dimension vectors $\underline{\mathbf{d}}=(a_1,\dots,a_r;n)$ of size $|\underline{\mathbf{d}}|=l$ and total dimension $\sum_{i=1}^r a_i\le n+l+1$ to the classification of all dense vectors in ambient dimension $l+1$. The following lemma shows that there is only a finite number of additional cases to consider.

Lemma 6.4. *For a given l, there are finitely many dense dimension vectors $(1^{e_1}, \ldots, l^{e_l}; n)$ having excess dimension $\sum_{i=1}^{l} ie_i - n \geq l+1$.*

Proof. We can assume that $n \geq 2l$, since there are finitely many dense dimension vectors in a given ambient dimension. Assume that $\sum_{i=1}^{l} ie_i = n + k + 1$ where $k \geq l$. By Lemma 3.1, if $k \geq 2l$, the dimension vector is trivially sparse. Hence, it suffices to prove that there are finitely many dense dimension vectors for each $l \leq k < 2l$. The dimension vector $(1^{e_1}, \ldots, l^{e_l}; n)$ is trivially sparse unless

$$\sum_{i=1}^{l} e_i i(n-i) \leq n^2 - 1.$$

Using $\sum_{i=1}^{l} ie_i = n + k + 1$, we can reexpress this inequality as

$$n(n+k+1) - \sum_{i=1}^{l} i^2 e_i = n^2 - (k+1)^2 + \sum_{i=1}^{l} (k+1-i) ie_i \leq n^2 - 1.$$

Hence,

$$\sum_{i=1}^{l} (k+1-i) ie_i \leq k(k+2).$$

Since the coefficient of each e_i is positive, there are finitely many such dense dimension vectors. ☐

We now give the full classification of dense dimension vectors of size 4.

$|\underline{\mathbf{d}}| = 4$: We first consider all cases for which Proposition 6.1 does not apply, namely when the ambient dimension n is 5, 6, or 7. Then we consider all possibilities for the length r of $\underline{\mathbf{d}}$.

- $n = 5$, $r \leq 4$: Theorem 5.1 implies that in this case all dimension vectors are dense except for $(1^2, 4^2; 5)$, $(1, 2, 3, 4; 5)$, and $(2^3, 4; 5)$.
- $n = 5$, $r = 5$: We consider those dimension vectors that are not trivially sparse (up to taking complement) one at a time in the following list and show how they can be reduced to smaller vectors.

$$(1^4, 4; 5) \xrightarrow{\text{Lemma 4.2}} (1^4, 3; 4) \text{ (dense)},$$

$$(1^3, 2, 4; 5) \xrightarrow{\text{complement}} (1, 3, 4^3; 5) \xrightarrow{\text{Lemma 4.2}} (1, 3^4; 4) \text{ (dense)},$$

$$(1^3, 3, 4; 5), (1^2, 2^2, 4; 5) \xrightarrow{\text{Lemma 4.3}} (1^2, 4^2; 5) \text{ (sparse)},$$

$$(1^2, a, 4^2; 5), \text{ for } 1 \leq i \leq 4, \text{ dominate } (1^2, 4^2; 5) \text{ (sparse)},$$

$$(1^2, 2, 3, 4; 5) \xrightarrow{\text{Lemma 4.2}} (1^2, 2^2, 3; 4) \text{ (trivially sparse)},$$

$$(1^2, 3^2, 4; 5) \xrightarrow{\text{Lemma 4.10}} (1^3, 2^2; 5) \text{ (dense)},$$

- $n = 5$, $r = 6$: All dimension vectors in this case are trivially sparse unless they have the form $(1^a, 4^b; 5)$ for $a + b = 6$. Also, if $2 \leq a, b$, then the resulting dimension vector dominates $(1^2, 4^2; 5)$ and hence is sparse. Thus the only dense dimension vectors in this case are $(1^5, 4; 5)$ (by Lemma 6.2), $(1, 4^5; 5)$, and $(4^6; 5)$ (consider the complement).
- $n = 6$, $r \leq 4$: By Theorem 5.1, all dimension vectors in this case are dense except for $(1, 3, 4^2; 6)$, $(2^2, 4^2; 6)$, and $(2, 3^2, 4; 6)$.

- $n = 6$, $r = 5$: Here is a list of dimension vectors that are not trivially sparse (up to taking complement) and how to reduce them to smaller ones which are considered in previous cases.

$$(1^4, 4; 6) \xrightarrow{\text{Lemma 4.2}} (1^4, 2; 4) \text{ (trivially sparse)},$$

$$(1^3, 2, 4; 6) \xrightarrow{\text{Lemma 4.2}} (1^3, 2, 3; 5) \text{ (dense by the case } (|\underline{\mathbf{d}}|, n, r) = (3, 5, 5)),$$

$$(1^3, 3, 4; 6) \xrightarrow{\text{complement}} (2, 3, 5^3; 6) \xrightarrow{\text{Lemma 4.2}} (2, 3, 4^3; 5) \text{ (dense by taking complement)},$$

$$(1^3, 4^2; 6) \xrightarrow{\text{complement}} (2^2, 5^3; 6) \xrightarrow{\text{Lemma 4.2}} (2^2, 3^3; 4) \text{ (trivially sparse)},$$

$$(1^2, 2^2, 4; 6) \xrightarrow{\text{Lemma 4.8}} (1^2, 2^2; 4) \text{ (dense by Theorem 5.1)},$$

$$(1^2, 2, 3, 4; 6) \xrightarrow{\text{complement}} (2, 3, 4, 5^2; 6) \xrightarrow{\text{Lemma 4.2}} (2, 3^2, 4^2; 5) \text{ (trivially sparse)},$$

$$(1^2, 2, 3, 4; 6), (1^2, 3^2, 4; 6) \text{ (sparse by Corollary 5.2)},$$

$$(1^2, 3, 4^2; 6) \xrightarrow{\text{Lemma 4.2}} (1^2, 3^3; 5) \text{ (trivially sparse)},$$

$$(1^2, 4^3; 6) \xrightarrow{\text{Lemma 4.10}} (1^2, 2^3; 6) \xrightarrow{\text{Lemma 4.6}} (1^2, 2^2; 4) \text{ (dense by Theorem 5.1)}.$$

- $n = 6$, $r = 6$: All dimension vectors in this case are trivially sparse except for $(1^5, 4; 6)$ which can be reduced by Lemma 4.2 to the smaller vector $(1^5, 3; 5)$.
- $n = 6$, $r = 7$: All dimension vectors in this case are trivially sparse.
- $n = 7$, $r \leq 4$: We know by Theorem 5.1 that all dimension vectors in this case are dense except for $(2, 4, 4, 4; 7)$ and $(3, 3, 4, 4; 7)$.
- $n = 7$, $r = 5$: Let $\underline{\mathbf{d}} = (d_1, \ldots, d_4, 4; 7)$, with $1 \leq d_i \leq 4$, be a vector in this category. It can be easily seen that either $\sum_{i=1}^{3} d_i < 7$ or $\underline{\mathbf{d}}$ is trivially sparse. If $\sum_{i=1}^{3} d_i < 7$, an argument analogous to the proof of Proposition 6.1 implies that the dimension vector $\underline{\mathbf{d}}$ can be reduced to a smaller one.
- $n = 7$, $r = 6$: All dimension vectors in this category are trivially sparse except for the following ones:

$$(1^5, 4; 7) \xrightarrow{\text{Lemma 4.2}} (1^5, 2; 5) \text{ (trivially sparse)},$$

$$(1^4, 2, 4; 7) \xrightarrow{\text{Lemma 4.2}} (1^4, 2, 3; 6) \text{ (trivially sparse)},$$

$$(1^4, 3, 4; 7) \xrightarrow{\text{Lemma 4.8}} (1^5; 4) \text{ (dense)},$$

$$(1^4, 4^2; 7) \xrightarrow{\text{complement}} (3^2, 6^4; 7) \xrightarrow{\text{Lemma 4.2}} (3^2, 5^4; 6) \text{ (trivially sparse)}.$$

- $n = 7$, $r = 7$: All dimension vectors in this case are trivially sparse except for $(1^6, 4; 7)$ which can be reduced by Lemma 4.2 to the smaller dimension vector $(1^6, 3; 6)$.
- $n = 7$, $r = 8$: All dimension vectors in this category are trivially sparse by Lemma 6.2.

We can now use Theorem 6.3 and Lemma 6.4 to give a complete list of dense dimension vectors of size 4, listed by excess dimension:

- $(1^a, 2^b, 3^c, 4^d; n)$ with $a + 2b + 3c + 4d \leq n + 1$.
- $(1^a, 2^b, 3^c, 4^d; n)$ with $a + 2b + 3c + 4d = n + 2$ and $a \leq 3$.
- $(1^a, 2^b, 3^c, 4^d; n)$ with $a + 2b + 3c + 4d = n + 3$, $a + b \leq 4$ and $(a, b) \neq (2, 2)$.
- $(1^a, 2^b, 3^c, 4^d; n)$ with $a + 2b + 3c + 4d = n + 4$ and such that $(1^a, 2^b, 3^c; 4)$ is dense. This means that either $a + b + c \leq 3$, or $a + b + c = 4$ and $a + 2b + 3c \neq 8$, or $a + c = 5$ and $b = 0$ with either $a \leq 1$ or $c \leq 1$.

- A finite set of dimension vectors $(1^a, 2^b, 3^c, 4^d; n)$ with $a + 2b + 3c + 4d \geq n + 5$. For $n \leq 7$ all these vectors are given above, and for $n \geq 8$ these vectors can be found by solving the inequality in Lemma 6.4 and using Proposition 6.1 to reduce to known cases.

We summarize the results of this section. To classify all dense vectors of size l, it is necessary to first classify all dense vectors of ambient dimension up to $l+1$. These vectors generate infinite families of dense vectors of size l by Theorem 6.3, having excess dimension at most l. There are finitely dense vectors having excess dimension greater than l. For ambient dimensions between $l+1$ and $2l-1$ these need to be found by hand, and the remaining ones can be found by Lemma 6.4 and Proposition 6.1.

7. The equidimensional case

In this section, using the reduction lemmas of Section 4, we characterize the density of the equidimensional vectors $\underline{\mathbf{d}} = (k^t; n)$. We will use the exponential notation for our dimension vectors. By taking complements if necessary, we can always assume that $n \geq 2k$. By the division algorithm, write $n = mk + r$, where $0 \leq r < k$. We then have the following theorem.

Theorem 7.1. *Let $k > 0$, $m \geq 2$ and $0 \leq r < k$ be nonnegative integers.*

(I) *The dimension vector $\underline{\mathbf{d}} = (k^t; mk + r)$ is dense if $t \leq m + 1$.*
(II) *The dimension vector $\underline{\mathbf{d}} = (k^t; mk + r)$ is trivially sparse if $t \geq m + 3$ or if $t = m + 2$ and $(m-2)k(k-r) > r^2 - 1$.*
(III) *The dimension vector $\underline{\mathbf{d}} = (k^{m+2}; mk+r)$ is dense if and only if the dimension vector $\underline{\mathbf{d}}' = ((k-r)^{m+2}; 2k-r)$ is dense.*

Proof. Suppose $t \geq m + 3$, then

$$tk(k(m-1)+r) \geq (m+3)k(km-k+r) \geq m^2k^2 + (2m-3)k^2 + (m+3)kr. \tag{4}$$

If $m = 2$, the right hand side of (4) is equal to $5k^2 + 5kr > 4k^2 + 4kr + r^2 - 1$ since $k > r$. Hence, the vector is trivially sparse. If $m \geq 3$, since $(m-3)k^2 + (m+3)kr \geq 2mkr$, the right hand side of (4) is greater than

$$m^2k^2 + 2mkr + mk^2 > m^2k^2 + 2mkr + r^2 - 1.$$

Hence, if $t \geq m + 3$, the dimension vector $\underline{\mathbf{d}} = (k^t; mk + r)$ is trivially sparse. If $t = m + 2$,

$$(m+2)k(mk-k+r) = m^2k^2 + (m-2)k^2 + (m+2)kr = m^2k^2 + 2mkr + (m-2)k(k-r).$$

Hence, the dimension vector $\underline{\mathbf{d}} = (k^t; mk+r)$ is trivially sparse if $t = m+2$ and $(m-2)k(k-r) > r^2 - 1$. This concludes the proof of part (2) of the theorem.

When $t = m + 2$, we may assume that $r > 0$. By Lemma 4.9, $\underline{\mathbf{d}} = (k^{m+2}; mk + r)$ is dense if the dimension vector $\underline{\mathbf{d}}' = ((k-r)^{m+2}; 2k-r)$ is dense. On the other hand, the converse is immediate. Therefore, we deduce part (3) of the theorem.

Finally, to prove part (1) of the theorem, consider the dimension vector $(1^r, k^{m+1}; mk+r)$. By Lemma 4.7, this vector is dense if $(1^r, k^m; mk - k + r)$ is dense. By induction on m, we conclude that it suffices to check the density of $(1^r, k^2; k + r)$. By another application of Lemma 4.7, this vector is dense if $(1^r; k)$ is dense. The latter vector is clearly dense since $r < k$. We conclude that $(1^r, k^{m+1}; mk+r)$ is dense. Since this vector dominates $(k^t; mk+r)$ when $t \leq m + 1$, we conclude part (1) of the theorem. $\square$

Remark 7.2. Theorem 7.1 leads to a complete characterization of equidimensional vectors. By Theorem 7.1, the vector $(k^t; mk+r)$ is dense if $t \leq m+1$ and trivially sparse if $t \geq m+3$. Hence, we only need to consider the case $t = m + 2$. Since Theorem 5.1 characterizes dense dimension vectors of length 4, we may assume $m \geq 3$. By Theorem 7.1 (2), $(k^{m+2}, mk+r)$ is trivially sparse if $(m-2)k(k-r) > r^2 - 1$. In particular, if $r = 0$, the dimension vector is trivially sparse. In fact, unless $r \geq \frac{\sqrt{5}-1}{2}k$, then the dimension vector is trivially sparse. Hence, Theorem 7.1 replaces checking the density of $(k^{m+2}; mk+r)$ to checking the density of $((k-r)^{m+2}; 2k-r)$, which is also an equidimensional vector where the dimensions are less than half the original dimension. Hence, Theorem 7.1 gives a very fast algorithm for checking the density of $(k^{m+2}; mk+r)$.

8. Further examples and questions

In the previous three sections, using the reduction techniques of Section 4, we classified dense and sparse dimension vectors of small length or size and dense equidimensional vectors. In this section, we show how these techniques produce examples of dense dimension vectors of large length and size. Lemma 4.6 and Lemma 4.7 proved in Section 4 have the special feature of reducing the length and the ambient dimension at the same time. This allows us to employ these lemmas in reverse and produce sequences of dense dimension vectors with large length and size out of a given dense dimension vector.

We start by applying Lemma 4.7. Recall that the Fibonacci sequence $\{F_i\}_{i \geq 0}$ is the sequence of nonnegative integers recursively defined by acquiring $F_0 = 0$, $F_1 = 1$, and $F_{i+2} = F_{i+1} + F_i$ for all $0 \leq i$.

Proposition 8.1. *Let $(a_1, \ldots, a_r; b)$ be a dense dimension vector such that for any $1 \leq t \leq r$, $b + a_t \leq n := \sum_{i=1}^r a_i$. Then for every $k \geq 0$, the dimension vector*

$$\underline{\mathbf{d}}_k := (a_1, \ldots, a_r, b, F_1.n + F_0.b, F_2.n + F_1.b, \ldots, F_k.n + F_{k-1}.b; F_{k+1}.n + F_k.b),$$

where $\{F_i\}_{0 \leq i}$ is the Fibonacci sequence, is dense.

Proof. The proof is by induction on k. For the base case of the induction, we need to show that the dimension vector $\underline{\mathbf{d}}_0 = (a_1, \ldots, a_r, b; n)$ is dense, which follows from our assumptions and Lemma 4.7. Assume that $\underline{\mathbf{d}}_k$ is dense. Lemma 4.7 implies that the dimension vector

$$(a_1, \ldots, a_r, b, F_1.n + F_0.b, \ldots, F_{k+1}.n + F_k.b; (1 + \sum_{i=0}^{k} F_i).n + (1 + \sum_{i=0}^{k-1} F_i).b)$$

is dense. Hence, it suffices to show that

$$F_{t+2} = 1 + \sum_{i=0}^{t} F_i \tag{5}$$

for any $t \geq 0$. We verify the identity (5) by induction on t. The case $t = 0$ is evident as $F_2 = 1 = 1 + F_0$. Assuming (5), we can compute

$$F_{t+3} = F_{t+2} + F_{t+1} = 1 + \sum_{i=0}^{t} F_i + F_{t+1} = 1 + \sum_{i=0}^{t+1} F_i.$$

This concludes the induction and the proof of the proposition. □

Example 8.2. If we apply the above proposition to the dense dimension vector $(1, 1, 1; 2)$, with $n = 3$ and $b = 2$, we get the sequence

$$\underline{\mathbf{d}}_k = (1, 1, 1, 2, 3F_1 + 2F_0, 3F_2 + 2F_1, \ldots, 3F_k + 2F_{k-1}; 3F_{k+1} + 2F_k)$$

of dense dimension vectors. On the other hand, one can easily check by induction on k that $3F_k + 2F_{k-1} = F_{k+3}$ (in fact, more generally, $F_tF_k + F_{t-1}F_{k-1} = F_{t+k-1}$ for all natural numbers t and k). Therefore, we obtain that for any $k \geq 0$ the dimension vector

$$\underline{\mathbf{d}}_k = (1, F_1, F_2, \ldots, F_{k+3}; F_{k+4})$$

is dense. Similarly, by applying the above proposition to the dense dimension vectors that we found in previous sections, we can construct infinitely many sequences of "Fibonacci type" that are dense.

Proposition 8.3. *Let $(a_1, \ldots, a_r, b; n)$ be a dense dimension vector with $n = \sum_{i=1}^r a_i$. Then, for every $k \geq 1$, the dimension vector*

$$\underline{\mathbf{d}}_k = (a_1, \ldots, a_r, b^k; n + (k-1)b)$$

is dense.

Proof. We prove the assertion by induction on k. The induction basis $k = 1$ is our hypothesis. If we assume that the dimension vector $\underline{\mathbf{d}}_k$ is dense, Lemma 4.6 implies that the dimension vector $\underline{\mathbf{d}}_{k+1}$ is dense, too. □

The reduction lemmas can be applied to study the density of arbitrary dimension vectors with size bounded by $n/2$. Unfortunately, after the reduction the dimension of some of the vector spaces might be more than half the ambient dimension. The same techniques can be applied even when some of the dimensions are greater than $n/2$. However, after the reduction, the new problem is no longer the density of the $\mathbb{P}GL(n)$ action on a product of Grassmannians, but the density of the action on a subvariety of a product of flag varieties. This was already encountered in the proof of Lemma 4.9. The following example is typical.

Example 8.4. We show that the dimension vector $(5, 5, 5, 5, 13; 14)$ is not dense. The reader can check that the same argument works for dimension vectors of the form $(k, k, k, k, 3k - 2; 3k - 1)$. Let U_1, U_2, U_3, U_4, W be general linear subspaces of a 14-dimensional vector space X, where $\dim U_i = 5$ for $1 \leq i \leq 4$ and $\dim W = 13$. We first reduce checking the density to checking the density of a configuration where the ambient vector space has dimension 10.

Let $V = U_1 + U_2$. Let $U_i' = U_i \cap V$ for $i = 3, 4$ and let $W_1 = W \cap V$. Finally, let

$$W_2 = ((U_3 \cap W) + (U_4 \cap W)) \cap V.$$

Then there is a natural restriction morphism

$$f : \mathrm{Stab}(U_1, U_2, U_3, U_4, W; X) \to \mathrm{Stab}(U_1, U_2, U_3', U_4', W_1, W_2; V).$$

The new twist in this case is that $W_2 \subset W_1$, hence the new data is not a point of a product of Grassmannians, but of a product of partial flag varieties. Since the expected dimensions of the stabilizers are equal and both configurations can be taken to be generic, the density of one configuration is equivalent to the density of the other configuration by Lemma 2.2.

Next we reduce the problem to one where the ambient dimension is 6. Set $V' = U_3' + U_4' + W_2$. Let $U_i' = U_i \cap V'$ for $i = 1, 2$, let $W_1' = W_1 \cap V'$ and

$$W_3 = ((U_1 \cap W_1) + (U_2 \cap W_1)) \cap V'.$$

There is a natural restriction morphism

$$f : \mathrm{Stab}(U_1, U_2, U_3', U_4', W_1, W_2; V) \to \mathrm{Stab}(U_1', U_2', U_3', U_4', W_1', W_2, W_3; V').$$

Again both configurations are generic subject to the restriction $W_2 \subset W_1$ and $W_2, W_3 \subset W_1'$ and by Lemma 2.2 the density of one is equivalent to the density of the other. The new twist at this stage is that both $W_2, W_3 \subset W_1'$. Hence, this is a configuration parameterized by a subvariety of a product of Grassmannians and flag varieties defined by imposing some linear conditions on the vector spaces. At this stage, it is clear that the configuration is not dense since the configuration $(1, 1, 4, 4; 5)$ exists as a subconfiguration by taking $T_1 = (U_1' + U_3') \cap W_1'$, $T_2 = (U_2' + U_4') \cap W_1'$.

We can speculate that whenever a dimension vector is not dense, there is always a configuration of vector spaces obtained by repeatedly taking spans, intersections and projections that gives a configuration which is trivially sparse. Based on the previous example and to get a better inductive set up, the following generalization of Question 1.1 may be more natural.

Question 8.5. *Let X be a subvariety of a product of flag varieties $\prod_{i=1}^{k} F(d_{i,1}, \dots, d_{i,j_i}; n)$ obtained by imposing linear relations on the vector spaces. When does the diagonal action of $\mathbb{P}GL(n)$ have a dense orbit on X?*

One can also generalize the question to other homogeneous varieties.

Question 8.6. *Let G be a semisimple linear algebraic group and let P_i, for $1 \leq i \leq n$, be parabolic subgroups. When does the diagonal action of G have a dense orbit on $\prod_{i=1}^{n} G/P_i$?*

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