THE ZERO SECTION OF THE UNIVERSAL SEMIABELIAN VARIETY, AND THE DOUBLE RAMIFICATION CYCLE

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Abstract. We study the Chow ring of the boundary of the partial compactification of the universal family of principally polarized abelian varieties. We describe the subring generated by divisor classes, and compute the class of the partial compactification of the universal zero section, which turns out to lie in this subring. Our formula extends the results for the zero section of the universal uncompactified family.

The partial compactification of the universal family of ppav can be thought of as the first two boundary strata in any toroidal compactification of $\mathcal{A}_g$. Our formula provides a first step in a program to understand the Chow groups of $\overline{\mathcal{A}}_g$, especially of the perfect cone compactification, by induction on genus. By restricting to the image of $\mathcal{M}_g$ under the Torelli map, our results extend the results of Hain on the double ramification cycle, answering Eliashberg's question.

Introduction

We are mainly interested in the Chow and cohomology groups of (compactified) moduli spaces of principally polarized abelian varieties (ppav). The tautological ring of the moduli space of ppav $\mathcal{A}_g$ is defined as the subring $R^*(\mathcal{A}_g) \subset A^*(\mathcal{A}_g)$ of the Chow ring (or of cohomology ring $RH^*(\mathcal{A}_g) \subset H^*(\mathcal{A}_g)$) generated by the Chern classes $\lambda_i := c_i(\mathbb{E})$ of the rank g Hodge bundle $\mathbb{E} \to \mathcal{A}_g$, with fiber over A being $H^{1,0}(\mathbb{C})$. Unlike the case of curves, this tautological ring is known completely. For a suitable toroidal compactification $\overline{\mathcal{A}}_g$ van der Geer [vdG99] proved that $RH^*(\overline{\mathcal{A}}_g)$ is generated by the λ_i with the only relations being the homogeneous degree pieces of the basic relation

$$(1) \qquad (1 + \lambda_1 + \cdots + \lambda_g)(1 - \lambda_1 + \cdots + (-1)^g \lambda_g) = 1.$$

In cohomology this relation follows from the triviality of $\mathbb{E} \oplus \overline{\mathbb{E}}$, the total space of the bundle of first cohomology. Esnault and Viehweg

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[EV02] proved the much more delicate result that it also holds in the Chow ring of $\overline{\mathcal{A}}_g$, which implies that $R^*(\overline{\mathcal{A}}_g) = RH^*(\overline{\mathcal{A}}_g)$.

Furthermore, van der Geer [vdG99] also proved that the tautological ring $R^*(\mathcal{A}_g) = RH^*(\mathcal{A}_g)$ is obtained from $R^*(\overline{\mathcal{A}}_g)$ by imposing one more relation $\lambda_g = 0$. Thus in $R^*(\overline{\mathcal{A}}_g)$ the class λ_g can be represented by a cycle supported on the boundary, and it is a natural question to find a suitable representative for it. A lot of progress on this was made by Ekedahl and van der Geer [EvdG05],[EvdG04]. In particular, in characteristic p suitable cycles were constructed, but over $\mathbb{C}$ this question remains open. Note that in characteristic zero Keel and Sadun [KS03] proved Oort's conjecture that $\mathcal{A}_g$ does not have complete subvarieties of codimension g.

One naturally defined geometric locus in $\overline{\mathcal{A}}_g$ is the locus δ_g, the closure of the locus of trivial extensions of semiabelic varieties of torus rank one. This locus was introduced and studied by Ekedahl and van der Geer [EvdG05]. We denote $\mathcal{A}'_g \supset \mathcal{A}_g$ Mumford's partial compactification, obtained by adding semiabelic varieties of torus rank one (compactifications of $\mathbb{C}^*$-extensions of $(g-1)$-dimensional ppav). The boundary $\mathcal{A}'_g \setminus \mathcal{A}_g$ is then the universal family of $(g-1)$-dimensional Kummer varieties (quotients of ppav by the -1 involution), and admits the zero section. The class δ_g is defined to be the class of the closure of the image of the zero section in a suitable toroidal compactification $\overline{\mathcal{A}}_g$ (recall that all toroidal compactifications contain $\mathcal{A}'_g$). Ekedahl and van der Geer show that on $\overline{\mathcal{A}}_g$ the class λ_g is equal to $(-1)^g\zeta(1-2g)\delta_g$ up to classes supported deeper in the boundary, on $\overline{\mathcal{A}}_g \setminus \mathcal{A}'_g$, in other words that on $\mathcal{A}'_g$ the class λ_g is proportional to δ_g. Thus understanding the class δ_g could lead to finding an explicit geometric cycle representing λ_g in characteristic zero. A study of the locus δ_g is also natural since Shepherd-Barron [SB06] showed that in the perfect cone toroidal compactification $\mathcal{A}_g^{\mathrm{Perf}}$ the normalization of the closure of the zero section is equal to $\mathcal{A}_{g-1}^{\mathrm{Perf}}$. Thus a full understanding of the class δ_g could provide an inductive approach for understanding the cohomology of the perfect cone compactification, for example addressing the conjecture of [EGH10] on intersection numbers of divisors on $\mathcal{A}_{g-1}^{\mathrm{Perf}}$. We note that for $g \le 3$ the locus δ_g was fully described, and its class in $\overline{\mathcal{A}}_g$ was computed completely by van der Geer in [vdG98], but nothing was previously known for higher g.

Denote $\mathcal{X}_g \to \mathcal{A}_g$ the universal family of ppav, and by $\mathcal{X}'_g \to \mathcal{A}'_g$ its partial compactification (note that the existence of a universal compactified family $\overline{\mathcal{X}}_g$ over a full toroidal compactification $\overline{\mathcal{A}}_g$ is only known for the second Voronoi toroidal compactification [Ale02]). Our main result is the computation of the class of the closure of the zero section $z'_g : \mathcal{A}'_g \to \mathcal{X}'_g$, which turns out to be a polynomial in divisor classes and a natural codimension two "gluing locus" in $\mathcal{X}'_g$, see Theorem 1.1. Moreover, we prove that the divisor classes and this codimension two class generate a certain geometric subring of the Chow ring of $\mathcal{X}'_g$, see Theorem 1.3. We also describe the algebraic cohomology of the universal family $\mathcal{X}_g \times_{\mathcal{A}_g} \mathcal{X}_g$ of products of ppavs, see Theorem 3.1. Since $\mathcal{X}_{g-1}$ is a cover of the boundary of $\mathcal{A}'_g$, our results mean that on $\overline{\mathcal{A}}_g$ we compute the class δ_g up to the stratum parameterizing semiabelic varieties of torus rank two.

Our results also have consequences for the moduli space of curves $\mathcal{M}_g$. Recall that the tautological ring $R^*(\mathcal{M}_g) \subset A^*(\mathcal{M}_g)$ (or similarly for the Deligne-Mumford compactification $\overline{\mathcal{M}}_g$ or the partial compactification by stable curves of compact type $\mathcal{M}_g^{ct}$) is defined to be the subring generated by the Mumford-Morita-Miller classes κ_i. Faber's conjecture [Fab99] states that the tautological ring of $\mathcal{M}_g$ (resp. of $\mathcal{M}_g^{ct}, \overline{\mathcal{M}}_g$) is Gorenstein with socle in dimension $g-2$ (resp. $2g-3, 3g-3$) — such rings are also called Poincaré duality rings. That is to say that the tautological ring is zero above the socle dimension, one-dimensional in the socle dimension, and has perfect pairing to the socle dimension. While the vanishing and the one-dimensionality are known, see [Loo95],[Fab97],[Ion02],[GV05], the perfect pairing part, and in general the structure of the ideal of relations among the κ_i remain mysterious, and are currently under intense investigation, see e.g. [Pan09]. Another interesting question is whether classes of various naturally defined geometric loci in $\mathcal{M}_g$ are tautological, see [FP05], and whether the classes of their closures in $\overline{\mathcal{M}}_g$ are tautological. For tautological classes on $\overline{\mathcal{M}}_g$ that vanish in $\mathcal{M}_g$ one can also ask to find explicit geometric representatives — in particular this question is of interest for the class λ_g, which is a pullback from the moduli space of ppav.

Our results on $\mathcal{X}'_g$ yield a further understanding of a natural codimension g class on $\mathcal{M}_{g,n}$: the two-branch-point locus, also called the double ramification locus [FSZ10]. It is defined as locus of $(C, p_1, \dots, p_n) \in \mathcal{M}_{g,n}$ such that a linear combination $\sum d_i p_i$ is a principal divisor (for some fixed $d_i \in \mathbb{Z}, \sum d_i = 0$). This is a natural "double Hurwitz" locus of curves admitting a map to $\mathbb{P}^1$ with prescribed preimages and ramification at 0 and ∞. Its class is also of interest in Gromov–Witten theory (see [FSZ10]), and the question of computing it is due to Eliashberg. The class of the closure of this locus in $\mathcal{M}_{g,n}^{ct}$ was recently computed by Hain [Hai11], and we extend his computation further into the boundary of $\overline{\mathcal{M}}_g$, to the open subset of the boundary δ_{irr} where the geometric genus is $g-1$ — this is precisely the preimage of the partial compactification of $\mathcal{A}_g$, and we obtain the result by pulling back our computations there.

While for $\overline{\mathcal{M}}_g$ the classes of the boundary divisors, and possibly the classes of closure of various geometric loci, are tautological [FP05], for $\overline{\mathcal{A}}_g$ already the boundary divisor(s) are non-tautological (as their classes are clearly not proportional to λ_1). Thus defining a suitable "extended" tautological subring of $A^*(\overline{\mathcal{A}}_g, \mathbb{Q})$ is a natural central further question to study; one could hope that such a ring would be defined geometrically, and would contain the classes of geometrically defined loci. Some results in this direction were obtained by the first author and Hulek in [GH11a], but the situation is far from clear, and studying natural geometric loci in $\overline{\mathcal{A}}_g$ is thus of particular interest.

Note also that since the Deligne-Mumford compactification $\overline{\mathcal{M}}_g$ admits a morphism to the second Voronoi [Nam80] and perfect cone [AB11] toroidal compactifications of $\mathcal{A}_g$, restricting a geometric cycle representing λ_g on $\overline{\mathcal{A}}_g$ to the image of the Torelli map would allow one to relate the tautological rings of $\overline{\mathcal{M}}_g$ and $\mathcal{M}_g^{ct}$ (which is the preimage of $\mathcal{A}_g$ under the Torelli map, as a stack), and perhaps to obtain a direct computational proof of the λ_g-conjecture (proven by Faber and Pandharipande [FP03], to which we also refer for a discussion).

1. Statement of results

The principal result of our paper is the computation of the class of the closure of the zero section of the universal abelian variety in the partial compactification $\mathcal{X}_g' \to \mathcal{A}_g'$.

Theorem 1.1. *Let $\mathcal{X}_g' \to \mathcal{A}_g'$ denote the partial compactification of the universal family of ppav $\mathcal{X}_g \to \mathcal{A}_g$, let $z_g : \mathcal{A}_g \to \mathcal{X}_g$ denote the zero section, let $z_g' : \mathcal{A}_g' \to \mathcal{X}_g'$ denote the closure of the zero section in the partial compactification, and let Z_g' denote its class in $A^g(\mathcal{X}_g', \mathbb{Q})$. Then we have*

$$Z_g' = \sum_{a+b+2c=g} \alpha_{a,b,c} (\Theta - D/8)^a D^b (\Delta - 2\Theta D)^c, \tag{2}$$

where the positive coefficients $\alpha_{a,b,c}$ are given by

$$\alpha_{a,b,c} = \frac{(-1)^{b+c+1}(2^{-b-c} - 2^{1-3b-3c})(2a+2b+2c-1)!!B_{2b+2c}}{(2a+2c-1)!!(2b+2c-1)!!a!b!c!}. \tag{3}$$

Here $\Theta \in A^1(\mathcal{X}_g', \mathbb{Q})$ denotes the class of the universal theta divisor trivialized along the zero section, $D \in A^1(\mathcal{X}_g', \mathbb{Q})$ denotes the class of *the boundary* $\mathcal{X}_g'\setminus\mathcal{X}_g$*, and* $\Delta\in A^2(\mathcal{X}_g',\mathbb{Q})$ *denotes the class of the gluing locus within* D*, where the* 0 *and* ∞ *sections of the universal Poincaré bundle that is the total space of* $\mathcal{X}_g'\setminus\mathcal{X}_g$ *are identified (all considered non-stacky, see below for details).*

Remark 1.2. The classes $\Theta-D/8$ and $\Delta-2\Theta D$ above may seem like a random choice, but in fact have a geometric significance. Indeed, $\Theta-D/8$ is in a sense the class of the theta divisor, with generic vanishing on the boundary taken out, and appears for example in Grothendieck-Riemann-Roch computations in [EvdG05], while $\Delta-2\Theta D$ is a natural "shift-invariant" class (see below), and corresponds to the Casimir tensor of $SL(2)$, acting on the fiberwise square of the universal family of ppav (this phenomenon will be explored in more generality in a forthcoming work of the authors).

Equivalently, the class of the partial compactification of the zero section can be written as

$$Z_g'=\sum_{a+b+2c=g}\eta_{a,b,c}\Theta^a D^b\Delta^c,$$

where the coefficients $\eta_{a,b,c}$ are equal to

$$\frac{(-1)^{b+c}(2c+2b-1)!!}{2^{3b+3c}a!c!}\sum_{x=0}^{b}\frac{(2-2^{2c+2x})B_{2c+2x}}{(2c+2b-2x-1)!!(2c+2x-1)!!(b-x)!x!}.$$

We note that as $\mathcal{X}_g/\pm 1$ is the boundary of the partial compactification $\mathcal{A}_{g+1}'$, we can interpret the above result as computing the class $\delta_{g+1}\in A^*(\overline{\mathcal{A}}_{g+1},\mathbb{Q})$ up to the second boundary stratum, of semiabelic varieties of torus rank two — see Remark 5.2 for more details on this.

The theorem above was surprising to us, as it claims that Z_g', which is a codimension g class, admits a polynomial expression in classes of degree 1 and 2. However, this turns out to be a fairly general phenomenon. Namely, we prove the following result.

Theorem 1.3. *Let* $\widetilde{Y}$ *denote the normalization of the boundary of the partial compactification* $\mathcal{X}_g'\to\mathcal{A}_g'$*. Any class in* $A^*(\mathcal{X}_g',\mathbb{Q})$ *whose pullback to* $\widetilde{Y}$ *is a polynomial in divisor classes on* $\widetilde{Y}$ *can be expressed on* $\mathcal{X}_g'$ *as a polynomial in the three classes* Θ*,* D *and* Δ*.*

Along the way of proving these results, we also further investigate the geometry and intersection theory of the total space of the universal Poincaré bundle (i.e. of $\mathcal{X}_g'\setminus\mathcal{X}_g$), which may be of independent interest. We also investigate the relations among Chow classes on the fiber square of a very general ppav, which may lead to further generalizations and questions.

Turning to the moduli space of curves, we apply the theorem above to obtain a partial answer to the following question of Eliashberg. Let $\underline{d} = (d_1, \ldots, d_n) \in \mathbb{Z}^n$ be integers summing to zero, and consider the locus $R_{\underline{d}}$ of curves $(X, p_1, \ldots, p_n) \in \mathcal{M}_{g,n}$ such that $\sum d_i p_i$ is a principal divisor on X. This locus is known as the *double ramification locus*, and the question is to compute the class of its closure in $\overline{\mathcal{M}}_{g,n}$.

On a smooth curve X, the divisor $\sum d_i p_i$ is principal if and only if its image in $\operatorname{Jac}(X)$ is zero. Therefore, the double ramification locus can be computed by pulling back the zero section of the universal Jacobian under the Abel–Jacobi map $s_{\underline{d}} : \mathcal{M}_{g,n} \to \operatorname{Jac}$ that sends $(X, p_1, \ldots, p_n)$ to $\sum d_i p_i \in \operatorname{Jac}(X)$. This map naturally extends to curves of compact type, since the Jacobians of such curves are abelian varieties, and was used by Hain in [Hai11] to compute the class of the closure of $R_{\underline{d}}$ in $\mathcal{M}^{ct}_{g,n}$.

In this paper, we take this approach one step further. The Abel–Jacobi map does not extend to a morphism over all of $\overline{\mathcal{M}}_{g,n}$, but it does extend to a morphism over the open part $\overline{\mathcal{M}}^o_{g,n}$ of $\overline{\mathcal{M}}_{g,n}$ parameterizing stable curves whose normalization has genus at least $g-1$. The compactified Jacobians of such curves are semiabelic varieties of torus rank one, and there exists an Abel–Jacobi map from $\overline{\mathcal{M}}^o_{g,n}$ to the partial compactification $\mathcal{X}'_g \to \mathcal{A}'_g$. Computing the class of the zero section in the partial compactification and pulling it back, we find the class of the closure of the double ramification locus in $\overline{\mathcal{M}}^o_{g,n}$.

Theorem 1.4. *Let $\overline{\mathcal{M}}^o_{g,n}$ be the open subset of $\overline{\mathcal{M}}_{g,n}$ parameterizing curves with geometric genus at least $g-1$. Let $\underline{d} = (d_1, \ldots, d_n) \in \mathbb{Z}^n$ be integers summing to zero, and let $R_{\underline{d}}$ denote the double ramification locus defined above. Then the class of the closure $\overline{R}_{\underline{d}}$ of $R_{\underline{d}}$ in $\overline{\mathcal{M}}^o_{g,n}$ is equal in $A^g(\overline{\mathcal{M}}^o_{g,n}, \mathbb{Q})$ to*

$$[\overline{R}_{\underline{d}}] = \sum_{a+b+2c=g} \eta_{a,b,c} (s^*_{\underline{d}}\Theta)^a \delta^b_{irr} (s^*_{\underline{d}}\Delta)^c.$$

Here $s^*_{\underline{d}}\Theta$ and $s^*_{\underline{d}}\Delta$ denote the pullbacks of the classes Θ and Δ to $\overline{\mathcal{M}}^o_{g,n}$ under the Abel–Jacobi map, and the coefficients $\eta_{a,b,c}$ are the same as in Theorem 1.1. These pullbacks can be expressed in terms of classes in $A^*(\overline{\mathcal{M}}^o_{g,n}, \mathbb{Q})$ in the following way:

$$s^*_{\underline{d}}\Theta = \frac{1}{2}\sum_{i=1}^n d_i^2 K_i - \frac{1}{2}\sum_{P\subseteq I}\left(d_P^2 - \sum_{i\in P} d_i^2\right)\delta_0^P - \frac{1}{2}\sum_{h>0, P\subseteq I} d_P^2 \delta_h^P,$$

$$s_{\underline{d}}^*\Delta = \sum_{i=1}^{n} |d_i|\xi_i.$$

Here K_i and δ_h^P are the standard divisor classes on $\overline{\mathcal{M}}_{g,n}^o$ (see Section 6 for details), $I = \{1,\ldots,n\}$ is the indexing set, $d_P = \sum_{i\in P} d_i$, and ξ_i is a codimension two class in $\overline{\mathcal{M}}_{g,n}^o$ whose generic point is a rational curve containing the i-th marked point attached at two nodes to a smooth genus $g-1$ curve containing the remaining marked points.

Remark 1.5. On the moduli space of curves of compact type this formula restricts to the result of Hain [Hai11], while on the moduli space of curves with rational tails (having a smooth component of maximum genus) this formula restricts to the result of Cavalieri, Marcus, and Wise [CMW11].

We would like to stress that while $\overline{\mathcal{M}}_{g,n}^o \setminus \mathcal{M}_{g,n}^{ct}$ is an irreducible divisor, computing a codimension g class on $\overline{\mathcal{M}}_{g,n}^o$ involves much more than computing it on $\mathcal{M}_{g,n}^{ct}$, and then computing one extra coefficient.

The structure of the paper is as follows. In Section 2 we introduce the notation, and review the known results on the geometric structure of the boundary of $\mathcal{X}_g'$ (which is also the second stratum of the boundary of $\overline{\mathcal{A}}_{g+1}$), mostly following [EGH10]. In Section 3 we study the subring of its Chow ring generated by the divisor classes. In Section 4 we study the normalization of the boundary of $\mathcal{X}_g'$ and describe the classes on the normalization that glue to classes on the actual boundary of $\mathcal{X}_g'$, culminating with a proof of Theorem 1.3. In Section 5 we study the closure of the zero section and obtain an expression for it, proving Theorem 1.1. Finally, in Section 6 we use this theorem, together with standard intersection techniques on $\overline{\mathcal{M}}_g$, to obtain an answer to Eliashberg's problem, proving Theorem 1.4.

2. Notation and known results

Throughout the text, we work with Chow groups with rational coefficients. The spaces that we work with are smooth Deligne-Mumford stacks, and thus the Chow groups admit a ring structure (below, we specifically avoid working with the Chow groups of the non-normal boundary $Y = \mathcal{X}_g' \setminus \mathcal{X}_g$).

We denote by $\pi : \mathcal{X}_g / \pm 1 \to \mathcal{A}_g$ the universal family of Kummer varieties (which are quotients of ppav by the involution ± 1), and denote by $z_g : \mathcal{A}_g \to \mathcal{X}_g$ its zero section. By abuse of notation, we also denote z_g the image of the zero section as a locus in $\mathcal{X}_g$, and denote Z_g its class in $A^g(\mathcal{X}_g)$. We denote by $T \subset \mathcal{X}_g$ the universal symmetric theta divisor trivialized along the zero section (so that $T|_{z_g}$ is trivial).

Our problem is motivated by the following result:

Theorem 2.1 ([Hai11] in homology, implied by the results of [DM91] in Chow, see [BL04])**.** *The class of the zero section in $A^g(\mathcal{X}_g)$ and in $H^{2g}(\mathcal{X}_g)$ is equal to*

$$Z_g = \frac{T^g}{g!}.$$

Remark 2.2. This result has a long history, and many approaches to it have been developed. We are grateful to Richard Hain, Claire Voisin, and Gerard van der Geer for discussions on these topics. Indeed, Hain [Hai11, Prop. 8.1] proves this result using Hodge-theoretic methods, while the argument in the Chow ring uses the Fourier transform on the Chow ring, and is based on ideas of Deninger and Murre, including [DM91, Cor. 2.22]. We refer to [BL04, Cor. 16.5.7] for a complete proof, which is also given in [vdGM12, Exercise 13.2]. We also refer to Section 3 for more results and a discussion of the relationship of the Poincaré bundle and the class T.

The goal of this paper is to extend this formula to Mumford's partial compactification of the moduli space of ppav, which we denote by $\mathcal{A}_g'$. In this section, we recall the construction of the universal family over the partial compactification.

The partial compactification is the blow-up of the partial Satake compactification $\mathcal{A}_g \sqcup \mathcal{A}_{g-1}$ along the boundary. The boundary of the partial compactification is the universal family $\mathcal{X}_{g-1}$:

$$\mathcal{A}_g' = \mathcal{A}_g \sqcup \mathcal{X}_{g-1}.$$

Geometrically, the boundary of the partial compactification parameterizes semiabelic varieties of torus rank one, described as follows. For a point $(B, b) \in \mathcal{X}_{g-1}$, where $B \in \mathcal{A}_{g-1}$ is an abelian variety of dimension $g-1$ and $b \in B$ a point on it, up to sign, the semiabelic variety corresponding to (B, b) is obtained by compactifying the $\mathbb{C}^*$-extension of B

$$1 \to \mathbb{C}^* \to G \to B \to 0$$

to a $\mathbb{P}^1$-bundle $\widetilde{G}$ over B by adding the 0- and ∞-sections, and then gluing these sections with a shift by b to obtain the non-normal variety $\overline{G} = \widetilde{G}/(\beta, 0) \sim (\beta + b, \infty)$ (we use β instead of the more standard notation z, to distinguish this from the zero section, and to emphasize that β and b are in a sense points of dual abelian varieties).

We extend the universal family $\pi : \mathcal{X}_g \to \mathcal{A}_g$ to a family over the partial compactification $\pi' : \mathcal{X}_g' \to \mathcal{A}_g'$ by globalizing the construction above. We follow the notation of [EGH10], the results and setup of which we now recall. We let $\mathcal{X}_{g-1}^2 = \mathcal{X}_{g-1} \times_{\mathcal{A}_{g-1}} \mathcal{X}_{g-1}$ be the fiberwise square, with $pr_i : \mathcal{X}_{g-1}^2 \to \mathcal{X}_{g-1}$ denoting the projections to the two factors. Let $\mathcal{P}$ denote the Poincaré bundle on $\mathcal{X}_{g-1}^2$, and let $\widetilde{Y} = \mathbb{P}(\mathcal{P} \oplus \mathcal{O})$ denote the projectivization of $\mathcal{P}$. We now define the extension Y of the universal family over the boundary by gluing the 0- and ∞-sections of $\widetilde{Y}$ with a shift by the second coordinate, and factorizing by the involution. In other words, we glue $(B, \beta, b, 0) \in \widetilde{Y}$ and $(B, \beta + b, b, \infty) \in \widetilde{Y}$, and then factorize by j, where j denotes the involution on the semiabelic variety fiber of $\widetilde{Y} \to \mathcal{X}_{g-1}$. We denote $\Delta \subset Y$ the gluing locus (and by abuse of notation its class in cohomology), i.e. Δ denotes the image of the glued 0- and ∞-sections of $\widetilde{Y}$. We summarize the geometry in the following diagram:

$$\begin{tikzcd}
 & & & (\widetilde{Y} = \mathbb{P}(\mathcal{P} \oplus \mathcal{O}))/j \ni (B, \beta, b, x) \arrow[dl] \arrow[d] \\
\mathcal{X}_g/\pm 1 \arrow[d] & \sqcup & Y \arrow[d] & \mathcal{X}_{g-1}^2 \ni (B, \beta, b) \arrow[dl, "pr_2"'] \arrow[d, "pr_1"] \\
\mathcal{A}_g \arrow[d] & \sqcup & \mathcal{X}_{g-1} \ni (B, b) \arrow[d] & \mathcal{X}_{g-1} \ni (B, \beta) \arrow[dl] \\
\mathcal{A}_g & \sqcup & \mathcal{A}_{g-1} \ni (B) &
\end{tikzcd}$$

We avoid working directly on the boundary family Y, because it is not normal and the Chow groups do not have an intersection product. We instead do all our computations on $\widetilde{Y}$, which is a $\mathbb{P}^1$-bundle over $\mathcal{X}_{g-1}^2$, and then only at the end take the involution and the gluing into account by requiring our computations to be invariant under them.

We now summarize the known results about the Chow rings of the various objects in the diagram.

The Picard group of $\mathcal{A}_{g-1}$ is equal to the first Chow group and is generated by the first Chern class λ_1 of the Hodge bundle. The Picard group and the first Chow group of the universal family is $\mathrm{Pic}_{\mathbb{Q}}(\mathcal{X}_{g-1}/\pm 1) = \mathbb{Q}\lambda_1 \oplus \mathbb{Q}T$, where T is the universal symmetric theta divisor trivialized along the zero section (note that it is only defined up to translation by two-torsion points, which is torsion, and thus gives a well-defined class over $\mathbb{Q}$).

The Picard group of the product family $\mathcal{X}_{g-1}^2$ is generated by the pullback of λ_1, which we denote by L, by the pullbacks $T_i = pr_i^* T$

of the theta divisors from the two factors, and by the class P of the universal Poincaré bundle, also trivialized along the zero section (see [EGH10]). By abuse of notation, we also use L, T_1, P and T_2 to denote the pullbacks of these classes to $A^1(\widetilde{Y})$. We recall that by the results of Deninger and Murre [DM91] (see also [Voi12a]), the direct image $R\pi_*\mathbb{Q}$ of a constant sheaf in any family of ppav admits a multiplicative decomposition. It follows, (see [Voi12, Prop. 4.3.6, Cor. 4.3.9] and Remark 2.2), that for classes T_1, P and T_2 on $\mathcal{X}^2_{g-1}$, all trivialized along the zero section by definition, a polynomial relation $f(T_1, P, T_2) = 0$ holds in $H^*(\mathcal{X}^2_{g-1})$ if and only if it holds in the Chow ring and if and only if it holds fiberwise. Along the way of our computation, we compute the relations between these classes on a very general ppav, and thus describe entirely the subring of $H^*(\mathcal{X}^2_{g-1})$ (and of the Chow ring) generated by these classes — the result is given in Theorem 3.1.

The Chow and the cohomology rings of $\mathcal{X}^2_{g-1}$ admit a natural automorphism which plays a key role in our computations. Let $s : \mathcal{X}^2_{g-1} \to \mathcal{X}^2_{g-1}$ denote the shift map defined by

$$s(B, \beta, b) = (B, \beta + b, b),$$

and let $s^* : A^*(\mathcal{X}^2_{g-1}) \to A^*(\mathcal{X}^2_{g-1})$ denote the induced map on the Chow ring. The action of s^* on the divisors T_1, P and T_2 was computed in [GL08],[EGH10] to be

$$s^*(T_2) = T_2; \quad s^*(P) = P + 2T_2; \quad s^*(T_1) = T_1 + P + T_2. \tag{4}$$

The Chow ring of $A^*(\widetilde{Y})$ is generated over the Chow ring $A^*(\mathcal{X}^2_{g-1})$ by one class ξ satisfying the relation $\xi^2 = \xi P$ (see [Ful98]). We think of ξ as the class of the 0-section, in which case $\xi - P$ is the class of the ∞-section, and the relation $\xi \cdot (\xi - P) = 0$ expresses the fact that these sections do not intersect.

The action of the involution j on $\widetilde{Y}$ is studied in detail in [GH11b, Sec. 4], where it is described globally in coordinates. It is easy to see that j interchanges the 0- and ∞-sections of $\widetilde{Y}$, and thus its action on $A^*(\widetilde{Y})$ interchanges ξ and $\xi - P$, which implies in particular that $j^*P = -P$. From the explicit description of the action we then also see that $j^*T_i = T_i$ (since the theta divisors are symmetric).

We also consider several cycles on the entire partial compactification $\mathcal{X}'_g$, and their pullbacks to $\widetilde{Y}$. The divisor T extends to a universal polarization divisor $\Theta \in A^1(\mathcal{X}'_g)$. The boundary of $\mathcal{X}'_g$ is an irreducible divisor, the class of which we denote D, therefore we have $\mathrm{Pic}_{\mathbb{Q}}(\mathcal{X}'_g) = \mathbb{Q}\lambda_1 \oplus \mathbb{Q}\Theta \oplus \mathbb{Q}D$. Finally, we consider the class of the gluing locus $\Delta \in A^2(\mathcal{X}'_g)$.

Finally, we need to know how these cycles restrict to the boundary. Let $(\cdot)|_{\widetilde{Y}} : A^*(\mathcal{X}'_g) \to A^*(\widetilde{Y})$ denote the pullback map. Then by [Mum83],[GL08], and [EGH10] we have

$$D|_{\widetilde{Y}} = -2T_2; \quad \Theta|_{\widetilde{Y}} = \xi + T_1 - P/2. \tag{5}$$

We compute the pullback of Δ to $\widetilde{Y}$ in Proposition 4.3 (note that $\Delta|_\Delta$ was computed in [EGH10]).

3. Intersection theory on $\mathcal{X}_{g-1} \times_{\mathcal{A}_{g-1}} \mathcal{X}_{g-1}$

We prove our main result by restricting the formula (2) to the boundary of the partial compactification of the universal family, and expressing all of the cycles involved in terms of the divisor classes ξ, T_1, P and T_2 defined in the previous section. To compare products of cycles on the boundary, we first need to understand the subring of the Chow ring generated by these divisors.

In this section, we compute the subring of $A^*(\mathcal{X}^2_{g-1}, \mathbb{Q})$ generated by the classes T_1, P and T_2. We show that this ring is a Gorenstein ring with socle in dimension $2g-2$, and that there are no relations in degrees up to and including the middle dimension. This calculation improves on the results of [EGH10], in particular on Theorem 7.1, which describes the pushforwards of products of T_1, P, and T_2 to the base $\mathcal{A}_{g-1}$.

Theorem 3.1. *Let R denote the subring of $A^*(\mathcal{X}^2_{g-1}, \mathbb{Q})$ generated by the classes T_1, P, and T_2, and let R^k denote the subspace of R spanned by monomials of degree k. Then*

(1) *The ideal of relations in R is generated by all the coefficients of the one basic relation*

$$(T_1 + nP + n^2T_2)^g = 0, \quad n \in \mathbb{Z} \tag{6}$$

considered as a polynomial in n (i.e. by all the homogeneous in n pieces of it). In particular, there are no non-trivial relations between T_1, P and T_2 in degree less than g.

(2) *R is a Gorenstein ring with socle in codimension $2g-2$, in other words,*

$$R^{2g-2} \cong \mathbb{Q}, \qquad \dim R^k = 0 \textit{ for } k > 2g-2,$$

and for any $0 \leq k \leq g-1$ the product map

$$R^{g-1-k} \times R^{g-1+k} \to R^{2g-2} \cong \mathbb{Q}$$

is a perfect pairing (in particular $\dim R^{g-1-k} = \dim R^{g-1+k}$). Moreover, the multiplication by $(T_1T_2)^k$ is an isomorphism from R^{g-1-k} to R^{g-1+k} .

Remark 3.2. The ring R introduced in the theorem is very natural: its restriction to the Cartesian square of a very general ppav generates its algebraic cohomology ring. As discussed above, the results of Deninger and Murre [DM91] on the multiplicative decomposition theorem imply that a relation among the classes T_1, P and T_2 holds if and only if it holds when restricted to a very general ppav. Therefore, the second part of the above statement admits a Hodge-theoretic interpretation. Moreover, the above statement can be combined with the Fourier transform $F : A^*(\mathcal{X}_g) \to A^*(\mathcal{X}_g^\vee)$ of families over $\mathcal{A}_g$ (the principal polarization identifies the universal family with its dual). Applying the Fourier transform provides an alternative approach to this theorem, but does not seem to yield the first part of the theorem, nor all of the statements in the second part of the theorem. We give a direct proof of the theorem below, and in a separate future work we will further investigate the geometry of this setup, and generalize the result to a higher fiberwise product (when the fiber over $A \in \mathcal{A}_{g-1}$ is $A^{\times k}$), where the theorem of the cube plays a crucial role.

We prove the theorem by considering the action of the shift operator s^* defined by (4) on the ring $A^*(\mathcal{X}^2_{g-1})$ and by using Theorem 2.1.

Proposition 3.3. *For any integer n relation (6) holds in the Chow ring, i.e. we have $(T_1 + nP + n^2 T_2)^g = 0$ in $A^g(\mathcal{X}^2_{g-1}, \mathbb{Q})$.*

Proof. The class T_1 is the pullback of the universal theta divisor T on $\mathcal{X}_{g-1}$. According to Theorem 2.1, $T^{g-1} = (g-1)! Z_{g-1} \in A^{g-1}(\mathcal{X}_{g-1})$. Multiplying both sides of this equality by T and recalling that T is trivial along the zero section, so that $TZ_{g-1} = 0$, we see that T^g is zero in the Chow ring. Pulling back this relation from $\mathcal{X}_{g-1}$ to $\mathcal{X}^2_{g-1}$ under pr_1^* we get that $T_1^g = 0$ in $A^g(\mathcal{X}^2_{g-1})$.

We now apply the shift operator to this relation. A direct calculation using (4) shows that $(s^*)^n(T_1) = T_1 + nP + n^2 T_2$. The shift operator s^* is an automorphism of $A^*(\mathcal{X}^2_{g-1})$, so the result follows. □

Proof of Theorem 3.1. The proof is direct and computational. We prove the statements of the theorem in the following order. First, we prove the vanishing of R^k for $k > 2g-2$ and show that multiplication by $(T_1T_2)^k$ is a surjective map from R^{g-1-k} to R^{g-1+k}. Then we use a pushforward calculation to prove that $R^{2g-2} \cong \mathbb{Q}$. We then show that the relations (6) generate the ideal of relations, and that multiplication by $(T_1T_2)^k$ is also injective. Finally, we prove the perfect pairing statement.

We define a second grading d on R by setting
$$d(T_1^a P^b T_2^c) := a - c.$$

This grading is motivated by the fact that all the summands of the n^{g-k} term of relation (6) have degree k in this grading. We now consider the decomposition of R with respect to d and the usual degree:

$$R = \bigoplus_{k=0}^{\infty} R^k = \bigoplus_{k=0}^{\infty} \bigoplus_{l=-k}^{k} R^k_l, \quad R^k_l = \{L \in R | \deg(L) = k,\, d(L) = l\}.$$

We consider relation (6) as a polynomial of degree $2g$ in a variable n. The coefficients of this polynomial give $2g+1$ relations in R^g, one in each R^g_l:

(7)
$$\sum_{m=0}^{\lfloor (g-l)/2 \rfloor} T_1^{l+m} P^{g-l-2m} T_2^m \frac{g!}{(l+m)!(g-l-2m)!m!} = 0 \in R^g_l,\ 0 \le l \le g,$$

(8)
$$\sum_{m=0}^{\lfloor (g+l)/2 \rfloor} T_1^{m} P^{g+l-2m} T_2^{m-l} \frac{g!}{m!(g+l-2m)!(m-l)!} = 0 \in R^g_l, -g \le l < 0.$$

We first prove by induction on p the following auxiliary statement:

$$T_1^{g-1-p} P^{2p+1} = 0 \text{ for } 0 \le p \le g-1.$$

The base case $p = 0$ is relation (7) for $l = g-1$. Now consider relation (7) for $l = g-1-2p$. Multiplying it by T_1^p, we see that it consists of terms of the form $T_1^{g-1-(p-m)} P^{2(p-m)+1} T_2^m$ with non-zero coefficients. All terms containing positive powers of T_2 are equal to zero by induction, hence so is $T_1^{g-1-p} P^{2p+1}$.

We now show that $R^k_l = 0$ for $l \ge 2g-k-1$. Let $T_1^a P^b T_2^c$ be an element of R^k_l, so that $a+b+c = k$ and $a-c \ge 2g-k-1$. It follows that $2a+b \ge 2g-1$. Let $l = \lfloor (b-1)/2 \rfloor$, then $a \ge g-1-l$ and $b \ge 2l+1$. It follows that $T_1^a P^b T_2^c$ is a multiple of $T_1^{g-1-l} P^{2l+1}$ and is therefore equal to zero. Hence $R^k_l = 0$ for $l \ge 2g-k-1$. A similar proof shows that $R^k_l = 0$ for $l \le -2g+k+1$.

We now use the above statement to describe the structure of the graded components R^k for $k \ge g$. First, we see that $R^k = 0$ for $k > 2g-2$, because if l is non-negative, then $l \ge 2g-k-1$, and if l is non-positive, then $l \le -2g+k+1$, and in either case $R^k_l = 0$ by the result of the previous paragraph.

We now show that the map from R^{g-1-k} to R^{g-1+k} given by multiplication by $(T_1T_2)^k$ is surjective for $1 \le k \le g-1$, in other words that every element of R^{g-1+k} can be written as a multiple of $(T_1T_2)^k$. We have already seen above that $R^{g-1+k}_l = 0$ for $|l| > g-1-k$. Now suppose $l \le g-1-k$, and assume without loss of generality that $l \ge 0$.

Let $T_1^aP^bT_2^c\in R_l^{g-1+k}$, so that $a+b+c=g-1+k$ and $a-c=l$. If $c\geq k$ then $a=c+l\geq k$, so $T_1^aP^bT_2^c$ is a multiple of $(T_1T_2)^k$. If $c<k$, then $a+b\geq g$, and we can use relation (7) to express $T_1^aP^b$ as a multiple of T_1T_2. Repeating this procedure if necessary, we can raise the exponent of T_2 to k and write $T_1^aP^bT_2^c$ as a multiple of $(T_1T_2)^k$, which proves the surjectivity of multiplication by $(T_1T_2)^k$.

We now show that $R^{2g-2}\cong\mathbb{Q}$. We have shown that in the subspace R^{2g-2} the only non-trivial d-graded piece is R_0^{2g-2}, consisting of elements of the form $T_1^{g-1-a}P^{2a}T_2^{g-1-a}$, and that all such elements are multiples of $T_1^{g-1}T_2^{g-1}$. In other words, R^{2g-2} is at most one-dimensional. To show that these classes are non-zero, we recall that in [EGH10, Theorem 7.1] the pushforwards of these classes to $A^*(\mathcal{A}_{g-1})$ were computed to be

$$h_*(T_1^{g-1-a}P^{2a}T_2^{g-1-a})=(-1)^a\frac{(g-1)!(2a)!(g-1-a)!}{a!}[\mathcal{A}_{g-1}]. \tag{9}$$

Therefore, all of the classes $T_1^{g-1-a}P^{2a}T_2^{g-1-a}$ are non-zero, and so R^{2g-2} has dimension one. Also, every class of the form $T_1^{g-1-a}P^{2a}T_2^{g-1-a}$ is in fact a non-zero multiple of $T_1^{g-1}T_2^{g-1}$, as proved above.

We have shown that R^k is spanned as a vector space by monomials that are multiples of $(T_1T_2)^{k-g+1}$. We now show that these monomials are linearly independent, by induction on k from $k=2g-2$ down to $k=g-1$. This will prove both that multiplication by $(T_1T_2)^{k-g+1}$ is an isomorphism from R^{2g-2-k} to R^k for $k\geq g$, and that there are no relations in degree less than g.

The base case, namely that $T_1^{g-1}T_2^{g-1}$ is non-zero, was established above. Now suppose that we have a linear relation in R^k for some $g-1\leq k<2g-2$:

$$(T_1T_2)^{k-g+1}\cdot\left(\sum_{l=1}^{2g-2-k}\sum_{m=0}^{g-1-\lceil(k+l)/2\rceil}A_{lm}T_1^{l+m}P^{2g-2-k-l-2m}T_2^m+\right.$$

$$+\sum_{m=0}^{g-1-\lceil k/2\rceil}B_mT_1^mP^{2g-2-k-2m}T_2^m+$$

$$\left.+\sum_{l=-1}^{-2g+2+k}\sum_{m=0}^{g-1-\lceil(k-l)/2\rceil}C_{lm}T_1^mP^{2g-2-k+l-2m}T_2^{-l+m}\right)=0.$$

Multiplying this relation by T_2, we get a relation in R^{k+1} of the form

$$(T_1T_2)^{k-g+2}\cdot\left(\sum_{l=1}^{2g-2-k}\sum_{m=0}^{g-1-\lceil (k+l)/2\rceil} A_{lm}T_1^{l+m-1}P^{2g-2-k-l-2m}T_2^m\right)+$$

$$+\text{ (terms with negative } d) = 0.$$

The left bracket only contains monomials with non-negative d that are multiples of $(T_1T_2)^{k-g+2}$. The terms in the right bracket can be written as multiples of $(T_1T_2)^{k-g+1}$ using relations (8). These relations preserve the second grading d, and hence it follows that the right and left brackets are both equal to zero. By induction, monomials that are multiples of $(T_1T_2)^{k-g+2}$ are linearly independent in R^{k+1}, hence all of the coefficients A_{lm} are zero. Similarly, multiplying by T_1 shows that $C_{lm}=0$ for all l and m.

It remains to show that all the B_m are zero, i.e. that there are no non-trivial relations in R_0^k. There are two cases to consider. If k is odd, we multiply by P to obtain a relation in R_0^{k+1}:

$$B_0(T_1T_2)^{k-g+1}P^{2g-1-k}+$$

$$+(T_1T_2)^{k-g+2}\cdot\left(\sum_{m=1}^{g-1-\lceil k/2\rceil} B_mT_1^{m-1}P^{2g-1-k-2m}T_2^{m-1}\right)=0.$$

All terms in the right bracket contain a positive even power of P. On the other hand, using relation (7) with $l=k+1-g$, we can express $P^{2g-1-k}(T_1T_2)^{k-g-1}$ as a linear combination of multiples of $(T_1T_2)^{k-g+2}$, including, with a non-zero coefficient, a term not containing P. This term cannot cancel any of the terms in the right bracket. Therefore $B_0=0$, and hence all of the other B_m are zero as well.

If k is even this reasoning does not work, because R_0^k has dimension one greater than R_0^{k+1}. Instead, if we have a relation in R_0^k, we multiply it by T_1 and P to obtain relations in R_1^{k+1} and R_0^{k+1}:

$$B_0T_1^{k-g}P^{2g-2-k}T_2^{k-g-1}+$$

$$+(T_1T_2)^{k-g}\cdot\left(\sum_{m=1}^{g-1-k/2} B_mT_1^{m}P^{2g-2-k-2m}T_2^{m-1}\right)=0,$$

$$B_0T_1^{k-g-1}P^{2g-1-k}T_2^{k-g-1}+$$

$$+(T_1T_2)^{k-g}\cdot\left(\sum_{m=1}^{g-1-k/2} B_mT_1^{m-1}P^{2g-1-k-2m}T_2^{m-1}\right)=0.$$

In each of these relations, we express the first term in terms of multiples of $(T_1T_2)^{k-g}$ by using (7) with $l = k-g+2$ and $l = k-g+1$. By induction, all of the coefficients of the obtained linear relations are zero. The coefficients in front of the top power of P in each relation are

$$B_1 - B_0\frac{(2g-2-k)(2g-3-k)}{k-g+3} = 0,$$
$$B_1 - B_0\frac{(2g-1-k)(2g-2-k)}{k-g+2} = 0.$$

It is easy to check that the determinant of this system cannot vanish for integer values of g. Therefore $B_0 = 0$, and hence all other B_m vanish as well. Therefore, multiples of $(T_1T_2)^{k-g+1}$ form a basis for R^k. This proves that the multiplication by $(T_1T_2)^{k-g+1}$ map from R^{2g-2-k} to R^k is an isomorphism, and that there are no other relations in the ring R.

Finally, we need to show that the product map defines a perfect pairing

$$R^k \times R^{2g-2-k} \to R^{2g-2} \simeq \mathbb{Q}.$$

First, it is clear that $R^k_l \times R^{2g-2-k}_m = 0$ unless $m = -l$, because R^{2g-2}_{l+m} is zero unless $l+m=0$. Therefore, it is sufficient to show that $R^k_l \times R^{2g-2-k}_{-l} \to R^{2g-2}_0$ is a perfect pairing. From the above discussion it is clear that $R^k_l = R^{k-l}_0 \cdot T_1^l$ and $R^{2g-2-k}_{-l} = R^{2g-2-k-l}_0 \cdot T_2^l$. If there exists an element $X \in R^{2g-2-k}_{-l}$ that pairs to zero with R^k_l, then the element $XT_1^l \in R^{2g-2-k+l}_0$ pairs to zero with R^{k-l}_0. Therefore, it is sufficient to prove that $R^k_0 \times R^{2g-2-k}_0 \to R^{2g-2}_0$ is a perfect pairing for $1 \le k \le g-1$. Finally, we can assume k to be even, because if k is odd then $R^k_0 = R^{k-1}_0 \cdot P$ and if $X \in R^{2g-2-k}_0$ pairs to zero with R^k_0 then $XP \in R^{2g-1-k}$ pairs to zero with R^{k-1}_0.

We now prove by induction on k that $R^{2k}_0 \times R^{2g-2-2k}_0 \to R^{2g-2}_0$ is a perfect pairing. For $k=1$, suppose that $X = aT_1^{g-2}T_2^{g-2} + bT_1^{g-3}P^2T_2^{g-3}$ pairs to zero with R^2_0. Multiplying X by T_1T_2 and P^2 and taking the pushforward to the base using (9), we get that

$$a(g-1)! - 2b(g-2)! = 0, \quad -2a(g-2)! + 12b(g-3)! = 0,$$

which implies that $a = b = 0$.

Now suppose that the element

$$X = T_1^{g-1-2k}T_2^{g-1-2k}\sum_{i=0}^{k} a_iT_1^iP^{2k-2i}T_2^i$$

in $R^{2g-2-2k}_0$ pairs to zero with R^{2k}_0. Then the elements XT_1T_2 and XP^2 in R^{2g-2k}_0 kill R^{2k-2}_0, so by induction they are zero. The element XT_1T_2

contains the term $a_0 T_1^{g-2k} P^{2k} T_2^{g-2k}$. Using relations (7) to express $T_1^{g-2k} P^{2k}$ as a multiple of T_2, and using the fact that multiples of $(T_1 T_2)^{g+1-2k}$ in R_0^{2g-2k} are linearly independent, we see that

$$a_i(g-2k+i)!(2k-2i)!i! = a_0(g-2k)!(2k)!0!$$

Similarly, the element XP^2 contains the terms $a_0 T_1^{g-2k-1} P^{2k+2} T_2^{g-2k+1}$ and $a_1 T_1^{g-2k} P^{2k} T_2^{g-2k}$. Expressing them as multiples of $T_1^{g-2k+1} T_2^{g-2k+1}$ and setting the coefficient of $T_1^{g-k} T_2^{g-k}$ to zero, we see that

$$a_1(g-2k)!(2k-1)!1! = a_0(g-2k-1)!(2k+1)!0!$$

This equation, together with the $i = 1$ equation above, are a system for a_0 and a_1 with determinant $2k-2g-1$. Hence $a_0 = a_1 = 0$, and the other equations imply that the remaining a_i are also zero. Therefore, the class X is zero, which proves the perfect pairing statement. $\square$

4. Shift-invariant classes

Our goal is to compute the restriction of the zero section of the universal semiabelian variety to the boundary Y in terms of products of pullbacks of geometric cycles defined on the whole family $\mathcal{X}'_g$. The boundary Y is not a normal variety, so the Chow group $A^*(Y)$ does not have an intersection product. To avoid this difficulty we instead work in the Chow ring $A^*(\widetilde{Y})$, where $\widetilde{Y}$ is a 2-to-1 cover of the normalization of the boundary. For this reason, we need to determine which cycles in $A^*(\widetilde{Y})$ are pullbacks of cycles from $A^*(Y)$, and in particular pullbacks of intersections of cycles on $\mathcal{X}'_g$ with Y. We denote by $(\cdot)|_{\widetilde{Y}} : A^*(\mathcal{X}'_g) \to A^*(\widetilde{Y})$ the composition of the restriction to Y with the pullback to $\widetilde{Y}$.

The Chow ring $A^*(\widetilde{Y})$ is generated over the Chow ring $A^*(\mathcal{X}^2_{g-1})$ by the class ξ of the zero section satisfying the relation $\xi^2 - \xi P = 0$. In the previous section we determined the subring R^* of $A^*(\mathcal{X}^2_{g-1})$ generated by the classes T_1, P and T_2. In this section, we describe the classes in $\widetilde{R}^* \subset A^*(\widetilde{Y})$ that are pullbacks of classes from Y, where $\widetilde{R}^* = R^*[\xi]/(\xi^2 - \xi P)$ denotes the subring of $A^*(\widetilde{Y})$ generated by T_1, P, T_2 and ξ. By abuse of notation, we will also use T_1, P and T_2 to denote the pullbacks of these classes to $A^1(\widetilde{Y})$.

The boundary Y is the quotient by the involution j of the $\mathbb{P}^1$-bundle $\widetilde{Y}$ over $\mathcal{X}^2_{g-1}$, with the zero section Δ_0 glued to the infinity section Δ_∞ by a shift, resulting in the locus $\Delta \subset Y$. The two sections Δ_0 and Δ_∞ define pullback maps $(\cdot)|_0$ and $(\cdot)|_\infty$ from $A^*(\widetilde{Y})$ to $A^*(\mathcal{X}^2_{g-1})$. By definition ξ is the class of the zero section Δ_0, hence

$$\xi \cdot \Delta_0 = \xi^2 = \xi \cdot P = P \cdot \Delta_0.$$

Therefore, the map $(\cdot)|_0 : A^*(\widetilde{Y}) \to A^*(\mathcal{X}^2_{g-1})$ consists in setting $\xi = P$. Similarly, the class of the infinity section Δ_∞ is $\xi - P$, hence

$$\xi \cdot \Delta_\infty = \xi(\xi - P) = 0,$$

and the map $(\cdot)|_0 : A^*(\widetilde{Y}) \to A^*(\mathcal{X}^2_{g-1})$ consists in setting $\xi = 0$.

Given a subvariety $V \subset Y$, the preimage of $V \cap \Delta$ in $\widetilde{Y}$ consists of two connected components, namely the preimages of V in $\widetilde{Y}$ intersected with Δ_0 and Δ_∞. Therefore, a class $X \in A^*(\widetilde{Y})$ is the pullback of a class from $A^*(Y)$ only if it is *shift-invariant*, in other words only if

$$s^*(X|_{\Delta_\infty}) = X|_{\Delta_0}, \tag{10}$$

where the above equality is in $A^*(\mathcal{X}^2_{g-1})$.

We also recall from [GH11b, Sec. 4] and from the discussion in Section 2 that the action of the involution j on the semiabelic fibers of the universal family induces the following action on the Picard group:

$$j^*\xi = \xi - P, \quad j^*P = -P, \quad j^*T_1 = T_1, \quad j^*T_2 = T_2.$$

We now describe the shift-invariant and j-invariant classes.

Proposition 4.1. *Let $\widetilde{R}$ denote the ring $\mathbb{Q}[\xi, T_1, P, T_2]/I$, where I is the ideal generated by $\xi^2 - \xi P$ and relations (6). Let $j : \widetilde{R} \to \widetilde{R}$ denote the automorphism defined on the generators by*

$$j(\xi) = \xi - P, \quad j(P) = -P, \quad j(T_1) = T_1, \quad j(T_2) = T_2,$$

and let s be the shift operator defined on the subring generated by T_1, P and T_2 as follows:

$$s(T_1) = T_1 + P + T_2, \quad s(P) = P + 2T_2, \quad s(T_2) = T_2.$$

Then the subset of elements $X \in \widetilde{R}$ that are j-invariant and that are shift-invariant:

$$j(X) = X, \quad s(X(0, T_1, P, T_2)) = X(\xi, T_1, P, T_2)$$

is the subring generated by the classes $\Theta := \xi + T_1 - P/2$, $D := -2T_2$, and $-4\xi T_2 - P^2 + 2PT_2$.

Remark 4.2. The notation Θ and D is due to the fact that these are in fact the restrictions of the corresponding classes on $\mathcal{X}'_g$, according to (5). The next proposition shows that the third class in in fact the restriction of Δ.

Proof. We first consider the automorphism j on the free polynomial ring $\mathbb{Q}[\xi, T_1, P, T_2]$. It is clear that j is an involution, and that the j-invariant subring is generated by $\xi - P/2$, P^2, T_1 and T_2:

$$\mathbb{Q}[\xi, T_1, P, T_2]^j = \mathbb{Q}[\xi - P/2, P^2, T_1, T_2].$$

Let $r : \mathbb{Q}[\xi, T_1, P, T_2] \to \widetilde{R}$ denote the projection map. First we note that j preserves the ideal I, hence j in fact descends to an involution of $\widetilde{R}$.

Suppose that $X \in \widetilde{R}$ satisfies $j(X) = X$. If $X = r(Y)$, then setting $Z = (Y + j(Y))/2$ we see that $X = r(Z)$ and $j(Z) = Z$. In other words, every j-invariant element in $\widetilde{R}$ is the image of a j-invariant element in $\mathbb{Q}[\xi, T_1, P, T_2]$. Since $(\xi - P/2)^2 = P^2/4$ in $\widetilde{R}$, we see that $\widetilde{R}^j$ is generated by $\xi - P/2$, T_1 and T_2.

The shift operator s does not extend to the entire ring $\widetilde{R}$, so we cannot compute the subring of shift-invariant classes in the same way, as an invariant subring of the action of a finite group. However, we make the following observation. Let $S \subset \mathbb{Q}[\xi, T_1, P, T_2]$ denote the subring generated by the classes $\Theta = \xi + T_1 - P/2$, $\mu = \xi - P/2$ and T_2. We have shown above that $r(S) = \widetilde{R}^j$. It turns out that the ring S admits an involution such that the subring of fixed elements is precisely the subring of shift-invariant classes.

Indeed, define an automorphism σ of S on the generators as follows:

$$\sigma(\Theta) = \Theta, \quad \sigma(\mu) = -\mu - T_2, \quad \sigma(T_2) = T_2.$$

The automorphism σ preserves the ideal $S \cap I$ and it is an involution, therefore σ descends to an involution on $\widetilde{R}^j$. Moreover, an element $X \in \widetilde{R}^j$ satisfies the gluing condition if and only if it is σ-invariant. Using the same reasoning as above, we see that subset of elements of $\widetilde{R}^j$ satisfying the gluing condition is the image under r of the invariant subring S^σ. The invariant subring S^σ is generated by Θ, D, and the class $\mu \cdot \sigma(\mu) = -\mu(\mu + T_2) = -(\xi T_2 + P^2/4 - PT_2/2)$, which proves the theorem. $\square$

We now give an interpretation for the third invariant class appearing in Proposition 4.1:

Proposition 4.3. *The pullback of Δ, considered as a class in $A^2(\mathcal{X}_g')$, to $\widetilde{Y}$ is equal to*

$$\Delta|_{\widetilde{Y}} = -4\xi T_2 - P^2 + 2PT_2 = (2\xi - P)(-2\xi + P - 2T_2) \in A^2(\widetilde{Y}).$$

Proof. The proof of this formula is a slight extension of the ideas of the proof of [EGH10, Prop. 4.3], where it is shown that $\Delta|_\Delta = P(-P - 2T_2)$. We note that the formula above restricts to this expression when we set $\xi = P$ (which we think of as restricting to the 0-section), while for $\xi = 0$ (the ∞-section) the above formula restricts to $-P^2 + 2PT_2$, which is obtained from $P(-P - 2T_2)$ by sending P to $-P$, which we know to be the action of the involution j on $\mathrm{Pic}(\widetilde{Y})$.

To prove the proposition we interpret the class Δ geometrically. Indeed, recall from [EGH10] that Δ is the locus where Y is not normal, and thus in a small neighborhood of itself Δ is the intersection of the two local irreducible components of the locus $Y \subset \mathcal{X}'_g$. Therefore the class of Δ is a product of divisors, and so lies in the ring generated by ξ, T_1, P and T_2. The class of Δ also satisfies the conditions of Proposition 4.1, hence it is a linear combination of Θ^2, ΘD, D^2 and $-4\xi T_2 - P^2 + 2PT_2$. Finally, Δ restricts to $-P^2 + 2PT_2$ when we set $\xi = P$, and it is easy to see that $-4\xi T_2 - P^2 + 2PT_2$ is the only class that satisfies this condition.

□

Proof of Theorem 1.3. The result now immediately follows from Proposition 4.1 and Proposition 4.3. □

Remark 4.4. In the next section, we show that the restriction of the zero section to the boundary is a polynomial in ξ and T_1, and therefore can be expressed as a polynomial in Θ, D and Δ. For now, we note two curious facts.

First, we note that the class $Q := \Delta - 2\Theta D = 4T_1T_2 - P^2$ does not contain ξ, and is therefore in the image of $A^2(\Delta)$ in $A^2(\widetilde{Y})$. We do not know a geometric explanation or meaning of why such a shift-invariant class should exist. However, the expression for the zero section in terms of the class Q is much simpler than in terms of Δ (see Theorem 1.1).

Second, we note that one can show that the subring of R^* invariant under gluing (i.e. under the involution σ) is generated by Θ, D, and Δ, together with one additional class, $\xi(6PT_2 + 12T_2^2) + P^3 - 4PT_2^2$, that satisfies a quadratic relation in Θ, D, and Δ. We do not know if this class has any geometric meaning.

5. Class of the partial boundary of the zero section

We now prove Theorem 1.1, obtaining an explicit expression for the class of the locus of the closure of the zero section in the partial compactification.

Our goal is to extend Theorem 2.1 to the partial compactification. Denote by $z'_g : \mathcal{A}'_g \to \mathcal{X}'_g$ the closure of the zero section in the partial compactification of the universal family, and denote, as above, by $\Theta \subset \mathcal{X}'_g$ the closure of the theta divisor and its class. In [vdG98] van der Geer computes the Chow rings of $\overline{\mathcal{A}}_3$ and $\mathcal{X}'_2$, and in particular shows that $Z'_g \neq [T^g]/g!$ in $A^2(\mathcal{X}'_2)$. It is easy to deduce that such an equality does not hold in any higher genus either. We now compute the difference.

We describe the locus z'_g explicitly using our description of the geometry of $\mathcal{X}'_g$, as the universal space of the universal Poincaré bundle over the universal fiberwise product $\mathcal{X}^2_{g-1} = \mathcal{X}_{g-1} \times_{\mathcal{A}_{g-1}} \mathcal{X}_{g-1}$. Indeed, the semiabelic variety of torus rank one is no longer a group, but is acted upon by the semiabelian variety (the $\mathbb{C}^*$-bundle over the same base B), which is a group. The zero for the group law of the semiabelian variety is the point $1 \in \mathbb{C}^*$ lying in the fiber over the zero in the base abelian variety B. The zero of the semiabelic variety becomes one of the limits of two-torsion points on it (as described in detail in [GH11b]) — which one, it does not matter for us, as their classes are all equivalent modulo torsion, and we are working in the Chow ring with rational coefficients. Thus the restriction of z'_g to the boundary $\mathcal{X}_{g-1}$ of $\mathcal{A}'_g$ is the map that associates to $(B, b) \in \mathcal{X}_{g-1}$ the point $(B, 0, b, 1) \in Y = \partial\mathcal{X}'_g$. This is of course a section of the universal Poincaré bundle restricted to the locus $\{(B, 0, b)\}$, and thus its class $\partial Z'_g := Z'_g|_Y$ is equal to ξ times the class of the locus $\{(B, 0, b)\} \subset \mathcal{X}^2_{g-1}$. However, this class is just the class of the zero section $z_{g-1} : \mathcal{A}_{g-1} \to \mathcal{X}_{g-1}$, pulled back to $\mathcal{X}^2_{g-1}$ under pr_1. By Theorem 2.1 discussed above, this is the pullback of the class $T^{g-1}/(g-1)!$ under the projection map pr_1, i.e. the class $T_1^{g-1}/(g-1)!$ in our notation. Therefore, we have proved the following result:

Proposition 5.1. *The class of the restriction to $\widetilde{Y}$ of the closure of the zero section Z'_g is equal to*

$$\partial Z'_g = \frac{\xi T_1^{g-1}}{(g-1)!} \in A^g(\widetilde{Y}).$$

Notice that there is an ambiguity here: we could have as well deduced the same formula with ξ replaced by $\xi + P$, by arguing that the 1-section of the $\mathbb{P}^1$-bundle is also a section over the B that is the ∞-section, instead of the 0-section, with the corresponding shift. This is consistent, as $T_1^{g-1}P = 0 \in A^g(\mathcal{X}^2_{g-1})$ by Proposition 3.3. Of course the zero section, being defined geometrically on Y, pulls back to a shift-invariant class on the normalization $\widetilde{Y}$ of Y, and Theorem 1.3 applies to show that $\partial Z'_g$ is a polynomial in the classes Θ, D, and Δ. It remains to compute the coefficients, proving our main result.

Proof of the main theorem 1.1. We first note that the class $\partial Z'_g = \frac{\xi T_1^{g-1}}{(g-1)!}$ satisfies the conditions of Proposition 4.1 (it is shift-invariant since $T_1^{g-1}P = 0$). Therefore, it can be written as a polynomial in Θ, D and Δ. It turns out that the formula for the zero section is simpler in terms of the alternative classes $\Theta - D/8$, D, and $\Delta - 2\Theta D$.

These three classes also generate the subring of shift-invariant polynomials, therefore there exists a formula

$$\frac{\xi T_1^{g-1}}{(g-1)!} = \sum_{a+b+2c=g} \alpha_{a,b,c}(\Theta - D/8)^a D^b (\Delta - 2\Theta D)^c, \tag{11}$$

where the classes Θ, D and Δ are given in terms of ξ, T_1, P and T_2 by

$$\Theta = \xi + T_1 - \frac{P}{2}, \quad D = -2T_2, \quad \Delta = -4\xi T_2 - P^2 + 2PT_2.$$

We first find the coefficients $\alpha_{a,0,c}$ not involving D.

In the main equation (11), set $T_2 = 0$, obtaining

$$\frac{\xi T_1^{g-1}}{(g-1)!} = \sum_{a+2c=g} \alpha_{a,0,c} \left(\xi + T_1 - \frac{P}{2}\right)^a (-P^2)^c.$$

For an arbitrary integer n we now formally set $T_1 = \left(n + \frac{1}{2}\right) P$. Using $\xi^2 = \xi P$ we then get

$$(\xi + nP)^a = n^a P^a + \sum_{i=1}^{a} n^{a-i} C_a^i \xi P^{a-1} = n^a P^a + [(n+1)^a - n^a]\xi P^{a-1}.$$

Therefore, equating the coefficients in front of ξP^{g-1} on both sides gives

$$\frac{\left(n + \frac{1}{2}\right)^{g-1}}{(g-1)!} = \sum_{a+2c=g} \alpha_{a,0,c}[(n+1)^a - n^a](-1)^c.$$

We now sum this equality from $n = 1$ to $n = N-1$, where N is another integer. The left hand side can be expressed in terms of Bernoulli numbers:

$$\sum_{n=1}^{N-1} \left(n + \frac{1}{2}\right)^{g-1} = \frac{1}{2^{g-1}} \left[\sum_{k=1}^{2N} k^{g-1} - \sum_{l=1}^{N} (2l)^{g-1} - 1\right] =$$

$$= \sum_{m=0}^{g-1} \frac{N^{g-m} B_m}{m!(g-m)!}(2^{1-m} - 1) - \frac{1}{2^{g-1}}.$$

Comparing this with the right hand side and equating coefficients of the powers of N yields

$$\alpha_{a,0,c} = \frac{(-1)^c}{a!(2c)!}(2^{1-2c} - 1)B_{2c},$$

as claimed by the theorem.

For the coefficients $\alpha_{a,b,c}$ with $b > 0$, we do not know an elegant derivation as above. Instead, we show that the remaining coefficients satisfy a triangular system of equations in terms of the coefficients

$\alpha_{a,0,c}$, and solve this system directly using Maple. We consider the main equation (11), and set $\xi = 0$:

$$\sum_{a+b+2c=g} \alpha_{a,b,c} \left(T_1 - \frac{P}{2} + \frac{T_2}{4}\right)^a (-2T_2)^b (4T_1T_2 - P^2)^c = 0.$$

Now formally apply the square root of the shift operator (4)

$$(s^*)^{1/2}(T_1) = T_1 + \frac{P}{2} + \frac{T_2}{4}, \quad (s^*)^{1/2}(P) = P + T_2, \quad (s^*)^{1/2}(T_2) = T_2,$$

to this equation. We get that

$$\sum_{a+b+2c=g} \alpha_{a,b,c} T_1^a (-2T_2)^b (4T_1T_2 - P^2)^c = 0.$$

This is a relation in the ring R^*, in other words this equation is a linear combination of relations (7)-(8). These relations are homogeneous with respect to the grading d, as well as the usual grading, so the d-homogeneous parts of the above equation vanish separately. The possible values of the grading d are $g-2h$, where $h = 0, \ldots, g$, so the above equation splits into the following system:

$$\sum_{c=0}^{\min(h,g-h)} \alpha_{g-h-c,h-c,c} T_1^{g-h-c} (-2T_2)^{h-c} (4T_1T_2 - P^2)^c = 0, \quad h = 0, \ldots, g.$$

First, assume that $g - h \geq h$. Expanding $(4T_1T_2 - P^2)^c$ and changing the order of summation, we can write the above as

$$\sum_{l=0}^{h} T_1^{g-h-l} P^{2l} T_2^{h-l} \frac{(-1)^{h+l} 2^{h-2l}}{l!} \sum_{c=l}^{h} \frac{c!}{(c-l)!} (-1)^c 2^c \alpha_{g-h-c,h-c,c} = 0.$$

This equation is satisfied if and only if the left hand side is a multiple of the corresponding relation (7). This gives us a triangular system of equations on the coefficients $\alpha_{g-h-c,h-c,c}$, and we have already determined the coefficient $\alpha_{g-2c,0,c}$ above, so the remaining coefficients are determined uniquely by this system.

Therefore, to prove Theorem 1.1 it is sufficient to substitute the coefficients (3) into the formula above and check that we get relation (7). Substituting and dividing out by a common multiple, we get

$$\sum_{l=0}^{h} T_1^{g-h-l} P^{2l} T_2^{h-l} \frac{(-1)^l 2^{-2l}}{l!} \sum_{c=l}^{h} \frac{(-1)^c 2^{2c} (2g-2c)!}{(g-c)!(c-l)!(g-h-c)!(h-c)!} = 0.$$

Using Maple, we evaluate the inside sum as

$$\sum_{c=l}^{h} \frac{(-1)^c 2^{2c}(2g-2c)!}{(g-c)!(c-l)!(g-h-c)!(h-c)!} = C_{g,h} \frac{(-1)^l 2^{2l} l!}{(g-l-h)!(h-l)!(2l)!},$$

where $C_{g,h}$ is a coefficient depending on g and h. Plugging this into the equation above, we see that it is indeed a multiple of (7), hence it is satisfied. This completes the calculation of the coefficients $\alpha_{g-h-c,h-c,c}$ for $g-h \geq h$, and the calculation of the other coefficients is virtually identical.

Finally, the coefficients $\eta_{a,b,c}$ are obtained by expanding formula (2) and using Maple to simplify. $\square$

Remark 5.2. Given the explicit formula we obtain for the extension of the zero section to the partial compactification, it is natural to ask whether a formula for the extension to the next boundary stratum (over the locus of torus rank two semiabelic varieties) may be possible. This locus of semiabelic varieties of torus rank two is the same for perfect cone, second Voronoi, and central cone (Igusa) toroidal compactifications — since all these compactifications coincide in genus 2, and restrict inductively to products. In principle it should be possible to describe explicitly the geometry of the universal family of semiabelic varieties of torus rank two (which can now be of two types, depending on whether the normalization is a $\mathbb{P}^1 \times \mathbb{P}^1$ bundle, or two copies of a $\mathbb{P}^2$ bundle). This computation would be very involved technically, but could shed further light on the class of the closure of the zero section in $\mathcal{A}_g^{\mathrm{Perf}}$, which would be instrumental in trying to inductively describe its cohomology. We note also that the fact that torus rank up to two strata of a toroidal compactification of $\mathcal{A}_g$ are closely related to the partial compactification of the universal family does not seem to extend deeper, as even the existence of a universal family over $\mathcal{A}_g^{\mathrm{Perf}}$ is not known globally.

6. Extension of the double ramification locus

In this section we extend Hain's formula for the double ramification locus from $\mathcal{M}_{g,n}^{ct}$ to $\overline{\mathcal{M}}_{g,n}^{o}$, i.e. to the entire preimage of $\mathcal{A}_g'$ under the Torelli map $\overline{\mathcal{M}}_g \to \overline{\mathcal{A}}_g$. We recall the setup. Fix a set of integers $\underline{d} = \{d_1, \ldots, d_n\}$ such that $\sum d_i = 0$. The *double ramification locus* $R_{\underline{d}}$ is defined as the locus in $\mathcal{M}_{g,n}$ consisting of $(X, p_1, \ldots, p_n)$ such that the sum $\sum d_i p_i$ is a principal divisor on X. The locus $R_{\underline{d}}$ is very natural from the point of view of Hurwitz theory. This locus, or related loci (see eg [Mül12]) also occurs naturally in various enumerative problems, and is also studied in Gromov–Witten theory [FSZ10].

We approach this locus in the following way. Denoting by AJ the Abel–Jacobi map, $R_{\underline{d}}$ is the locus in $\mathcal{M}_{g,n}$ where $\sum d_i AJ(p_i) = 0 \in \mathrm{Pic}^0(X)$, so we can compute $R_{\underline{d}}$ by pulling back the zero section of the universal abelian variety under the Abel–Jacobi map. The Jacobian of a curve of compact type is an abelian variety, so the Abel–Jacobi map extends to the moduli space $\mathcal{M}_{g,n}^{ct}$.

To compute the closure of the double ramification locus in $\overline{\mathcal{M}}_{g,n}^o$, we need to understand how the Abel–Jacobi map extends to $\overline{\mathcal{M}}_{g,n}^o$. Let X_t be a family of smooth curves of genus g with n marked points $p_1, \ldots, p_n$ degenerating to an irreducible nodal curve X_0. Choose a basis A_i, B_i of $H_1(X_t, \mathbb{Z})$ such that the degeneration corresponds to contracting the cycle A_1, and let ω_i be a basis for $H^0(X_t, \Omega)$ dual to the cycles A_i. Then the Abel–Jacobi map is obtained by integrating the basis ω_i between the marked points, i.e.

$$AJ\left(X_t, p_1, \ldots, p_n\right) = \sum_{i=1}^{n} d_i \left(\int_q^{p_i} \omega_1, \ldots, \int_q^{p_i} \omega_g \right) \in \mathrm{Jac}(X_t),$$

where $q \in X_t$ is an arbitrary base point.

Let $\widetilde{X}_0$ be the normalization of X_0, and let $p_\pm$ be the preimages of the node. The differentials $\omega_2, \ldots, \omega_g$ degenerate to holomorphic differentials on $\widetilde{X}_0$ normalized along the periods $A_2, \ldots, A_g$, while the differential ω_1 degenerates to a meromorphic differential on $\widetilde{X}_0$ having zero A-periods and having simple poles with residues ± 1 at $p_\pm$.

A line bundle of degree zero on X_0 is given by the data of a degree zero line bundle on $\widetilde{X}_0$ and a non-zero complex number ξ that defines an isomorphism of the stalks over the preimages $p_\pm$ of the nodes. For the line bundle that is the limit of the divisor $\sum d_i p_i$, this number is

$$x = \exp\left(\sum_{i=1}^{g} d_i \int_q^{p_i} \omega_1 \right),$$

which we think of as giving the coordinate on the $\mathbb{C}^*$-fiber of the corresponding semiabelic variety. Therefore, for a curve $(X, p_1, \ldots, p_n)$ of geometric genus $g-1$ the Abel–Jacobi map is given by the line bundle $\sum d_i p_i$ on the normalization of X and by the gluing parameter x given by the formula above.

Proof of Theorem 1.4. We denote, following [GZ12], by $s_{\underline{d}}$ the map $\overline{\mathcal{M}}_{g,n}^o \to \mathcal{X}_g'$ sending $(X, p_1, \ldots, p_n)$ to $(\mathrm{Pic}^0(X), \sum d_i AJ(p_i))$. The closure of the double ramification locus $\overline{R}_{\underline{d}} \subset \overline{\mathcal{M}}_{g,n}^o$ is the pullback of the zero section $s_{\underline{d}}^*(z_g')$. The class of the zero section Z_g' is given by a polynomial in Θ, D and Δ by Theorem 1.1, so to compute the class $[\overline{R}_{\underline{d}}]$ we need to compute the pullbacks of Θ, D, and Δ under $s_{\underline{d}}$.

The pullback $s_{\underline{d}}^*\Theta$ on $\overline{\mathcal{M}}_{g,n}$ was computed by Hain in [Hai11], and an alternative computation of it is one of the main results of [GZ12] (note also that a closely related divisor class was computed recently by Müller [Mül12], and a computation on the moduli space of curves with rational tails was done by Cavalieri, Marcus, and Wise in [CMW11]). This class is expressed in terms of the standard divisor classes on $\overline{\mathcal{M}}_{g,n}$ in the following way:

$$s_{\underline{d}}^*\Theta = \frac{1}{2}\sum_{i=1}^n d_i^2 K_i - \frac{1}{2}\sum_{P\subseteq I}\left(d_P^2 - \sum_{i\in P} d_i^2\right)\delta_0^P - \frac{1}{2}\sum_{h>0, P\subseteq I} d_P^2\delta_h^P.$$

Here K_i denotes the pullback of the relative dualizing sheaf of the universal curve $\overline{\mathcal{M}}_{g,1}\to\overline{\mathcal{M}}_g$ under the projection map $\pi_i : \overline{\mathcal{M}}_{g,n}\to\overline{\mathcal{M}}_{g,1}$ forgetting all but the i-th marked point, $I = \{1,\ldots,n\}$ denotes the indexing set, $d_P = \sum_{i\in P} d_i$, and δ_h^P denotes the class of the boundary divisor whose generic point is a reducible curve consisting of a smooth genus h component containing the marked points indexed by P joined at a node to a smooth genus $g-h$ component containing the remaining marked points.

The preimage of D is the locus of curves whose Jacobian is a semiabelic variety. Since D is a pullback of the boundary of $\mathcal{A}_g'$, the map $s_{\underline{d}} : \delta_{irr}\to D$ factors through a lift of $\overline{\mathcal{M}}_g^o\to\mathcal{A}_g'$, and the multiplicity is thus one, so we have $s_{\underline{d}}^*D = \delta_{irr}$.

To finish the proof of the theorem it remains to compute the pullback of the class Δ. Geometrically, $s_{\underline{d}}(X, p_1, \ldots, p_n)\in\Delta$ if the curve X is in δ_{irr} and if the Abel–Jacobi map of the divisor $\sum d_i p_i$ is a torsion-free sheaf that is not a line bundle. This happens precisely when the gluing parameter x discussed above tends to zero or to infinity, in other words when the limit of the integral of the differential ω_1 is infinite. The limit of the differential ω_1 has single poles at the nodes and no other singularities, therefore the limit of the integral of ω_1 is infinite if and only if one of the marked points p_i approaches the node. The resulting stable curve is the normalized curve of genus $g-1$ containing the remaining marked points, and the marked point p_i on a rational bridge connecting the preimages of the node. Therefore, the pullback of Δ is set-theoretically the locus ξ_i of curves in $\overline{\mathcal{M}}_{g,n}^o$ of this form, and the multiplicity is equal to the number of copies of p_i in the divisor, which is $|d_i|$. This finishes the proof of the theorem. $\square$

Remark 6.1. It is natural to try to extend this computation deeper into the boundary of $\overline{\mathcal{M}}_{g,n}$. Of course if one has a complete formula for the class δ_g on a bigger partial toroidal compactification of $\mathcal{A}_g$ (see Remark 5.2 about the difficulties of this), this would suffice, but in principle it could be that a computation on the moduli space of curves is easier than on the moduli space of abelian varieties, as the image of the Torelli map only hits some of the strata of a toroidal compactification, see [AB11],[MV11].

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