THE TROPICAL n-GONAL CONSTRUCTION

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Abstract. We give a purely tropical analogue of Donagi's n-gonal construction and investigate its combinatorial properties. The input of the construction is a harmonic double cover of an n-gonal tropical curve. For $n = 2$ and a dilated double cover, the output is a tower of the same type, and we show that the Prym varieties of the two double covers are dual tropical abelian varieties. For $n = 3$ and a free double cover, the output is a tetragonal tropical curve with dilation profile nowhere $(2, 2)$ or (4), and we show that the construction can be reversed. Furthermore, the Prym variety of the double cover and the Jacobian of the tetragonal curve are isomorphic as principally polarized tropical abelian varieties. Our main tool is tropical homology theory, and our proofs closely follow the algebraic versions.

Contents

1. Introduction

Tropical geometry aims to find polyhedral, piecewise-linear analogues of the objects studied in algebraic geometry. There are two kinds of algebraic objects for which this correspondence is particularly well-developed. The tropical analogues of algebraic curves are metric graphs, which are the subject of an extensive theory, starting with the seminal paper [MZ08]. The tropical analogue of an abelian variety is a real torus with additional integral structure, and tropical abelian varieties have perhaps received less attention.

There are two standard ways to associate principally polarized abelian varieties (ppavs) to algebraic curves, and both of these constructions carry over to the tropical setting. The Jacobian variety $\mathrm{Jac}(C)$ of a smooth algebraic curve C of genus g is a ppav of dimension g. The corresponding Torelli map $\mathcal{M}_g \to \mathcal{A}_g$ on the moduli spaces is injective, so a smooth algebraic curve can be recovered from its Jacobian. The tropical Jacobian of a metric graph was already introduced in [MZ08]. The tropical Torelli map $\mathcal{M}_g^{\mathrm{trop}} \to \mathcal{A}_g^{\mathrm{trop}}$ is no longer injective, and its non-injectivity locus was completely described in [CV10].

The Prym variety $\mathrm{Prym}(\widetilde{C}/C)$ is a ppav of dimension g associated to an étale double cover $\widetilde{C} \to C$ of algebraic curves of genera $2g + 1$ and $g + 1$, respectively. It is defined as the connected

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component of the identity of the kernel of the norm map $\mathrm{Nm} : \mathrm{Jac}(\widetilde{C}) \to \mathrm{Jac}(C)$, and carries a principal polarization that is half of the polarization induced from $\mathrm{Jac}(\widetilde{C})$. The Prym–Torelli map $\mathcal{R}_{g+1} \to \mathcal{A}_g$ on the corresponding moduli spaces is no longer injective (for example, it has positive-dimensional fibers for $g \leqslant 4$), and its fibers have been extensively studied [Don92]. The tropical Prym variety $\mathrm{Prym}(\widetilde{\Gamma}/\Gamma)$ associated to a harmonic double cover of metric graphs $\widetilde{\Gamma} \to \Gamma$ is defined in a completely analogous manner (see [JL18], [LU21] and [LZ22]). To the authors' knowledge, the question of describing the fibers of the tropical Prym–Torelli map $\mathcal{R}_{g+1}^{\mathrm{trop}} \to \mathcal{A}_g^{\mathrm{trop}}$ has not been considered.

There are several remarkable constructions concerning Prym varieties of double covers of curves of small gonality. Let $p : \widetilde{C} \to C$ be a (possibly ramified) double cover of a smooth curve C admitting a degree n map $f : C \to \mathbb{P}^1$. As x varies over $\mathbb{P}^1$, the sections of the fiber maps $(f \circ p)^{-1}(x) \to f^{-1}(x)$ glue together into a 2^n-gonal curve $\widetilde{D} \to \mathbb{P}^1$, which carries two additional structures. First, exchanging the sheets of p defines an involution on $\widetilde{D}$, hence a double cover $\widetilde{D} \to D$ of a 2^{n-1}-gonal curve $D \to \mathbb{P}^1$. Second, if p is étale, then sections have a well-defined parity and $\widetilde{D}$ decomposes as a disjoint union of two 2^{n-1}-gonal curves (if n is odd then they are exchanged by the involution, while if n is even then each is itself a double cover of a 2^{n-2}-gonal curve). For $n \leqslant 4$, and under certain restrictions on the ramification, the ppavs associated to these double covers satisfy a number of natural isomorphisms (due ultimately to the symmetries of the associated Weyl groups):

$n = 2$: Given a double cover $p : \widetilde{C} \to C$ of a hyperelliptic curve C, we obtain another such double cover $\widetilde{D} \to D$, and applying the construction to $\widetilde{D} \to D$ recovers the original double cover. The double cover p is étale if and only if $\widetilde{D} \to D$ is split. If p is ramified, then the Prym varieties $\mathrm{Prym}(\widetilde{C}/C)$ and $\mathrm{Prym}(\widetilde{D}/D)$ (which are not in general principally polarized) are dual to one another [Pan86].

$n = 3$: This case was the first to be described [Rec74]. Given an étale double cover $p : \widetilde{C} \to C$ of a trigonal curve, the trigonal construction produces a tetragonal curve D (the double cover $\widetilde{D} \to D$ being split), which is *generic* in the sense that the tetragonal map has no fibers with ramification profile (4) or $(2, 2)$. This construction can be reversed, and the Prym variety $\mathrm{Prym}(\widetilde{C}/C)$ is isomorphic to the Jacobian $\mathrm{Jac}(D)$. The Prym variety $\mathrm{Prym}(\widetilde{C}/C)$ is also principally polarized when p is ramified at two points, and the ramified trigonal construction was described by Dalalyan in [Dal79] and [Dal84], and was recently rediscovered in [LO19].

$n = 4$: This is the general construction, from which the other two may be derived (see [Don81] and [Don92]). Given an étale double cover $p : \widetilde{C} \to C$ of a generic tetragonal curve, we obtain two more such double covers $\widetilde{C}' \to C'$ and $\widetilde{C}'' \to C''$, and the Prym varieties $\mathrm{Prym}(\widetilde{C}/C)$, $\mathrm{Prym}(\widetilde{C}'/C')$, and $\mathrm{Prym}(\widetilde{C}''/C'')$ are isomorphic. This implies that the Prym–Torelli map is never injective, and Donagi conjectures in [Don92] that the tetragonal construction fully accounts for the non-injectivity of the Prym–Torelli map in $g \geqslant 7$.

The purpose of our paper is to give tropical versions of the bigonal and trigonal constructions (we outline the tropical tetragonal construction as well, but leave the details and proofs to a future paper). To describe our results, we first discuss the tropical notion of gonality. Baker [Bak08] defines the *combinatorial rank* $r(D)$ of a divisor D on a metric graph, and shows that it satisfies a Riemann–Roch theorem. One may therefore say that a metric graph Γ is n-gonal if it carries

a divisor D of degree n and rank $r(D) \geqslant 1$. This definition is not appropriate in our setting. Instead, following the paper [CD18], we say that a tropical curve Γ is n-*gonal* if it admits a finite harmonic morphism $\Gamma \to K$ of degree n, where K is a metric tree. Any fiber of such a map is a divisor of rank at least one (see [ABBR15], Proposition 4.2), so this definition of gonality is more restrictive. We note that [CD18] further require the n-gonal map to be *effective*, which is a numerical condition imposed on the vertices where $d_f(v) \geqslant 2$. This condition does not play a role in the n-gonal construction, and we do not impose it.

In complete analogy to the algebraic case, we consider a tower

$$\widetilde{\Gamma} \xrightarrow{\pi} \Gamma \xrightarrow{f} K$$

of harmonic morphisms of metric graphs, where K is a metric tree and the degrees are $\deg \pi = 2$ and $\deg f = n$, respectively. To this tower we associate, in a purely combinatorial way, a metric graph $\widetilde{\Pi}$ together with a harmonic map $\widetilde{\Pi} \to K$ of degree 2^n. This map factors as $\widetilde{\Pi} \to \widetilde{K} \to K$, where the *orientation double cover* $\widetilde{K} \to K$ is free (and hence split because K is a tree) if and only if π is free. In addition, there is a natural involution on $\widetilde{\Pi}$ with quotient map $\widetilde{\Pi} \to \Pi$. For $n = 2$ we have $\Pi = \widetilde{K}$ and hence a tower $\widetilde{\Pi} \to \Pi \to K$ of the same kind as the original tower, which we call the *tropical bigonal construction*. For $n = 3$ and π free, we instead obtain (a split double cover of) a tetragonal curve $\Pi \to K$, which we call the *tropical trigonal construction*. This construction can be inverted by the *tropical Recillas construction*, under certain restrictions on the fibers of the tetragonal map.

In order to state our results, we first clarify the issue of principal polarizations on tropical abelian varieties, and give a modified definition of the tropical Prym variety. Given an integral torus Σ with a polarization, we canonically construct in Lemma 4.10 an integral torus Σ^{pp} and a map $f : \Sigma^{pp} \to \Sigma$ such that the induced polarization on Σ^{pp} is principal. The map f is bijective on points but is dilated, hence not in general invertible as a map of integral tori. Now let $\pi : \widetilde{\Gamma} \to \Gamma$ be a harmonic double cover of tropical curves, then the kernel of the norm map $\mathrm{Nm} : \mathrm{Jac}(\widetilde{\Gamma}) \to \mathrm{Jac}(\Gamma)$ two connected components if π is free and one if π is dilated (see [JL18]). The even connected component $(\mathrm{Ker}\,\mathrm{Nm})_0$ has a polarization induced from $\mathrm{Jac}(\widetilde{\Gamma})$, which is known to not be principal in general (see [LU21]), and as an auxiliary result we compute its polarization type in Proposition 4.21. We then define the Prym variety $\mathrm{Prym}(\widetilde{\Gamma}/\Gamma)$ as the principally polarized tropical abelian variety $(\mathrm{Ker}\,\mathrm{Nm})_0^{pp}$, and we work with both objects: the torus $(\mathrm{Ker}\,\mathrm{Nm})_0$ (which is not principally polarized in general) and the principally polarized torus $\mathrm{Prym}(\widetilde{\Gamma}/\Gamma)$.

The main results of our paper are the following exact analogues of the results of [Pan86] and [Rec74]:

Theorem 1.1 (Theorem 5.7)**.** *Let $\widetilde{\Gamma} \xrightarrow{\pi} \Gamma \xrightarrow{f} K$ be a tower of harmonic morphisms of metric graphs of degrees $\deg \pi = \deg f = 2$, where K is a metric tree. Assume that there is no point $x \in K$ with the property that $|f^{-1}(x)| = 2$ and $|(f \circ \pi)^{-1}(x)| = 2$. Then the output $\widetilde{\Pi} \xrightarrow{\pi'} \Pi \xrightarrow{f'} K$ of the bigonal construction has the same property, and applying the bigonal construction to it produces the original tower. If moreover $\widetilde{\Gamma}$ and $\widetilde{\Pi}$ are both connected, then*

$$\big(\mathrm{Ker}\,\mathrm{Nm}(\pi)\big)_0 \cong \big(\mathrm{Ker}\,\mathrm{Nm}(\pi')\big)_0^{\vee},$$

where $(\cdot)^{\vee}$ denotes the dual tropical abelian variety.

Theorem 1.2 (Theorem 5.1)**.** *Let* K *be a metric tree. The tropical trigonal and Recillas constructions establish a one-to-one correspondence*

$$\left\{\begin{array}{c}\textit{Tropical curves } \Pi \textit{ with a}\\ \textit{harmonic map of degree 4 to } \mathsf{K}\\ \textit{with dilation profiles nowhere}\\ (4) \textit{ or } (2,2).\end{array}\right\} \underset{\textit{trigonal construction}}{\overset{\textit{Recillas construction}}{\rightleftarrows}} \left\{\begin{array}{c}\textit{Free double covers } \widetilde{\Gamma} \to \Gamma \textit{ with}\\ \textit{a harmonic map of degree 3}\\ \textit{from } \Gamma \textit{ to } \mathsf{K}.\end{array}\right\}$$

and under this correspondence, the Prym variety $\mathrm{Prym}(\widetilde{\Gamma}/\Gamma)$ *of a double cover and the Jacobian* $\mathrm{Jac}(\Pi)$ *of the associated tetragonal curve are isomorphic as principally polarized tropical abelian varieties.*

We would like to highlight the techniques that we use. Abelian varieties and maps between them are strongly constrained by intersection theory: for example, one may check that an isogeny is an isomorphism by computing its degree in homology. The tropical analogue of singular homology for rational polyhedral spaces was introduced in [IKMZ19]. Tropical homology was first applied to tropical abelian varieties by Gross and Shokrieh [GS19b], who established a number of fundamental results about tropical abelian varieties and proved a tropical version of the Poincaré formula for the class of a metric graph in the homology of its Jacobian.

The techniques of tropical homology, at least as they apply to abelian varieties, turn out to be quite powerful: in the proof of our main Theorem 1.2, we are able to translate the corresponding algebraic proof (see [Rec74] or [BL04, Theorem 12.7.2]) nearly line-by-line into the tropical setting. We are confident that our techniques also work for the tetragonal construction, but a number of additional results in tropical intersection theory will first need to be established.

Organization of the article. We start by introducing the tropical n-gonal construction in Section 2. The construction is purely combinatorial and can be understood without any prior knowledge of tropical geometry or the algebraic n-gonal construction. To simplify the exposition, we work with graphs without edge lengths, and passing to metric graphs involves nothing more than equipping the target tree with an edge length function. We conclude by proving the first parts of Theorems 1.1 (Propositions 2.7 and 2.8) and 1.2 (Proposition 2.15).

To establish the isomorphisms of the tropical abelian varieties, we first introduce the necessary background on tropical curves, rational polyhedral spaces and tropical homology in Section 3. Section 4 is devoted to tropical abelian varieties. We speak extensively about real tori with integral structure and develop a theory of morphisms between such objects. In particular, we refine the definition of the tropical Prym variety compared to the existing literature (see e.g. [LU21]), and calculate the class of $\widetilde{\Gamma}$ in the tropical homology of $\mathrm{Prym}(\widetilde{\Gamma}/\Gamma)$ (Theorem 4.27, which is the Prym version of the tropical Poincaré formula proved in [GS19b]). We believe this section to be of independent interest.

Finally, we prove the main parts of Theorems 1.1 and 1.2 in Section 5. The structure of the proof of Theorem 1.2 closely follows the original proof of the algebraic statement in [Rec74], using tropical instead of singular homology. The original proof of the algebraic bigonal construction in [Pan86] is based on the tetragonal construction, which we have not yet established, hence to prove Theorem 1.1 we instead adapt the arguments that are used in Theorem 1.2.

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2. Graphs and the tropical n-gonal construction

In this section we recall a number of standard definitions from graph theory, and define the n-gonal construction for double covers of degree n harmonic covers of trees. While we are primarily interested in the construction applied to metric graphs, the idea behind the definition as well as basic properties are best explained in the language of graphs, with an integer-valued local degree function recording the dilation factors. The description can then be lifted to metric graphs without much effort: assigning edge lengths to the base tree automatically determines edge lengths for all covers via the local degree function.

After describing the construction in general, we give more details on the *bigonal*, *trigonal*, and *tetragonal constructions*, which are the special cases for $n = 2, 3, 4$, respectively.

2.1. **Graphs and harmonic morphisms.** A *graph* G consists of a finite set of *vertices* $V(G)$, a finite set of *half-edges* $H(G)$, a root map $r : H(G) \to V(G)$, and a fixed-point-free involution $H(G) \to H(G)$ denoted $h \mapsto \overline{h}$. The set of *points* of G is $V(G) \cup H(G)$, so a point is either a vertex or a half-edge. An orbit $e = \{h, \overline{h}\}$ of the involution is an *edge* of G, and the set of edges is denoted $E(G)$. We allow graphs with loops and multiple edges between a pair of vertices. An *orientation* on G is a choice of ordering $(h, \overline{h})$ of the half-edges for each edge of G. The *tangent space* $T_v G = r^{-1}(v)$ to a vertex $v \in V(G)$ is the set of half-edges rooted at v, and the *valency* is $\mathrm{val}(v) = |T_v G|$ (so each loop at v contributes twice). The *genus* of a connected graph is defined as $g(G) = |E(G)| - |V(G)| + 1$, and a *tree* is a connected graph of genus zero.

A *morphism* of graphs $f : G \to K$ is a pair of maps $f : V(G) \to V(K)$ and $f : H(G) \to H(K)$ commuting with the root and involution maps. In particular, we only consider *finite* morphisms, i.e. vertices are sent to vertices and edges to edges, and no edge is contracted. A *harmonic morphism* is a pair consisting of a morphism $f : G \to K$ and a function $d_f : V(G) \cup H(G) \to \mathbb{Z}_{>0}$, called the *local degree* (for vertices) or *dilation factor* (for edges), satisfying the following properties:

(1) The degrees on the two half-edges comprising an edge $e = \{h, \overline{h}\} \in E(G)$ are equal, and we denote this quantity by $d_f(e) = d_f(h) = d_f(\overline{h})$.
(2) For any vertex $v \in V(G)$ and any half-edge $h' \in T_{f(v)}K$ we have

$$d_f(v) = \sum_{h \in T_v G \cap f^{-1}(h')} d_f(h). \tag{1}$$

When K is connected, the sum

$$\deg(f) = \sum_{y \in f^{-1}(x)} d_f(y) \tag{2}$$

is the same for any point x (vertex or half-edge) of K and is called the *(global) degree* of f. Given harmonic morphisms $f : G \to K$ and $g : K \to L$ of graphs with degree functions d_f and d_g, the

composition $g \circ f$ is a harmonic morphism with degree function $d_{g\circ f}$ given by

$$d_{g\circ f}(x) = d_f(x) d_g(f(x))$$

for any point x of G.

We say that a harmonic morphism $f : G \to K$ is *free* if it has local degree 1 everywhere; such morphisms are covering spaces in the topological sense. A harmonic morphism that is not free is called *dilated*. A *double cover* is a harmonic morphism of degree 2. Note that a double cover $\widetilde{G} \to G$ is equivalently expressed by an involution $\widetilde{G} \to \widetilde{G}$. A double cover is free if and only if the corresponding involution is fixed-point-free.

2.2. **The n-gonal construction for free covers.** We now define the n-gonal construction for free morphisms of graphs. This is done by Galois theory and is a special case of the n-gonal construction for harmonic morphisms described in Section 2.3. Our exposition is indebted to and closely follows [Don92].

Let $f : G \to K$ be a degree n cover of a connected graph K. Choosing a spanning tree $T \subset K$, we can trivialize the preimage as $f^{-1}(T) = T_1 \sqcup \cdots \sqcup T_n$, where each T_j is a copy of T. Let $\{e_1, \ldots, e_g\} = E(K) \backslash E(T)$ be the complementary edges, which we orient as $e_i = (h_i, k_i)$ (so $k_i = \overline{h_i}$). For each $j = 1, \ldots, n$ let $h_{ij} \in H(G)$ and $k_{ij} \in H(G)$ be the preimages of the h_i and k_i, respectively, labeled in such a way that $r(h_{ij}), r(k_{ij}) \in V(T_j)$. The matching of the h_{ij} and k_{ij} into edges is determined by certain *monodromy elements* $m_1, \ldots, m_g \in S_n$ by the rule

$$\overline{h_{ij}} = k_{i,m_i(j)}, \quad \text{for } i = 1, \ldots, g,$$

and conversely any set of g elements of S_n determines a degree n cover in this way. The subgroup of S_n generated by the m_i is the *monodromy group* of the cover, and the cover is connected if and only if the monodromy group acts transitively on the underlying n-element set.

Now let $\widetilde{G} \to G \to K$ be a tower consisting of a free double cover $\pi : \widetilde{G} \to G$ followed by a free degree n cover $f : G \to K$. The double cover π determines a fixed-point-free involution $\iota : \widetilde{G} \to \widetilde{G}$, and the monodromy action commutes with the involution. Hence the monodromy elements $m_1, \ldots, m_g \in S_{2n}$ associated to the cover $f \circ \pi$ lie in the *signed permutation group* $S_n^B \subset S_{2n}$ (which is the Weyl group of the root system B_n). Identifying the underlying set with $\{\pm 1, \ldots, \pm n\}$ and the involution with $j \mapsto -j$, this is the group

$$S_n^B = \left\{ m \in S_{2n} \;\middle|\; m(-j) = -m(j) \text{ for all } j = 1, \ldots, n \right\} \subset S_{2n}.$$

Conversely, a set of g elements of S_n^B determines a tower $\widetilde{G} \to G \to K$.

The n-*gonal construction* associates to the tower $\widetilde{G} \to G \to K$ a degree 2^n cover $\widetilde{p} : \widetilde{P} \to K$, whose points correspond to the local sections of π. It is convenient to describe this cover in terms of notation that we will later use for harmonic covers. The set of 2^n sections of the absolute value map $\{\pm 1, \ldots, \pm n\} \to \{1, \ldots, n\}$ (which is a local model for π) can be identified with the set of formal expressions

$$W = \left\{ \sum_{j=1}^{n} \left[a_j \cdot j + a_{-j} \cdot (-j) \right] \;\middle|\; a_{\pm j} \in \{0, 1\}, a_j + a_{-j} = 1 \right\}.$$

The group S_n^B acts on W by the rule

$$m\left(\sum \left[a_j \cdot j + a_{-j} \cdot (-j) \right] \right) = \sum \left[a_j \cdot m(j) + a_{-j} \cdot m(-j) \right],$$

giving a natural embedding $S_n^B \subset S_{2^n}$.

We now construct a degree 2^n free cover $\widetilde{p} : \widetilde{P} \to K$ in the following way. As above, pick a spanning tree $T \subset K$ and orient the complementary edges $e_i = (h_i, k_i)$, where $i = 1, \dots, g$. Let $m_1, \dots, m_g \in S_n^B$ be the monodromy elements of the cover $\widetilde{G} \to K$ with respect to these choices. For each $w \in W$ let T_w be a copy of T, and for each $v \in V(T) = V(K)$ denote the corresponding vertex in T_w by $v_w \in V(T_w)$. Similarly, for each $i = 1, \dots, g$ and each $w \in W$ let h_{iw} and k_{iw} be half-edges rooted on T_w, specifically at $r(h_{iw}) = r(h_i)_w$ and $r(k_{iw}) = r(k_i)_w$. The graph $\widetilde{P}$ is the disjoint union of the T_w over $w \in W$, with the half-edges h_{iw} and k_{iw} attached. To complete the definition, we need to specify how the h_{iw} and k_{iw} pair up into edges via the involution map, which we do using the action of the monodromy elements on W:

$$\overline{h_{iw}} = k_{im_i(w)}, \quad i = 1, \dots, g, \quad w \in W.$$

The map $\widetilde{p}$ sends each T_w isomorphically to T, and each h_{iw} and k_{iw} to h_i and k_i, respectively.

Let $\widetilde{K} \to K$ be the *orientation double cover* associated to the cover $\widetilde{P} \to K$, obtained as the quotient by the normal subgroup $A_{2^n} \subset S_{2^n}$. The tower $\widetilde{G} \to G \to K$ is said to be *orientable* if the orientation cover is trivial (i.e. split), or, equivalently, if the monodromy group lies in the *even signed permutation group* $S_n^D = S_n^B \cap A_{2n}$ (the Weyl group of the root system D_n, consisting of those permutations that change an even number of signs). The n-gonal construction $\widetilde{p} : \widetilde{P} \to K$ associated to an orientable $(2, n)$-tower $\widetilde{G} \to G \to K$ splits as a disjoint union of two copies of a degree 2^{n-1} cover $p : P \to K$. The groups S_n^B and S_n^D are the Weyl groups of the root systems B_n and D_n, respectively, and the bigonal, trigonal, and tetragonal constructions arise from the symmetries of these root systems for $n = 2, 3, 4$. We will review each of these constructions below.

2.3. **The tropical n-gonal construction.** We now extend the n-gonal construction to a tower $\widetilde{G} \to G \to K$, where the harmonic, not necessarily free morphisms $\pi : \widetilde{G} \to G$ and $f : G \to K$ have degrees 2 and n, respectively. We first summarize the idea.

Let $x \in K$ be a point. If the morphisms π and f have degree one at points over x, then the fiber $(f \circ \pi)^{-1}(x)$ can be identified with $\{\pm 1, \dots, \pm n\}$, and the n-gonal construction $\widetilde{p} : \widetilde{P} \to K$ is defined (over x) to be the set of sections of the absolute value map $\{\pm 1, \dots, \pm n\} \to \{1, \dots, n\}$ (which gives 2^n points over x). Now if f is not free, we think of each point $y \in f^{-1}(x)$ as $d_f(y)$ actual points, which are infinitesimally packaged together. More formally, we identify $f^{-1}(x)$ with a *partition* of $\{1, \dots, n\}$ into sets of sizes $d_f(y)$ for every $y \in f^{-1}(x)$, so that the points of $f^{-1}(x)$ correspond to the elements of the partition. The corresponding description of the fiber $(f \circ \pi)^{-1}(x)$ is given by *signed partitions*, and the fiber $\widetilde{p}^{-1}(x)$ is the set of *multisections* of the signed partition.

Definition 2.1. A *signed partition* is a partition of the set $\{\pm 1, \dots, \pm n\}$ that is stable under the involution $\iota(k) = -k$. Such a partition has the following form. First, we choose a partition of the set $\{1, \dots, n\}$:

$$\{1, \dots, n\} = D_1 \sqcup \cdots \sqcup D_k,$$

where $|D_i| = d_i$ and $d_1 + \cdots + d_k = n$. We then partition each preimage $f^{-1}(D_i) \subset \{\pm 1, \dots, \pm n\}$ in one of two possible ways:

(1) $f^{-1}(D_i)$ consists of two parts, namely $D_{+i} = \{+k \mid k \in D_i)$ and $D_{-i} = \{-k \mid k \in D_i\}$, each mapping bijectively onto D_i. We say that D_i is *free*.

(2) $f^{-1}(D_i)$ is a single $2n$-element subset, which we denote both D_{+i} and D_{-i} by abuse of notation. We say that D_i is *dilated*.

We say that the partition $\{D_{\pm 1}, \dots, D_{\pm k}\}$ is *free* if each D_i is free (i.e. if $D_{+i} \neq D_{-i}$ for all i) and *dilated* otherwise.

Thinking of the fibers $(f \circ \pi)^{-1}(x)$ as signed partitions, the corresponding description of the fibers of the n-gonal construction is given as follows.

Definition 2.2. A *multisection* Σ of a signed partition $\{D_{\pm i}\}$ is an expression of the form

$$\Sigma = a_{+1}D_{+1} + a_{-1}D_{-1} + \cdots + a_{+k}D_{+k} + a_{-k}D_{-k},$$

where the $a_{\pm i}$ are nonnegative integers satisfying $a_{+i} + a_{-i} = d_i$ for $i = 1, \dots, k$ (in particular, the $a_{\pm i}$ sum to n).

For each free D_i there are $d_i + 1$ choices for the $a_{\pm i}$, while for each dilated D_i all choices give the same result, since $D_{+i} = D_{-i}$. Hence the number of multisections of the partition is equal to

$$\left|\left\{a_{+1}D_{+1} + a_{-1}D_{-1} + \cdots + a_{+k}D_{+k} + a_{-k}D_{-k}\right\}\right| = \prod_{\text{free } D_i} (d_i + 1).$$

Suppose that the signed partition $\{E_{\pm 1}, \dots, E_{\pm m}\}$ is a *refinement* of $\{D_{\pm 1}, \dots, D_{\pm k}\}$, so that each of the parts $E_{\pm j}$ of the first partition lies in one of the parts of the second, which we denote by $r(E_{\pm j})$. Given a multisection Σ of $\{E_{\pm j}\}$, we define the *induced multisection* $r(\Sigma)$ of the partition $\{D_{\pm i}\}$ by

$$r\left(\sum_{j=1}^{m} \left(b_{+j}E_{+j} + b_{-j}E_{-j}\right)\right) = \sum_{j=1}^{m} \left(b_{+j}r(E_{+j}) + b_{-j}r(E_{-j})\right).$$

In particular, a section σ of the map $f : \{\pm 1, \dots, \pm n\} \to \{1, \dots, n\}$ (which is a multisection corresponding to the trivial partition) induces a multisection $r(\sigma)$ by the rule $a_{+i} = |\sigma(D_i) \cap D_{+i}|$ and $a_{-i} = d_i - a_{+i}$. The *degree* of a multisection is the number of sections inducing it, and is equal to

$$\deg\left(\sum_{i=1}^{k} \left(a_{+i}D_{+i} + a_{-i}D_{-i}\right)\right) = \prod_{\text{free } D_i} \binom{d_i}{a_{+i}} \times \prod_{\text{dilated } D_i} 2^{d_i}. \tag{3}$$

If the partition $\{D_{\pm i}\}$ is free, we define the *sign* of a multisection as

$$\operatorname{sgn}\left(\sum_{i=1}^{k} \left(a_{+i}D_{+i} + a_{-i}D_{-i}\right)\right) = (-1)^{a_{+1}+\cdots+a_{+k}}.$$

It is clear that $\operatorname{sgn}\Sigma = \operatorname{sgn} r(\Sigma)$ for any multisection if both partitions are free, and $\operatorname{sgn} r(\sigma) = \operatorname{sgn}(\sigma(1)\cdots\sigma(n))$ for any section σ.

We now apply this construction to each fiber of the tower $\widetilde{G} \to G \to K$. Recall that by a *point* of a graph we mean either a vertex or a half-edge.

Definition 2.3 (Tropical n-gonal construction)**.** Let K be a connected graph, and let $\widetilde{G} \to G \to K$ be a tower consisting of a double cover $\pi : \widetilde{G} \to G$ followed by a degree n harmonic morphism $f : G \to K$. The n-*gonal construction* $\widetilde{p} : \widetilde{P} \to K$ is a harmonic morphism of degree 2^n constructed as follows.

We define the graph morphism $\widetilde{p} : \widetilde{P} \to K$ by specifying the fiber over each point x of K (the points over a vertex are vertices, and the points over a half-edge are half-edges). Denote the

preimages of x in G by $f^{-1}(x) = \{x_1, \dots, x_k\}$. Each x_i is either *free*, having two preimages in $\widetilde{G}$ that we label $\pi^{-1}(x_i) = \{\widetilde{x}_{+i}, \widetilde{x}_{-i}\}$, or *dilated*, having a unique preimage that we label $\pi^{-1}(x_i) = \widetilde{x}_{+i} = \widetilde{x}_{-i}$ by abuse of notation. We then view the x_i and the $\widetilde{x}_{\pm i}$ as defining respectively a partition of the set $\{1, \dots, n\}$ and a signed partition of the set $\{\pm 1, \dots, \pm n\}$, where each x_i corresponds to a subset of size $d_f(x_i)$ (the local degree of f at x_i). We define the points of $\widetilde{P}$ lying above x as the set of multisections

$$\widetilde{p}^{-1}(x) = \left\{ a_{+1}\widetilde{x}_{+1} + a_{-1}\widetilde{x}_{-1} + \cdots + a_{+k}\widetilde{x}_{+k} + a_{-k}\widetilde{x}_{-k} \;\middle|\; a_{\pm i} \in \mathbb{Z}_{\geqslant 0},\ a_{+i} + a_{-i} = d_f(x_i) \right\}$$

of the signed partition $\{\widetilde{x}_{\pm i}\}$. The number of points in $\widetilde{p}^{-1}(x)$ is equal to

$$\left|\widetilde{p}^{-1}(x)\right| = \prod_{\text{free } x_i} \left(d_f(x_i) + 1\right).$$

Now let $h \in H(K)$ be a half-edge rooted at a vertex $v \in V(K)$. The root map allows us to view the signed partition $\{\widetilde{h}_{\pm j}\} = (f \circ \pi)^{-1}(h)$ as a refinement of the signed partition $\{\widetilde{v}_{\pm i}\} = (f \circ \pi)^{-1}(v)$, and the root map on $\widetilde{P}$ is defined by inducing the corresponding multisections:

$$r\left[\sum \left(a_{+i}\widetilde{h}_{+i} + a_{-i}\widetilde{h}_{-i}\right)\right] = \sum \left[a_{+i} r\left(\widetilde{h}_{+i}\right) + a_{-i} r\left(\widetilde{h}_{-i}\right)\right].$$

The involution on the half-edges is induced from $\widetilde{G}$:

$$\overline{\sum \left(a_{+i}\widetilde{h}_{+i} + a_{-i}\widetilde{h}_{-i}\right)} = \sum \left(a_{+i}\overline{\widetilde{h}_{+i}} + a_{-i}\overline{\widetilde{h}_{-i}}\right).$$

This defines the graph $\widetilde{P}$ and the morphism $\widetilde{p} : \widetilde{P} \to K$. Finally, we define the local degree of $\widetilde{p}$ on a point x of $\widetilde{P}$ as the degree (3) of the multisection:

$$d_{\widetilde{p}}\left(\sum \left(a_{+i}\widetilde{x}_{+i} + a_{-i}\widetilde{x}_{-i}\right)\right) = \prod_{\text{free } x_i} \binom{d_f(x_i)}{a_{+i}} \times \prod_{\text{dilated } x_i} 2^{d_f(x_i)}. \tag{4}$$

We note that this construction specializes to the one described in Section 2.2 when π and f are free. Indeed, in this case for each point x of K the partition $\{\widetilde{x}_{\pm i}\}$ is trivial, the set of multisections is identified with the set W, and the local degrees (4) are equal to one. The involution on the half-edges is trivial over an edge $e \in E(T)$ of the spanning tree, and over a complementary edge is induced by the corresponding monodromy element.

We define the *orientation double cover* $k : \widetilde{K} \to K$ of the tower $\widetilde{G} \to G \to K$ as a quotient of $\widetilde{p} : \widetilde{P} \to K$ by the fiberwise equivalence relation induced by signs of multisections. More precisely, we say that a point $x \in K$ is *dilated* if any point of $f^{-1}(x)$ is dilated, and *free* if all points $f^{-1}(x)$ are free. For a dilated point x we set $k^{-1}(x) = \{\widetilde{x}\}$ and $d_k(\widetilde{x}) = 2$. For x free define $k^{-1}(x) = \{\widetilde{x}^+, \widetilde{x}^-\}$ with two points, corresponding to the two possible signs of the multisections in $\widetilde{p}^{-1}(x)$, and we set $d_k(\widetilde{x}^{\pm}) = 1$. This defines a harmonic double cover $k : \widetilde{K} \to K$. Indeed, if $h \in H(K)$ is a free half-edge rooted at a free vertex $v = r(h) \in V(K)$, then $r(\widetilde{h}^{\pm}) = \widetilde{v}^{\pm}$ because the root map on multisections preserves sign. There is a natural quotient morphism $q : \widetilde{P} \to \widetilde{K}$ sending a multisection to its sign, and we set

$$d_q(y) = \begin{cases} d_{\widetilde{p}}(y)/2, & \text{if } \widetilde{p}(y) \text{ is dilated,} \\ d_{\widetilde{p}}(y), & \text{if } \widetilde{p}(y) \text{ is free,} \end{cases} \tag{5}$$

where $d_{\tilde{p}}$ is given by (4). The orientation double cover is free if and only if the double cover $\pi : \tilde{G} \to G$ is free, in which case we say that the tower $\tilde{G} \to G \to K$ is *orientable* if $\tilde{K} \to K$ is a trivial double cover.

We need to verify that the morphisms $\tilde{p} : \tilde{P} \to K$ and $q : \tilde{P} \to \tilde{K}$ are harmonic.

Proposition 2.4. *The morphisms $\tilde{p} : \tilde{P} \to K$ and $q : \tilde{P} \to \tilde{K}$ are harmonic of degrees 2^n and 2^{n-1}, respectively.*

Proof. Since $\tilde{p}$ is the composition of q with the degree 2 morphism $k : \tilde{K} \to K$, it is sufficient to verify the statement for q.

Pick a vertex $v \in V(K)$ and a half-edge $h \in H(K)$ rooted at v. The root map defines a refinement of signed partitions $r : \{\tilde{h}_j^{\pm}\} \to \{\tilde{v}_i^{\pm}\}$, where $\{\tilde{v}_i^{\pm}\} = (f \circ \pi)^{-1}(v)$ and $\{\tilde{h}_j^{\pm}\} = (f \circ \pi)^{-1}(h)$. A vertex $w \in V(\tilde{P})$ lying over v is a multisection of the signed partition $\{\tilde{v}_i^{\pm}\} = (f \circ \pi)^{-1}(v)$, while the half-edges $l \in H(\tilde{P})$ lying over h and rooted at v are those multisections of the signed partition $\{\tilde{h}_j^{\pm}\}$ that induce the multisection w.

Since we are mapping to $\tilde{K}$, we need to consider signs. There are three possibilities:

(1) Both v and h are dilated, having unique preimages $\tilde{v}$ and $\tilde{h}$ in $\tilde{K}$. The local degree at w is $d_q(w) = d_{\tilde{p}}(w)/2$, where $d_{\tilde{p}}(w)$ is the number of sections σ that induce the multisection w of the signed partition $\{\tilde{v}_i^{\pm}\}$. For each half-edge $l \in T_w\tilde{P}$ mapping to $\tilde{h}$, the local degree is $d_q(l) = d_{\tilde{p}}(l)/2$, where $d_{\tilde{p}}(l)$ is the number of sections inducing the multisection l of $\{\tilde{h}_j^{\pm}\}$. Since $T_w\tilde{P} \cap q^{-1}(\tilde{h})$ is the set of multisections of $\{\tilde{h}_j^{\pm}\}$ inducing w, it follows that

$$d_q(w) = \sum_{l \in T_w\tilde{P} \cap q^{-1}(\tilde{h})} d_q(l),$$

as required.

(2) The vertex v is dilated, while the half-edge h is free, having preimages $\tilde{h}^{\pm}$ in $\tilde{K}$. As before, $d_q(w) = d_{\tilde{p}}(w)/2$ is half the total number of sections inducing w. The half-edges $l \in T_w\tilde{P}$ mapping to $\tilde{h}^+$ (respectively $\tilde{h}^-$) are the even (respectively odd) multisections of $\{\tilde{h}_j^{\pm}\}$ inducing w, and now $d_q(l)$ counts sections without a $1/2$ factor. Hence

$$d_q(w) = \sum_{l \in T_w\tilde{P} \cap q^{-1}(\tilde{h}^+)} d_q(l) = \sum_{l \in T_w\tilde{P} \cap q^{-1}(\tilde{h}^-)} d_q(l),$$

since the number of even and odd sections inducing w is the same.

(3) Both v and h are free. The argument is identical to 1, except that we are now counting sections with a fixed parity.

To finish the proof, we note that $\deg(q)$ is half the total number of sections (over a dilated point), or the number of even or odd sections (over a free point), and hence is equal to 2^{n-1}. □

We introduce one additional structure on $\tilde{P}$. The involution $\iota : \tilde{G} \to \tilde{G}$ associated to the double cover $\pi : \tilde{G} \to G$ induces an involution $\iota_P : \tilde{P} \to \tilde{P}$ which acts by exchanging all signs:

$$\iota_P\left(\sum \left(a_{+i}\tilde{x}_{+i} + a_{-i}\tilde{x}_{-i}\right)\right) = \sum \left(a_{+i}\tilde{x}_{-i} + a_{-i}\tilde{x}_{+i}\right).$$

Denoting the quotient by $P = \tilde{P}/\iota_P$, we obtain a double cover $\tilde{P} \to P$ (having local degrees 1 and 2 corresponding to the orbits of size 2 and 1, respectively). The involution ι_P is fixed-point-free

if and only if over every vertex $v \in V(K)$ there is a free vertex $v_i \in f^{-1}(v)$ having odd local degree $d_f(v_i)$. Hence the double cover $\widetilde{P} \to P$ need not be free if $\widetilde{G} \to G$ is free, and vice versa.

Since ι_P preserves the local degrees of $\widetilde{p}$, there is an induced harmonic morphism $p : P \to K$ of degree 2^{n-1}. If n is even, then the involution ι_p preserves signs, and our morphisms factor into a tower

$$\widetilde{P} \xrightarrow{2} P \xrightarrow{2^{n-2}} \widetilde{K} \xrightarrow{2} K.$$

On the other hand, if n is odd then the involution ι_p exchanges the equivalence classes in every fiber (if they are distinct), and we have a diagram

$$\begin{tikzcd}
 & \widetilde{P} \arrow[dl, "2"'] \arrow[dr, "2^{n-1}"] & \\
P \arrow[dr, "2^{n-1}"'] & & \widetilde{K} \arrow[dl, "2"] \\
 & K &
\end{tikzcd}$$

In particular, if the double cover $\widetilde{G} \to G \to K$ is orientable, then the n-gonal construction $\widetilde{p} : \widetilde{P} \to K$ splits as two isomorphic copies of the degree 2^{n-1} cover $p : P \to K$ that are exchanged by ι_P.

For $n = 2$, 3, and 4, the construction is called the *bigonal*, *trigonal*, and *tetragonal construction*, respectively. We now work out in detail these special cases.

2.4. **The tropical bigonal construction.** We first recall the bigonal construction for free covers (see Section 2.3 in [Don92]). Let $\widetilde{G} \to G \to K$ be a tower of free double covers, defined by monodromy elements $m_1, \ldots, m_g \in S_2^B$. The outcome of the bigonal construction is a tower $\widetilde{P} \to P \to K$ of the same type, where $P = \widetilde{K}$ coincides with the orientation double cover. The group S_2^B has an external automorphism: identifying S_2^B with the dihedral group D_4, it is conjugation by a $\pi/4$ rotation. The monodromy elements of the tower $\widetilde{P} \to P \to K$ are obtained by applying the external homomorphism to the m_i. Iterating the bigonal construction reproduces the original cover, because the square of the external automorphism is an internal automorphism. It is elementary to show by looking at the monodromy group that if $\widetilde{G}$ is connected, then $\widetilde{P}$ is connected if and only if the tower $\widetilde{G} \to G \to K$ is not orientable.

We now consider a tower $\widetilde{G} \to G \to K$ of harmonic double covers of graphs. We first introduce a classification system for points of K:

Definition 2.5. Let $\widetilde{G} \to G \to K$ be a tower of harmonic double covers. A point $x \in V(K) \cup H(K)$ is called

(1) *Type I* if it has one dilated preimage in P.
(2) *Type II* if it has one free preimage in P.
(3) *Type III* if it has two preimages in P, one free and one dilated.
(4) *Type IV* if it has two preimages in P, both free.
(5) *Type V* if it has two preimages in P, both dilated.

Let $\widetilde{P} \to P \to K$ be the bigonal construction (see Definition 2.3 with $n = 2$) associated to the tower $\widetilde{G} \to G \to K$. The types of points of K change as follows (see Figure 1):

$$\mathrm{I} \to \mathrm{I}, \quad \mathrm{II} \to \mathrm{III}, \quad \mathrm{III} \to \mathrm{II}, \quad \mathrm{IV} \to \mathrm{IV}, \quad \mathrm{V} \to \mathrm{I}.$$

$\widetilde{G} \longrightarrow G \longrightarrow K$				$\widetilde{P} \longrightarrow P \longrightarrow K$			
Type I	$x_1^{\pm}$ ●	x_1 •	x •	Type I	$2x_1^{\pm}$ ●	•	x •
Type II	x_1^{+} • x_1^{-} •	x_1 •	x •	Type III	$2x_1^{+}$ • $2x_1^{-}$ • $x_1^{+} + x_1^{-}$ •	• •	x •
Type III	x_1^{+} • x_1^{-} • $x_2^{\pm}$ •	x_1 • x_2 •	x •	Type II	$x_1^{+} + x_2^{\pm}$ • $x_1^{-} + x_2^{\pm}$ •	•	x •
Type IV	x_1^{+} • x_1^{-} • x_2^{+} • x_2^{-} •	x_1 • x_2 •	x •	Type IV	$x_1^{+} + x_2^{+}$ • $x_1^{-} + x_2^{-}$ • $x_1^{+} + x_2^{-}$ • $x_1^{-} + x_2^{+}$ •	• •	x •
Type V	$x_1^{\pm}$ • $x_2^{\pm}$ •	x_1 • x_2 •	x •	Type I	$x_1^{\pm} + x_2^{\pm}$ ●	•	x •

FIGURE 1. The structure of the bigonal construction locally over a point $x \in K$. The size of points indicates the dilation factor with respect to K.

We immediately observe that the tropical bigonal construction is not invertible, since Type V and Type I points both produce Type I points. This phenomenon, which we call *dilation collapse*, also occurs for the trigonal construction (see Remark 2.17) and forces us to introduce restrictions on the dilation of the n-gonal map.

Definition 2.6. A tower $\widetilde{G} \to G \to K$ of harmonic double covers is called *generic* if K has no points of Type V, in other words if no point of K has two preimages in G, each having a single preimage in $\widetilde{G}$.

Note that in a generic tower $\widetilde{G} \to G \to K$, the type of a point of K is the number of preimages in $\widetilde{G}$. Restricted to generic towers, the bigonal construction is an involution, and dilation behavior in fibers is exchanged (this is the tropical analogue of Lemma 2.7 in [Don92]).

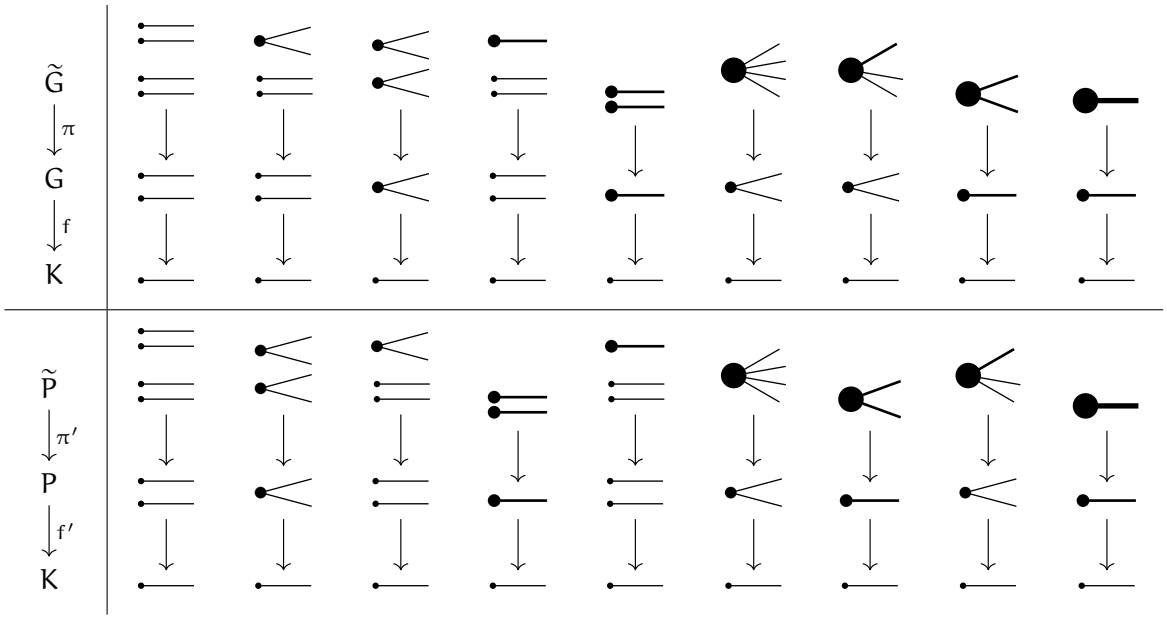

FIGURE 2. Half-edge local pictures of the tropical bigonal construction. Thickness of edges and vertices corresponds to dilation.

Proposition 2.7. *Let $\widetilde{G} \to G \to K$ be a generic tower of harmonic double covers, and let $\widetilde{P} \to P \to K$ be the bigonal construction.*

(1) The tower $\widetilde{P} \to P \to K$ is also generic.
(2) Points of K that are dilated with respect to $G \to K$ are in 1:1-correspondence with points of P that are dilated with respect to $\widetilde{P} \to P$, and the same is true for $\widetilde{G} \to G$ and $P \to K$.
(3) The bigonal construction applied to $\widetilde{P} \to P \to K$ reproduces the original tower.

Proof. The first two statements follow directly from the type classification shown on Figure 1. For the last part, let $\widetilde{G}' \to G' \to K$ be the bigonal construction of $\widetilde{P} \to P \to K$. Again it is obvious from Figure 1 that the fibers of $\widetilde{G} \to G \to K$ and $\widetilde{G}' \to G' \to K$ are the same over every $x \in K$. To complete the proof it remains to check that this identification commutes with the root map. This can be done case by case: all possible half-edge local pictures of the bigonal construction are shown in Figure 2. The computation is left to the avid reader. □

We now restrict our attention to generic towers $\widetilde{G} \to G \to K$, where the graph K is a tree, so that the double cover $f : G \to K$ is a *hyperelliptic graph*.

Proposition 2.8. *Let $\widetilde{G} \to G \to K$ be a generic tower of harmonic double covers, where $\widetilde{G}$ is connected and K is a tree, and let $\widetilde{P} \to P \to K$ be the bigonal construction. Then $\widetilde{P}$ is connected if and only if the double cover $\pi : \widetilde{G} \to G$ is not free. Furthermore, in this case, the genera of the graphs are related as follows:*

$$g(\widetilde{G}) - g(G) = g(\widetilde{P}) - g(P). \tag{6}$$

Proof. If the double cover π is free, then so is the orientation double cover $\widetilde{K} \to K$, which is then trivial since K is a tree. Therefore $P = \widetilde{K}$ is disconnected, and hence so is $\widetilde{P}$.

Conversely, suppose that π is not free. If K contains a point x of type I with respect to the tower $\widetilde{G} \to G \to K$, then x has type I with respect to the tower $\widetilde{P} \to P \to K$ as well, and therefore

$\widetilde{P}$ is connected. If K contains no points of type I, then it must contain a point of type II (otherwise $G \to K$ is free and hence disconnected) and a point of type III (otherwise π is free). The bigonal construction exchanges types II and III, hence the tower $\widetilde{P} \to P \to K$ also contains both type II and III points. Therefore both $\widetilde{P} \to P$ and $P \to K$ are dilated, so $\widetilde{P}$ is connected.

To determine the relationship between the genera of the curves $\widetilde{G}$, G, $\widetilde{P}$, P (assuming that they are all connected), we look at the local structure (see Figure 1). The vertex counts satisfy

$$\begin{aligned} |V(\widetilde{G})| - |V(G)| &= 2 \cdot \#\{\text{type IV vertices}\} + \#\{\text{type II vertices}\} + \#\{\text{type III vertices}\} \\ &= |V(\widetilde{P})| - |V(P)|, \end{aligned}$$

and a similar relation holds for the number of edges:

$$\begin{aligned} |E(\widetilde{G})| - |E(G)| &= 2 \cdot \#\{\text{type IV edges}\} + \#\{\text{type II edges}\} + \#\{\text{type III edges}\} \\ &= |E(\widetilde{P})| - |E(P)|. \end{aligned}$$

The relation (6) immediately follows. □

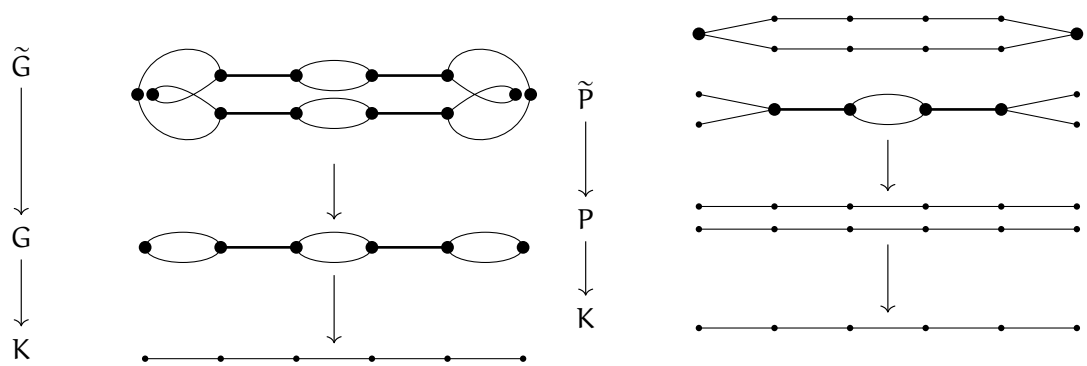

FIGURE 3. Example of the bigonal construction with thickness indicating dilation with respect to the base tree K. The involution on $\widetilde{G}$ is reflection along the horizontal axis and similarly for each of the two components of $\widetilde{P}$.

Example 2.9. In Figure 3 we see an example of the bigonal construction. Note that $\widetilde{G} \to G$ is a free double cover and therefore $\widetilde{P}$ is necessarily disconncted. Nevertheless, one can check that applying the bigonal construction to the tower $\widetilde{P} \to P \to K$ reproduces the input tower. Let us modify this example by contracting the extremal edges of K and everything lying above them. The result is shown in Figure 4. This time the input (and hence the output as well) contains a vertex of Type I. In particular, input and output are connected. Again one can check that the bigonal construction applied to $\widetilde{P} \to P \to K$ reproduces the original tower.

2.5. **The tropical trigonal and Recillas construction.** Let $\widetilde{G} \to G \to K$ be a tower of free covers of a graph K, of degrees 3 and 2. The topological trigonal construction associates to the tower $\widetilde{G} \to G \to K$ a tower $\widetilde{P} \to P \to K$ of free covers, of degrees 4 and 2. If the tower $\widetilde{G} \to G \to K$ is orientable, then the double cover $\widetilde{P} \to P$ is trivial. The *Recillas construction* inverts this correspondence, by associating to a degree 4 free cover $P \to K$ an orientable tower $\widetilde{G} \to G \to K$ of free covers of degrees 3 and 2.

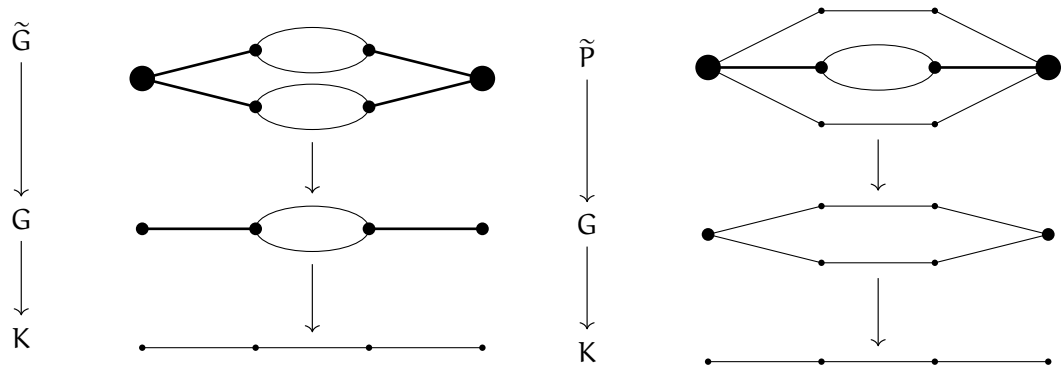

FIGURE 4. Example of the bigonal construction with connected input and output. Thickness corresponds to dilation and the involutions are given by reflection along the horizontal axis.

The trigonal and Recillas constructions arise from the group isomorphism $S_3^D \simeq S_4$, which corresponds to the equality $A_3 = D_3$ of root systems. We recall this isomorphism. The map $|\cdot| : \{\pm 1, \pm 2, \pm 3\} \to \{1, 2, 3\}$ has four even sections (such that the product of the images is positive), and the group S_3^D acts on this four-element set. This defines a homomorphism $S_3^D \to S_4$. Conversely, each pair of even sections corresponds a unique element of $\{\pm 1, \pm 2, \pm 3\}$ (the intersection of their images), and this gives the inverse homomorphism. Hence we obtain a bijection between orientable free towers $\widetilde{G} \to G \to K$ of degrees 3 and 2 and free degree 4 covers $P \to K$.

At this point the graphs $\widetilde{G}$ and P are not necessarily connected, and for future use we determine when they are. We say that a subgroup $H \subset S_4$ is *4-transitive* if it acts transitively on the underlying 4-element set. Similarly, we say that $H \subset S_4$ is *6-transitive* if it acts transitively on the set $\{\pm 1, \pm 2, \pm 3\}$ under the isomorphism $S_3^D \simeq S_4$. The graphs $\widetilde{G}$ and P are connected if and only if the monodromy group is 6-transitive or 4-transitive, respectively.

Lemma 2.10. *A 6-transitive subgroup of S_4 is 4-transitive. A 4-transitive subgroup of S_4 is 6-transitive if and only if it is generated by 2-cycles and 3-cycles.*

Proof. The 4-transitive subgroups of S_4 are, up to conjugacy, the cyclic subgroup generated by a 4-cycle, the Klein 4-group, the dihedral subgroup (generated by $(abcd)$ and (ac)), A_4, and S_4 itself. Of these, only A_4 and S_4 are generated by 2-cycles and 3-cycles. A 6-transitive subgroup has order divisible by 6, ruling out all subgroups except for S_3, A_4, and S_4. A direct verification shows that A_4 is 6-transitive while S_3 is not, which completes the proof. □

We now describe the trigonal and Recillas constructions for harmonic covers. Let $\widetilde{G} \to G \to K$ be an orientable double cover of a degree 3 harmonic morphism (which means, in particular, that $\pi : \widetilde{G} \to G$ is free). The output of the trigonal construction is (a trivial double cover over) a harmonic morphism $p : P \to K$ of degree 4. Similarly to the bigonal case, we classify the points of K by the structure of the fibers:

Definition 2.11. Let $\pi : \widetilde{G} \to G$ be an orientable double cover of a degree 3 harmonic morphism $f : G \to K$, and let $p : P \to K$ be the associated harmonic morphism of degree 4. A point $x \in V(K) \cup H(K)$ is said to be

(1) *Type A* if it has three preimages x_1, x_2, and x_3 in G. The corresponding points of P over x are $x_{+1}+x_{+2}+x_{+3}$, $x_{+1}+x_{-2}+x_{-3}$, $x_{-1}+x_{+2}+x_{-3}$, and $x_{-1}+x_{-2}+x_{+3}$. The local degrees of f and p are all equal to one.
(2) *Type B* if it has two preimages in G, namely x_1 with $d_f(x_1) = 1$ and x_2 with $d_f(x_2) = 2$. The corresponding points of P are $x_{+1} + 2x_{+2}$, $x_{+1} + 2x_{-2}$, and $x_{-1} + x_{+2} + x_{-2}$, at which p has degrees 1, 1, and 2, respectively.
(3) *Type C* if it has a single preimage x_1 in G with $d_f(x_1) = 3$. The corresponding points of P are $3x_{+1}$ and $x_{+1} + 2x_{-1}$, at which p has degrees 1 and 3, respectively.

A half-edge of type A may be rooted at a vertex of any type, but a half-edge of type B may be rooted at a type B or C vertex, while half-edge of type C may only be rooted at a type C vertex. There are thus six possible pairings of a half-edge and a vertex, and the local structure of the tower and the degree four map are shown on Figure 5. We note that the letters A, B, and C do not refer to root systems.

The outcome of the trigonal construction is a degree 4 harmonic morphism $p : P \to K$ with the property that the fibers of p do not have degree profiles (4) or $(2, 2)$. We observe that, in the algebraic setting, an identical restriction is imposed on the *ramification* profile of the degree four map. We give a corresponding definition:

Definition 2.12. A degree 4 harmonic morphism $p : P \to K$ is called *generic* if every point $x \in K$ has a preimage in P at which the local degree of p is equal to one. Given a generic $p : P \to K$, a point x of K is said to be of *type A, B, or C* if the degree profile of the fiber is $(1, 1, 1, 1)$, $(2, 1, 1)$, or $(3, 1)$, respectively.

For a generic degree 4 harmonic morphism $p : P \to K$, we can invert the tropical trigonal construction.

Definition 2.13 (The tropical Recillas construction)**.** Let $p : P \to K$ be a generic degree 4 harmonic morphism. The *Recillas construction* associated to $p : P \to K$ is a tower $\widetilde{G} \to G \to K$ of harmonic morphisms of degrees 2 and 3, where $\widetilde{G} \to G$ is free, constructed as follows.

For each point $x \in V(K) \cup H(K)$, consider a 4-element set $c(x) = \{x_1, x_2, x_3, x_4\}$ together with a partition whose components correspond to the points of $p^{-1}(x)$, and have sizes given by the local degrees of p. For each half-edge $h \in H(K)$ with root vertex $v = r(h)$, the root map $r : p^{-1}(h) \to p^{-1}(v)$ defines (non-uniquely in general) a bijective map $r : c(h) \to c(v)$ respecting the partitions. Similarly, for every edge $e = \{h, h'\} \in E(K)$ the involution $p^{-1}(h) \to p^{-1}(h')$ can be recorded (again, not necessarily uniquely) with a bijection $\{h_1, h_2, h_3, h_4\} \to \{h'_1, h'_2, h'_3, h'_4\}$ denoted $h_i \mapsto \overline{h_i}$.

We now consider, for each point x of K, the set $t(x)$ of two-element subsets of $\{x_1, x_2, x_3, x_4\}$, whose elements we denote by $x_i + x_j = \{x_i, x_j\}$. Taking the complement defines a fixed-point-free involution on $t(x)$. We define on each $t(x)$ the structure of a signed partition, in such a way that the equivalence classes fit together into a degree 6 harmonic morphism $\widetilde{G} \to K$ that factors into a tower $\widetilde{G} \to G \to K$. Indeed, the chosen partition on $c(x) = \{x_1, x_2, x_3, x_4\}$ defines an equivalence relation on $t(x)$ as follows: $x_i + x_j \sim x_k + x_l$ if $x_i \sim x_k$ and $x_j \sim x_l$, or if $x_i \sim x_l$ and $x_j \sim x_k$. For example, if the partition is $\{x_1, x_2, x_3\} \cup \{x_4\}$ (so $x_1 \sim x_2 \sim x_3$), then the equivalence relations on $t(x)$ are

$$x_1 + x_2 \sim x_1 + x_3 \sim x_2 + x_3, \quad x_1 + x_4 \sim x_2 + x_4 \sim x_3 + x_4.$$

It is clear that the equivalence relation on $t(x)$ respects the involution, hence defines the structure of a signed partition on $t(x)$. The assumption that p is generic implies (indeed, is equivalent to assuming that) these signed partitions are free for all x in K

We now let $\widetilde{f}^{-1}(x)$ be the set of components of the signed partition, and define the local degree of $\widetilde{f}$ to be the size of the equivalence class. Given $h \in H(K)$ rooted at $v = r(h) \in V(K)$, the root maps on the chosen 4-element sets induce maps

$$r : \widetilde{f}^{-1}(h) \to \widetilde{f}^{-1}(v), \quad r(h_i + h_j) = r(h_i) + r(h_j)$$

by passing to equivalence classes. Similarly, for each edge $e = \{h, h'\} \in E(K)$ there is an involution $h_i + h_j \mapsto \overline{h_i} + \overline{h_j}$ connecting the half-edges $\widetilde{f}^{-1}(h)$ and $\widetilde{f}^{-1}(h')$ into edges of $\widetilde{G}$. The fibers $\widetilde{f}^{-1}(x)$ therefore form a graph $\widetilde{G}$ together with a morphism $\widetilde{f} : \widetilde{G} \to K$, which is readily verified to be a harmonic morphism of degree 6. The morphism $\widetilde{f}$ factors as a free double cover $\widetilde{G} \to G$ (because each signed partition is free) and a harmonic morphism $G \to K$ of degree 3.

To summarize, a point $y \in \widetilde{G}$ represents a pair of points $x_1, x_2 \in P$, not necessarily distinct and mapping to the same point of K, and the degrees of $\widetilde{f}$ and p at these points are related as follows:

$d_p(x_1)$	$d_p(x_2)$	point type	$d_{\widetilde{f}}(y)$
1	1	$x_1 \neq x_2$	1
2	1	$x_1 \neq x_2$	2
2	2	$x_1 = x_2$	1
3	1	$x_1 \neq x_2$	3
3	3	$x_1 = x_2$	3

Remark 2.14. Generalizing Definition 2.13, we can naturally associate to a harmonic morphism $P \to K$ of degree n a degree $\binom{n}{k}$ harmonic morphism $\widetilde{G} \to K$, for any $k \leqslant n$.

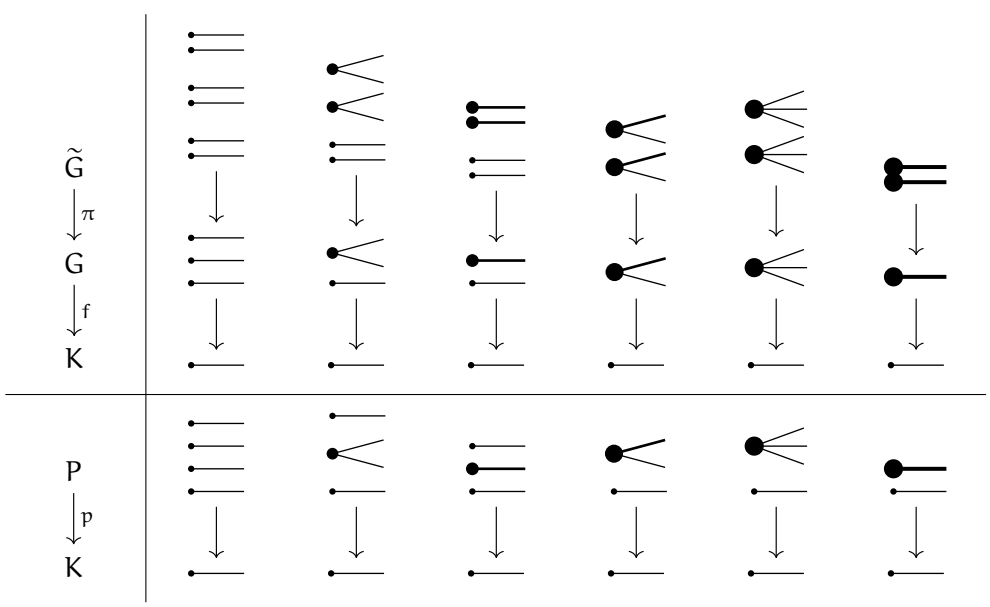

FIGURE 5. Overview of the trigonal construction and its inverse, locally over a half-edge of K. Thickness of edges and vertices corresponds to dilation.

We now assume that K is a tree, and refer to a harmonic morphism $f : G \to K$ as a *trigonal graph* (we note that a metric graph carrying a g^1_3 in the sense of Baker and Norine [BN07] is not necessarily a trigonal graph, but the converse is true [ABBR15]).

We are now ready to prove the first part of Theorem 1.2.

Proposition 2.15. *Let K be a tree. The trigonal construction and the Recillas construction establish a bijection between free double covers of trigonal curves $\widetilde{G} \to G \to K$ and generic tetragonal curves $P \to K$. The graph $\widetilde{G}$ is connected if and only if P is connected, in which case*

$$g(P) = g(G) - 1. \tag{7}$$

Example 2.16. Consider the tower $\widetilde{G} \to G \to K$ on the left of Figure 6. Applying the tropical trigonal construction to it produces the generic tetragonal curve on the right. Conversely, applying the tropical Recillas construction to $P \to K$ recovers the original tower. We will verify by hand in Example 5.2 that the Prym variety $\mathrm{Prym}(\widetilde{G}/G)$ and the Jacobian variety $\mathrm{Jac}(P)$ (which are defined after K having been equipped with arbitrary edge lengths) are isomorphic.

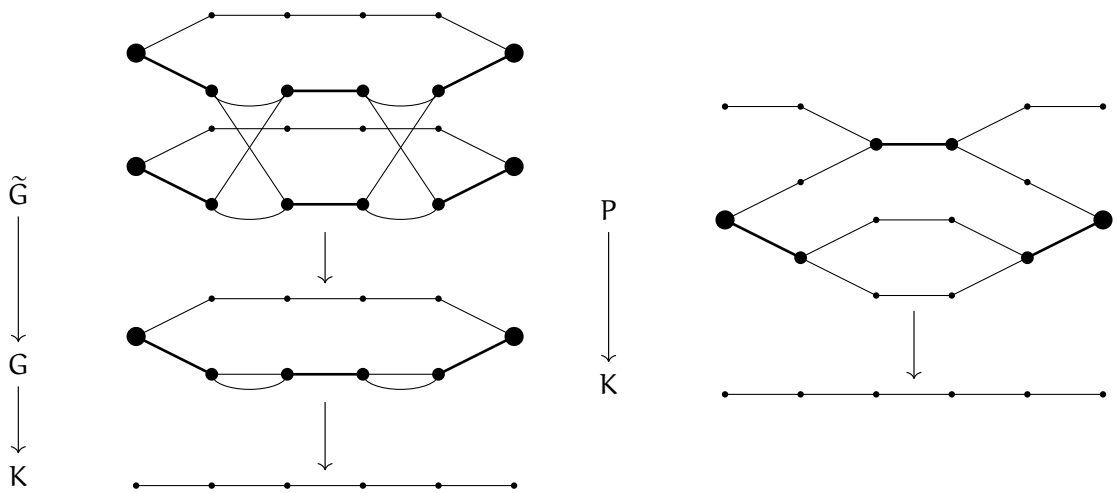

FIGURE 6. Example of a tower and a tetragonal curve corresponding to each other under the tropical trigonal construction. Thickness indicates dilation factors.

Proof of Prop. 2.15. We first note that a free double cover $\pi : \widetilde{G} \to G$ of a trigonal curve $f : G \to K$ forms an orientable tower $\widetilde{G} \to G \to K$, because the orientation double cover $\widetilde{K} \to K$ is free and hence trivial. The verification that the trigonal and Recillas constructions are inverses is straightforward and is left to the avid reader (this is first verified fiberwise, and then one checks that this identification commutes with the root maps in all six possible local situations depicted in Figure 5).

Given a *free* cover (or a tower of free covers), one can verify whether it is connected by checking the transitivity of the action of the monodromy group. The theory of harmonic covers does not have a well-developed Galois correspondence. Instead, we proceed as follows. Let $\widetilde{G} \to G \to K$ be a free double cover of a trigonal graph. We construct a tower $\widetilde{G}'' \to G'' \to K''$ of *free* harmonic morphisms of degrees 2 and 3, where K'' is a graph obtained by attaching loops to the tree K. Similarly, to a generic tetragonal graph $P \to K$ we associate a free cover $P'' \to K''$ of degree 4. The

original objects $\widetilde{G} \to G \to K$ and $P \to K$ are recovered from the free covers by *edge contraction*, an operation that preserves connectivity (so $\widetilde{G}''$ and P'' are connected if and only if respectively $\widetilde{G}$ and P are). Finally, this operation commutes with the trigonal and Recillas constructions, allowing us to reduce to the free case and apply Lemma 2.10.

We recall the operation of edge contraction for graphs and harmonic morphisms. Let $f : G \to K$ be a harmonic morphism, and let $e \in E(K)$ be an edge with root vertices u and v (which may be the same). Let K_e be the graph obtained from K by contracting e, i.e. by removing it and identifying u and v to a new vertex w. Now let G_e be the graph arising from G by contracting every edge $e' \in f^{-1}(e)$. The result is that we replace each connected component G_k of the preimage of the subgraph $\{u, v, e\} \subseteq K$ by a separate vertex $v_k \in V(G_e)$. We now define the *contraction* $f_e : G_e \to K_e$ of f along e by mapping each v_k to the new vertex w with local degree $d_{f_e}(v_k) = \deg(f|_{G_k})$, and equal to f elsewhere. It is elementary to verify that f_e is again a harmonic morphism.

Let $\widetilde{G} \to G \to K$ and $P \to K$ be related by the trigonal construction. We first remove dilated edges from $\widetilde{G}$, G, and P as follows. Given an edge $e \in E(G)$ with $d_f(e) > 1$, we replace it with two (if $d_f(e) = 2$) or three (if $d_f(e) = 3$) parallel undilated edges with the same root vertices as e. We similarly replace each of the two preimages of e in $\widetilde{G}$ with two or three parallel edges, preserving root vertices. The fiber of P over the edge $f(e)$ contains a dilated edge of the same degree $d_f(e)$, and we similarly replace this edge with $d_f(e)$ parallel edges. Performing this operation on all dilated edges of G, we obtain a tower $\widetilde{G}' \to G' \to K'$ (where $K' = K$ to preserve notation) and a tetragonal curve $P' \to K'$, which are related by the trigonal construction, such that each edge of K' has type A. Since the new edges have the same root vertices as the old edges, the graph $\widetilde{G}'$ is connected if and only if $\widetilde{G}$ is connected, and similarly for P and P'.

We now resolve the dilation above vertices of types B and C. We construct an orientable tower $\widetilde{G}'' \to G'' \to K''$ of *free* covers of degrees 3 and 2, and a degree 4 free cover $P'' \to K''$, where K'' is obtained by attaching a loop at each type B and C vertex of K', and such that $\widetilde{G}' \to G' \to K'$ and $P' \to K'$ are recovered by contracting these new loops. Specifically, for each vertex $v \in V(K')$ of type B or C, we perform the following operation:

(1) Let $v \in K'$ be a type B vertex, having preimages v_1 and u in G' (with $d_f(v_1) = 1$ and $d_f(u) = 2$), $v_1^{\pm}$ and $u^{\pm}$ in $\widetilde{G}'$, and $\tilde{v}_1^+ + 2\tilde{u}^+$, $\tilde{v}_1^- + \tilde{u}^+ + \tilde{u}^-$, and $\tilde{v}_1^+ + 2\tilde{u}^-$ in P'. We split u into two new vertices v_2 and v_3 and set $d_f(v_2) = d_f(v_3) = 1$. We similarly split $u^{\pm}$ into $\tilde{v}_2^{\pm}$ and $\tilde{v}_3^{\pm}$ in $\widetilde{G}'$, and split the degree two vertex $\tilde{v}_1^- + \tilde{u}^+ + \tilde{u}^-$ of P' into two degree one vertices $\tilde{v}_1^- + \tilde{v}_2^+ + \tilde{v}_3^-$ and $\tilde{v}_1^- + \tilde{v}_2^- + \tilde{v}_3^+$ (the other two vertices are relabeled).

We now attach a loop e at v. This loop lifts to a loop at v_1 and a pair of parallel edges between v_2 and v_3, and similarly in $\widetilde{G}'$. In P', the loop e lifts to loops at the old degree one vertices, and a pair of parallel edges joining the new degree one vertices. Finally, we reattach all loose edges to the new vertices $\widetilde{G}'$ and G' in any way that preserves harmonicity, and perform consistent attachments in P' (see Figure 7).

(2) Let $v \in K'$ be a type C vertex, having one preimage u in G' (with $d_f(u) = 3$), preimages $\tilde{u}^{\pm}$ in $\widetilde{G}'$, and preimages $3\tilde{u}^+$ and $\tilde{u}^+ + 2\tilde{u}^-$ in P'. We split u into three vertices v_1, v_2, and v_3 of degree one, and similarly split the $\tilde{u}^{\pm}$. The degree one vertex $3\tilde{u}^+$ of P' is relabeled $\tilde{v}_1^+ + \tilde{v}_2^+ + \tilde{v}_3^+$, while the degree three vertex $\tilde{u}^+ + 2\tilde{u}^-$ splits into three degree one vertices of the form $\tilde{v}_i^+ + \tilde{v}_j^- + \tilde{v}_k^-$.

We now attach a loop e to v that lifts to a triangle connecting the three new degree one vertices in G', and to a pair of triangles in $\widetilde{G}'$. Similarly, the loop e lifts to a loop at the old degree one vertex of P', and a triangle around the three new ones. We again reattach all edges in a way that preserves harmonicity (see Figure 8).

The outcome of resolving each vertex in this manner is an orientable tower $\widetilde{G}'' \to G'' \to K''$ of free covers and a degree 4 free cover $P'' \to K''$. To check that these two are related by the trigonal construction, it is sufficient to verify that, under the isomorphism $S_3^D \simeq S_4$, an element of the form $(ab)(-a-b)$ corresponds to a 2-cycle, and an element of the form $(abc)(-a-b-c)$ corresponds to a 3-cycle. Furthermore, the pair $\widetilde{G}' \to G' \to K'$ and $P' \to K'$ is recovered from $\widetilde{G}'' \to G'' \to K''$ and $P'' \to K''$ by contracting the added loops e in K''. Finally, we note that $\widetilde{G}''$ is connected if and only if $\widetilde{G}'$ is (because contracting a connected subgraph does not change connectedness), and similarly for P'' and P'.

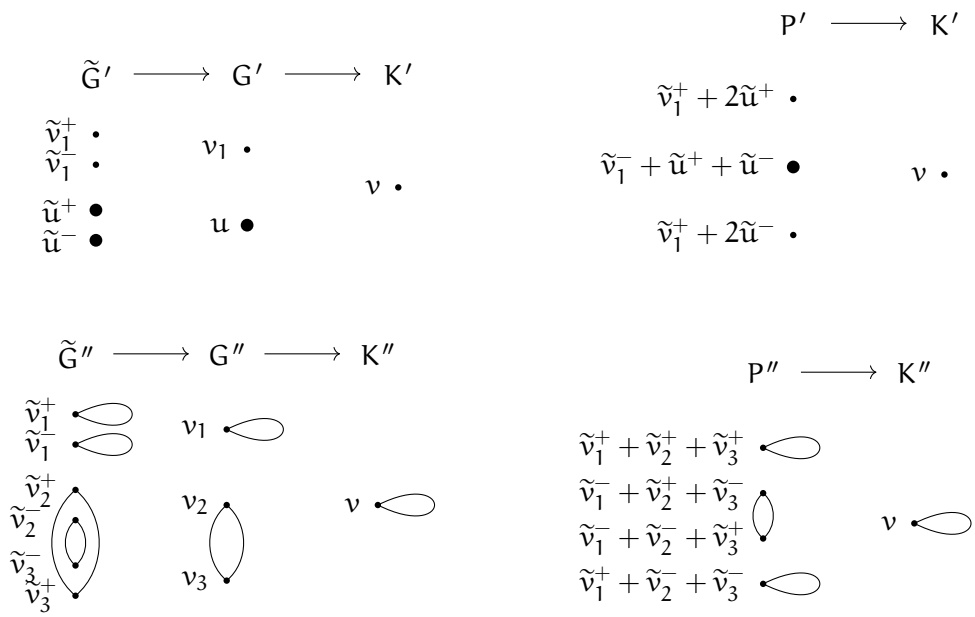

FIGURE 7. Resolution of a type B vertex

We now consider the monodromy group of the cover $P'' \to K''$. The tree K' is naturally a spanning tree for K'', and, as it is clear from Figures 7 and 8, the monodromy elements corresponding to the complementary loops are 2-cycles (when resolving a type B vertex) and 3-cycles (when resolving a type C vertex). The cover P'' is connected if and only if the monodromy group is 4-transitive. By Lemma 2.10 this happens if and only if it is 6-transitive, i.e. if and only if $\widetilde{G}''$ is connected. Hence $\widetilde{G}$ is connected if and only if P is connected.

To complete the proof, we determine the relationship between the genera. Looking at the local structure in Figure 5, we see that $p^{-1}(x)$ has one more element than $f^{-1}(x)$ for each point x of K (vertices and half-edges). It follows that

$$g(P) = |E(P)| - |V(P)| + 1 = |E(G)| + |E(K)| - |V(G)| - |V(K)| + 1 = g(G) - 1,$$

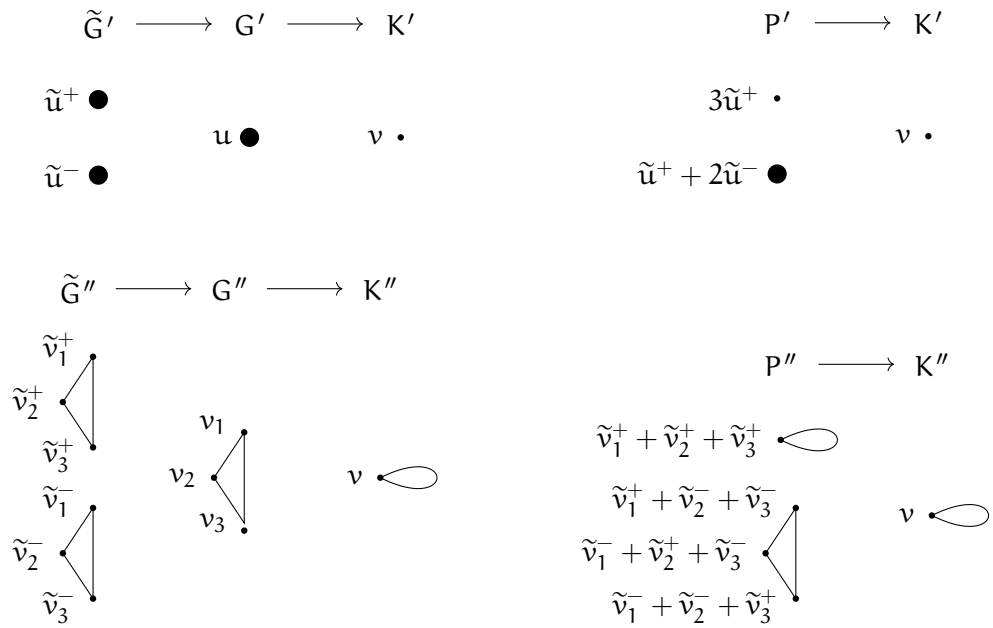

FIGURE 8. Resolution of a type C vertex

because K is a tree and hence $|E(K)| - |V(K)| = -1$. This completes the proof. □

Remark 2.17. The trigonal construction can be applied to towers $\widetilde{G} \to G \to K$ where $\widetilde{G} \to G$ is not free. The result is a tower $\widetilde{P} \to P \to K$ where, depending on the degree profiles of the first tower, $\widetilde{P} \to P$ may be dilated, $P \to K$ non-generic, or both. Conversely, the Recillas construction can be extended to non-generic tetragonal curves $P \to K$ to produce towers $\widetilde{G} \to G \to K$ where $\widetilde{G} \to G$ is not free (for example, a point x of K with degree profile $(2, 2)$ has two preimages in G, one having degree 2 and a single preimage in $\widetilde{G}$, the other having degree 1 and two preimages). However, applying the trigonal construction to the resulting tower $\widetilde{G} \to G \to K$ does not reproduce the original tetragonal curve $P \to K$. Hence the bijective correspondence fails for these generalizations.

2.6. **The tetragonal construction.** In this section, we briefly summarize the harmonic tetragonal construction, which we plan to study in detail in a future paper.

Let $\widetilde{G} \to G \to K$ be a free double cover of a free degree four cover of a graph K. Applying the tetragonal construction, we obtain a $(2, 8)$-tower of free covers $\widetilde{P} \to P \to K$. Since 4 is even, this factors through the orientation double cover

$$\widetilde{P} \longrightarrow P \longrightarrow \widetilde{K} \longrightarrow K.$$

If $\widetilde{G} \to G \to K$ is orientable, then by definition $\widetilde{K} \to K$ is trivial and thus the outcome splits as a pair of $(2, 4)$-towers $\widetilde{P}_i \to P_i \to K$. Applying the tetragonal construction to any of the $\widetilde{P}_i \to P_i \to K$ produces the other two towers, which corresponds to the triality of the root system D_4.

Now let $\widetilde{G} \to G \to K$ be a tower of harmonic morphisms of degrees 2 and 4, and let $\widetilde{P} \to K$ be the outcome of the tetragonal construction. In general, the graph $\widetilde{P}$ does not split (if $\widetilde{G} \to G$ is not

orientable), and the involution $\iota : \widetilde{P} \to \widetilde{P}$ may have fixed points. However, imposing some natural restrictions on the dilation produces an outcome that exactly corresponds to the free case, and mirrors the algebraic construction.

Proposition 2.18. *Let K be a tree, and let $\widetilde{G} \to G \to K$ be a free double cover of a generic (in the sense of Definition 2.12) tetragonal graph $G \to K$. The tetragonal construction applied to $\widetilde{G} \to G \to K$ splits as a disjoint union of $\widetilde{G}_i \to G_i \to K$ for $i = 1, 2$, where each tower is a free double cover of a generic tetragonal graph.*

Proof. If $\widetilde{G} \to G$ is free then so is the orientation double cover $\widetilde{K} \to K$, which is then trivial since K is a tree. Hence $\widetilde{P} \to K$ splits as a disjoint union of morphisms $\widetilde{P}_1 \to K$ and $\widetilde{P}_2 \to K$ of degree eight. Since $G \to K$ is generic, each point x of K has a preimage at which f has odd degree (specifically, equal to one) and which has two preimages in $\widetilde{G}$ (since $\widetilde{G} \to G$ is free). Hence the sign involution $\iota : \widetilde{P} \to \widetilde{P}$ acts without fixed points. Since the involution restricts to each connected component, we take quotients and obtain two towers $\widetilde{P}_i \to P_i \to K$, for $i = 1, 2$. It is then a direct verification to show that if a point $x \in K$ is of type A, B, or C with respect to the original tower, then x is of the same type with respect to each of the two new towers. In particular, the $P_i \to K$ are generic tetragonal curves. □

3. Rational polyhedral spaces and tropical homology

In this section, we review a number of standard notions of tropical geometry. The ambient category containing all tropical objects that we consider in this article is the category of *rational polyhedral spaces*. In particular, tropical curves are purely 1-dimensional rational polyhedral spaces satisfying a smoothness condition. We rephrase the tropical n-gonal construction and the tropical Recillas construction for tropical curves (thus justifying the name). Finally, we summarize some basic properties of tropical cycles and tropical homology that will serve as essential tools to prove Theorems 1.1 and 1.2. Tropical homology was introduced in [IKMZ19], but our exposition closely follows the sheaf-theoretic approach [GS19a].

3.1. **Rational polyhedral spaces.** A *(rational) polyhedron* in $\mathbb{R}^n$ is a finite intersection of half-spaces of the form

$$\{x \in \mathbb{R}^n \mid \langle m, x \rangle \leqslant a\}$$

for some $m \in (\mathbb{Z}^n)^*$ and $a \in \mathbb{R}$. Consider the partial compactification $\overline{\mathbb{R}}^n$ of $\mathbb{R}^n$, where $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$, endowed with the order topology. It is stratified by sets of the form

$$\mathbb{R}_I = \left\{(x_i)_{i=1,\dots,n} \;\middle|\; x_i = \infty \text{ if and only if } i \in I\right\}$$

for $I \subset \{1, \dots, n\}$. A *polyhedron* in $\overline{\mathbb{R}}^n$ is the closure of a polyhedron in one of the $\mathbb{R}_I$. A *polyhedral subset* $X \subseteq \overline{\mathbb{R}}^n$ is a finite union of polyhedra.

Let $X \subseteq \overline{\mathbb{R}}^n$ be a polyhedral subset. An *integral affine linear function* on X, or *affine function* for short, is a function $f : X \to \mathbb{R}$ such that locally at every point of X it is of the form $x \mapsto \langle m, x \rangle + a$ for some $m \in (\mathbb{Z}^n)^*$ and $a \in \mathbb{R}$. An affine function is not allowed to take an infinite value, so the local expression $\langle m, x \rangle + a$ is required to satisfy $m_i = 0$ near any point x of X with $x_i = \infty$. In other words, affine functions are locally constant at infinity. Affine functions form a sheaf on X, denoted Aff_X.

A *rational polyhedral space* X is a second countable Hausdorff topological space together with a sheaf of continuous functions Aff_X such that for every point $x \in X$ there is an open neighborhood

$x \in U \subseteq X$, an open subset $V \subseteq Y$ of a polyhedral set $Y \subseteq \overline{\mathbb{R}}^n$, and a homeomorphism $\phi : V \to U$ such that pullback of affine functions along ϕ is an isomorphism $\phi^{-1}\operatorname{Aff}_V \to \operatorname{Aff}_U$. If all these polyhedral sets Y can be taken to live in $\mathbb{R}^n$, then X is called *boundaryless*. A point $x \in X$ is called *regular* if it has an open neighborhood isomorphic to an open subset of $\mathbb{R}^n$, where n is the *local dimension* at x. The subset of regular points in X is denoted X^{reg}. A rational polyhedral space X is said to be *purely n-dimensional* if each point of X^{reg} has local dimension n.

A *morphism of rational polyhedral spaces* is a continuous map $f : X \to Y$ that induces a morphism $f^{-1}\operatorname{Aff}_Y \to \operatorname{Aff}_X$. It is *proper* if the preimage of every compact subset of Y is compact. In particular, if X is compact, then f is proper.

The *cotangent sheaf* Ω^1_X of a rational polyhedral space X is the quotient of the sheaf Aff_X by the subsheaf of locally constant functions. A morphism $f : X \to Y$ of rational polyhedral spaces induces a morphism of cotangent sheaves

$$f^{-1}\Omega^1_Y \longrightarrow \Omega^1_X.$$

For $x \in X$, the dual $T^{\mathbb{Z}}_x X = \operatorname{Hom}_{\mathbb{Z}}(\Omega^1_{X,x}, \mathbb{Z})$ of the stalk of the cotangent sheaf is the *integral tangent space* at x. Dualizing the morphism of cotangent sheaves gives the differential $d_x f : T^{\mathbb{Z}}_x X \to T^{\mathbb{Z}}_{f(x)} Y$.

3.2. **Tropical curves and harmonic morphisms.** Let Γ be a connected and compact purely 1-dimensional rational polyhedral space. The underlying topological space of Γ has the combinatorial structure of a connected and finite graph G. Furthermore, we can define an edge length function $\ell : E(G) \to (0, \infty]$ by setting $\ell(e)$ to be the smallest positive increment of an affine function along e. The pair (G, ℓ), which is a *metric graph*, is called a *model* for Γ. We say that Γ is a *tropical curve* if all edges of infinite length are extremal (in other words, there are finitely many vertices at infinity, and they are univalent). We always implicitly choose a model when talking about a tropical curve. The genus of Γ is the genus of any graph model. By a point x of Γ we mean a point in the metric space, which may correspond to either a vertex or an interior point of an edge with respect to a chosen model (an edge $e \in E(G)$ of the model may be viewed as a generic point of the corresponding edge in Γ).

An affine function f on a tropical curve Γ has a well-defined *slope* along each oriented edge e of Γ. A tropical curve Γ is called *smooth* if locally around every finite vertex it is isomorphic to

$$\bigcup_{i=0,\dots,n} e_i \mathbb{R}_{\geqslant 0} \subseteq \mathbb{R}^{n+1}/(1,\dots,1)\mathbb{R} \tag{8}$$

for some $n \geqslant 1$. This condition ensures that Γ has sufficiently many affine functions, in the following sense: given an affine function defined near a point $x \in \Gamma$, the only condition on the slopes along the outgoing edges is that they sum to zero. Furthermore, the univalent vertices of a smooth tropical curve Γ are at infinity. We note that an arbitrary metric graph can be augmented to produce a smooth tropical curve in a canonical way, by attaching a compact infinite ray to each finite univalent vertex.

Conversely, let G be a finite graph and let $\ell : E(G) \to (0, \infty]$ be an edge length assignment such that an edge is infinite if and only if it is extremal. We construct a smooth tropical curve Γ with model (G, ℓ) by gluing real intervals $[0, \ell(e)]$ for every edge $e \in E(G)$ according to the adjacency determined by G, and choosing the smooth local model (8) at each vertex of valency 3 or higher (cf. Proposition 3.6 in [MZ08]). We henceforth assume that all tropical curves are smooth, in other words we consider metric graphs (G, ℓ) having finitely many univalent points at infinity. In particular, trees are assumed to have all of their leaf vertices at infinity.

Let Γ be a (smooth) tropical curve with a chosen orientation. A *harmonic 1-form* is a global section of Ω^1_Γ. More explicitly, a harmonic 1-form $\omega = \sum_{e \in E(\Gamma)} a_e de$ is given by the choice of a coefficient $a_e \in \mathbb{Z}$ subject to the condition

$$\sum_{e \text{ entering } v} a_e - \sum_{e \text{ leaving } v} a_e = 0$$

at every vertex v of Γ.

A *harmonic function* on Γ is a section of Aff_Γ, i.e. a continuous function $f : U \to \mathbb{R}$ on an open subset $U \subset \Gamma$ that is linear with integer slope on every edge, and such that the sum of outgoing slopes of f at every vertex is zero. Recording the slopes of f, we obtain a harmonic 1-form on U. We note that harmonic functions are constant near infinite extremal vertices.

Let $\Gamma_1 = (G_1, \ell_1)$ and $\Gamma_2 = (G_2, \ell_2)$ be tropical curves, and let $f : G_1 \to G_2$ be a harmonic morphism of graphs with degree function d_f, where we recall that we do not allow graph morphisms to contract edges. If $d_f(e) = \frac{\ell_2(f(e))}{\ell_1(e)}$ for each edge $e \in E(G_1)$, then we define the associated *harmonic morphism of tropical curves* $f : \Gamma_1 \to \Gamma_2$, which is an affine linear map on each edge e of Γ_1 with slope or *dilation factor* equal to $d_f(e)$. It is elementary to verify that f is a morphism of rational polyhedral spaces. Conversely, a surjective morphism of rational polyhedral spaces $f : \Gamma_1 \to \Gamma_2$ induces a harmonic morphism of graphs (with respect to appropriately chosen models) if it does not contract any edges, and such a morphism has a well-defined global degree $\deg f$. Given a harmonic morphism $f : \Gamma_1 \to \Gamma_2$ and a point $x \in \Gamma_1$, we denote $d_f(x) = d_f(e)$ if x lies in the interior of the edge e, and $d_f(x) = d_f(v)$ if x corresponds to a vertex v (with respect to an appropriate model). We observe that, given a harmonic morphism of graphs $f : G_1 \to G_2$, a choice of edge lengths on G_2 uniquely determines edge lengths on G_1 in such a way that f induces a harmonic morphism of tropical curves.

In parallel to the graph case, we say that a harmonic morphism $f : \Gamma_1 \to \Gamma_2$ of tropical curves is *free* if $d_f(x) = 1$ for all $x \in \Gamma_1$ (equivalently, if it is a covering isometry), *dilated* if it is not free, and a *double cover* if it has global degree 2.

We note that an arbitrary harmonic morphism $f : (G_1, \ell_1) \to (G_2, \ell_2)$ of metric graphs can be augmented to a harmonic morphism of smooth tropical curves in the following way. For each univalent vertex $v_2 \in V(G_2)$, attach a compact infinite ray l to v_2, then for each $v_1 \in f^{-1}(v_2)$ attach $d_f(v_1)$ compact infinite rays to v_1 and map them with degree 1 to l.

Remark 3.1. A tropical curve Γ arising as the tropicalization of an algebraic curve naturally comes with a *vertex weight function*, recording the genera of the irreducible components of the special fiber. These vertex weights appear to play no role in the tropical n-gonal construction, and we do not consider them.

3.3. **Divisors on tropical curves.** Let Γ be a tropical curve. A *divisor* D on Γ is a finite formal $\mathbb{Z}$-linear combination of points on Γ, i.e. $D = \sum a_x \cdot x$ with $a_x = 0$ for almost all $x \in \Gamma$. Denote the group of divisors on Γ by $\mathrm{Div}(\Gamma)$. The *degree* of a divisor D is $\deg D = \sum a_x$. A divisor is called *effective* if $a_x \geqslant 0$ for all $x \in \Gamma$, in which case we write $D \geqslant 0$. Denote the set of effective divisors of degree n by $\mathrm{Div}^+_n(\Gamma)$.

A *rational function* $f : \Gamma \to \mathbb{R}$ is a piecewise linear function with integer slopes. A rational function induces a divisor as follows

$$\operatorname{div} f = \sum_{x \in \Gamma} \big(\text{sum of outgoing slopes at } x\big) \cdot x.$$

Any divisor of the form $\operatorname{div} f$ is called a *principal divisor*, and the subgroup of principal divisors is denoted by $\operatorname{Prin}(\Gamma) \subseteq \operatorname{Div}(\Gamma)$. Two divisors D_1 and D_2 are *linearly equivalent* if $D_1 - D_2$ is a principal divisor, in which case we write $D_1 \sim D_2$. The *Picard variety* of Γ is defined as

$$\operatorname{Pic}(\Gamma) = \operatorname{Div}(\Gamma)/\operatorname{Prin}(\Gamma) \qquad \text{and} \qquad \operatorname{Pic}_k(\Gamma) = \big\{[D] \in \operatorname{Pic}(\Gamma) \mid \deg D = k\big\}.$$

In degree 0, the *Picard group* $\operatorname{Pic}_0(\Gamma)$ is a group, while every $\operatorname{Pic}_k(\Gamma)$ is a torsor over $\operatorname{Pic}_0(\Gamma)$.

3.4. **The n-gonal and Recillas construction for tropical curves.** In this section, we extend the n-gonal and Recillas construction to tropical curves. We recall that our definition of gonality involves maps to trees instead of Baker–Norine rank.

Definition 3.2. An n-*gonal* tropical curve is a tropical curve Γ together with a harmonic map $f : \Gamma \to K$ of degree n to a metric tree K. For $n = 2, 3, 4$ we will also use the terms *hyperelliptic*, *trigonal*, and *tetragonal*, respectively. A tetragonal curve is called *generic* if for all $x \in K$ the fiber $f^{-1}(x)$ has dilation profile $(3, 1)$, $(2, 1, 1)$, or $(1, 1, 1, 1)$. A double cover $\widetilde{\Gamma} \to \Gamma \to K$ of a hyperellipic curve is called *generic* if K has no point which has two preimages in Γ, each of which has a unique preimage in $\widetilde{\Gamma}$.

A harmonic morphism of tropical curves is uniquely specified by giving a harmonic morphism of graphs together with an edge length function on the target. This observation allows us to directly lift the tropical n-gonal construction and the tropical Recillas construction from graphs to tropical curves:

Definition 3.3. Let $\widetilde{\Gamma} \to \Gamma \to K$ be a double cover of an n-gonal tropical curve. The n-*gonal construction* is the degree 2^n harmonic morphism of tropical curves $\widetilde{\Pi} \to K$ that arises by running Construction 2.3 on the underlying tower of graphs, and then endowing $\widetilde{\Pi}$ with the edge-length function induced by K. Similarly, the tropical Recillas associates to a generic tetragonal curve $\Pi \to K$ a tower $\widetilde{\Gamma} \to \Gamma \to K$ consisting of a free double cover of a trigonal curve.

All results from Section 2 carry over to the setting of tropical curves. In particular, the fiberwise description over a point $x \in K$ still holds, where x may now correspond to a vertex or an interior edge point. We may view an edge e of a graph model of K as a generic point for the points of that edge in K. In this sense our definition of the construction for generic fibers ensures that the fibers of n-gonal construction depend continuously on the metric realization. With this point of view, the root map at the level of graphs can now be understood as the continuous limit for $x \in K$ approaching a vertex.

Remark 3.4. The language of divisors allows us to define the n-gonal and Recillas constructions directly for tropical curves (bypassing the graph case) in exact parallel to the algebraic setting. Let $\pi : \widetilde{\Gamma} \to \Gamma$ be a free double cover of an n-gonal curve $f : \Gamma \to K$, and let $\widetilde{\Pi} \to K$ be the output of the n-gonal construction. The graph-theoretic fibers of $\widetilde{\Pi}$ are multisections of π, which we can reformulate in terms of divisors as

$$\widetilde{\Pi} = \Big\{x_1 + \cdots + x_n \in \operatorname{Div}_n^+(\widetilde{\Gamma}) \;\Big|\; \exists x \in K : \pi(x_1) + \cdots + \pi(x_n) = \sum_{y \in f^{-1}(x)} d_f(y) \cdot y\Big\}. \tag{9}$$

Even more is true: since K is a tree, the natural graph model for $\widetilde{\Gamma}$ is loop-free and hence [BU22] gives a polyhedral structure on $\operatorname{Div}_n^+(\widetilde{\Gamma})$. The graph structure of $\widetilde{\Pi}$ is precisely the restriction of this polyhedral structure. A similar description can be given for the tropical Recillas construction.

Let $k : \Pi \to K$ be a generic tetragonal tropical curve, and let $\widetilde{\Gamma} \to \Gamma \to K$ be the output of the tropical Recillas construction. Then on the level of graphs

$$\widetilde{\Gamma} = \left\{ x_1+x_2 \in \mathrm{Div}_2^+(\Pi) \;\middle|\; \exists x_3+x_4 \in \mathrm{Div}_2^+(\Pi) \text{ and } x \in K \text{ such that } x_1+x_2+x_3+x_4 = \sum_{y \in k^{-1}(x)} d_k(y) \cdot y \right\}, \tag{10}$$

and the involution whose quotient is the double cover $\widetilde{\Gamma} \to \Gamma$ is the one sending $x_1 + x_2$ to $x_3 + x_4$. However, it turns out that defining $\widetilde{\Pi}$ and $\widetilde{\Gamma}$ in this way does not naturally induce the correct edge lengths. For example, in the context of the trigonal construction, consider a $(2,3)$-tower $\widetilde{\Gamma} \to \Gamma \to K$ with an edge $e \in E(K)$ of Type C (in the language of Definition 2.11). Then both edges $\widetilde{e}^+$ and $\widetilde{e}^-$ in $\widetilde{\Gamma}$ above e are of the same length, say a. For the construction of Π there are two relevant cells in $\mathrm{Div}_3^+(\widetilde{\Gamma})$, namely

$$(\widetilde{e}^+ \times \widetilde{e}^+)/S_2 \times \widetilde{e}^- \qquad \text{and} \qquad (\widetilde{e}^- \times \widetilde{e}^- \times \widetilde{e}^-)/S_3.$$

The two edges of Π arising from $\widetilde{e}^+$ and $\widetilde{e}^-$ are given as the diagonals of these cells, both of which have lattice length a, in contrast to the correct edge lengths which would have been a and $3a$. This is one of the main reasons that we define the n-gonal and Recillas constructions for combinatorial graphs first, and then import the construction into the setting of tropical curves.

3.5. **Tropical cycles.** We will now recall the definition of the tropical cycle class groups $Z_k(X)$ associated to a rational polyhedral space X. We first recall the definition in affine space, following [GS19a].

Definition 3.5. Let $\mathbb{R}^n$ be endowed with the integral structure $\mathbb{Z}^n \subseteq \mathbb{R}^n$. A *tropical fan* k*-cycle* on $\mathbb{R}^n$ is a function $A : \mathbb{R}^n \to \mathbb{Z}$ satisfying the following properties:

(1) For all $x \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}_{>0}$ we have $A(\lambda x) = A(x)$.
(2) The *support* $|A| = \overline{\{x \in \mathbb{R}^n \mid A(x) \neq 0\}}$ is a purely k-dimensional rational polyhedral set.
(3) A is locally constant on $|A|^{\mathrm{reg}}$ and is equal to 0 on $|A| - |A|^{\mathrm{reg}}$.
(4) A satisfies the so-called *balancing condition*. This is a condition at every codimension 1 cell τ of the rational polyhedral structure of $|A|$ and requires the sum of the outwards facing lattice normal vectors of incident maximal cells, weighted by the values of A, to be contained in the tangent space of τ. The condition does not depend on the chosen fan structure of $|A|$ and we will only need to check balancing when $k = 1$, in which case the only codimension one cell is the origin $\tau = 0$. Let e range over the 1-dimensional cones in $|A|$ and for each e let $\eta_e \in e \cap \mathbb{Z}^n$ be an outwards facing primitive tangent vector of e. The balancing condition at 0 is
$$\sum_e A(\eta_e)\eta_e = 0. \tag{11}$$
We note that for a 1-cycle A (whose support $|A|$ is a graph), verifying balancing does not involve the values at the vertices (points of $|A| - |A|^{\mathrm{reg}}$), and we will often ignore condition (3) and allow A to have arbitrary vertex values.

The idea is now to define a tropical k-cycle on an arbitrary rational polyhedral space by requiring it to look locally like a tropical fan k-cycle.

Definition 3.6. Let X be a rational polyhedral space. A *local face structure* at a point $x \in X$ is a finite polyhedral complex Σ such that

(1) x is contained in the topological interior of $|\Sigma|$,
(2) there exists a chart $U \to V \subseteq \overline{\mathbb{R}}^n$ of X such that $|\Sigma| \subseteq U$,
(3) x is contained in every inclusion maximal cell of Σ.

Face structures are higher-dimensional analogues of graph models for tropical curves.

Definition 3.7. Let X be a rational polyhedral space. A *tropical* k*-cycle* is a function $A : X \to \mathbb{Z}$ such that the following properties hold.

(1) A is *locally constructible*, i.e. for every $x \in X$ there is a local face structure Σ at x such that the restriction to the relative interior $A|_{\mathrm{relint}(\sigma)}$ is locally constant for every $\sigma \in \Sigma$.
(2) For every $x \in X$ the germ of A at x (extended to be constant along all lines through the origin) defines a tropical fan k-cycle on the real tangent space $T_x^{\mathbb{Z}}(X) \otimes_{\mathbb{Z}} \mathbb{R}$ at x.

The set of tropical k-cycles on X is denoted $Z_k(X)$.

Example 3.8. Let Γ be a smooth tropical curve. A function $A : \Gamma \to \mathbb{Z}$ with value 0 on every vertex of valence > 2 is balanced if and only if its value on every edge is the same. If this value is 1 everywhere, then this defines the *fundamental cycle* $[\Gamma] \in Z_1(\Gamma)$ of Γ.

We emphasize that if Γ is not smooth then it does not admit a fundamental cycle. For example, the balancing condition (11) cannot be satisfied at a finite 1-valent vertex. At a 1-valent vertex at infinity, however, the balancing condition is trivially satisfied, since all affine functions are locally constant and hence the tangent space is 0. In particular, the primitive tangent vector of an infinite edge at the vertex at infinity is 0 and Equation (11) is satisfied.

Example 3.9. Let Γ be a smooth tropical curve and define $[\Delta_\Gamma] : \Gamma^2 \to \mathbb{Z}$ by $[\Delta_\Gamma](x, y) = 1$ if $x = y$ and x is not a vertex of valence > 2 and $[\Delta_\Gamma](x, y) = 0$ otherwise. We claim that this is a tropical 1-cycle on Γ^2, the *diagonal cycle*.

A choice of graph model on Γ provides a polyhedral complex structure on Γ^2. Subdividing cells of the form $e \times e$ for any edge e of Γ provides a face structure that shows that $[\Delta_\Gamma]$ is (locally) constructible. We check balancing. Locally at a point (x, x), where x not a vertex of Γ, the support of $[\Delta_\Gamma]$ looks like the diagonal in $\mathbb{R}^2$. This is clearly balanced because the sum of outwards facing primitive tangent vectors at the origin is 0. Now assume x is a vertex of Γ. Denote the primitive tangent vectors of the edges of Γ incident to x by $\eta_1, \ldots, \eta_n$. Then the primitive tangent vectors of $[\Delta_\Gamma]$ at (x, x) are

$$\begin{pmatrix} \eta_i \\ \eta_i \end{pmatrix} \in T^{\mathbb{Z}}_{(x,x)}(\Gamma^2) \cong T^{\mathbb{Z}}_x(\Gamma) \times T^{\mathbb{Z}}_x(\Gamma) \qquad \text{for } i = 1, \ldots, n,$$

and again we see that the sum is 0 because Γ was assumed smooth, i.e. $\sum_i \eta_i = 0$.

Let $A, B : X \to \mathbb{Z}$ be tropical k-cycles on X. The sum function $A + B : X \to \mathbb{Z}$ is not, in general, a tropical k-cycle. However, there exists a tropical k-cycle agreeing with the algebraic sum $A + B$ away from the non-regular locus $|A| \setminus |A|^{\mathrm{reg}} \cup |B| \setminus |B|^{\mathrm{reg}}$. Denoting this cycle $A + B$ by abuse of notation, we obtain a group structure on $Z_k(X)$.

Now let $f : X \to Y$ be a proper and surjective morphism of k-dimensional rational polyhedral spaces. There is a *pushforward* $f_* : Z_k(X) \to Z_k(Y)$ defined as follows. Let $A \in Z_k(X)$. At $y \in Y^{\mathrm{reg}} \setminus f(X \setminus X^{\mathrm{reg}})$ define

$$f_*A(y) = \sum_{x \in f^{-1}(y)} \left[T^{\mathbb{Z}}_y Y : d_x f(T^{\mathbb{Z}}_x X)\right] A(x), \tag{12}$$

where the lattice index is taken to be 0 if it is not finite. Extending this to all of Y with constant value 0 gives a well-defined tropical k-cycle on Y. Similar to the case of addition of tropical cycles, this push-forward is in general not a tropical cycle, but there is a tropical cycle (also denoted f_*A) that coincides with f_*A away from a locus of dimension at most $k-1$.

Example 3.10. Let Γ and Π be smooth tropical curves and let $\pi : \Gamma \to \Pi$ be a harmonic map of degree d. Then $\pi_*[\Gamma] = d[\Pi]$. To see this, it suffices to check that $\pi_*[\Gamma](y) = d$ for every y in the interior of an edge of Π. By definition and (2) we have

$$\pi_*[\Gamma](y) = \sum_{x \in \pi^{-1}(y)} \left[T_y^{\mathbb{Z}}\Pi : d_x\pi(T_x^{\mathbb{Z}}\Gamma)\right] = \sum_{x \in \pi^{-1}(y)} d_\pi(x) = n,$$

where $d_\pi(x)$ is the dilation factor on the edge to which x belongs and $d_x\pi$ is the differential of π at x.

3.6. **Tropical homology.** Finally, we give a brief introduction to tropical homology groups (which were introduced in [IKMZ19]), following the paper [GS19a]. Let X be a rational polyhedral space, possibly with boundary.

Definition 3.11. Let $p \geqslant 1$. Define the sheaf Ω_X^p of *tropical* p*-forms* to be the image of

$$\bigwedge^p \Omega_X^1 \longrightarrow i_*\Big(\bigwedge^p \Omega_X^1|_{X^{\mathrm{reg}}}\Big),$$

where $i : X^{\mathrm{reg}} \to X$ is the inclusion.

There is a maximal stratification of X such that Ω_X^1 is locally constant on every stratum. A singular q-simplex, i.e. a continuous map $\sigma : \Delta^q \to X$, is *allowable* if every open face of Δ^q maps into a single stratum of X. Denote by $\mathbb{Z}_{\sigma(\Delta^q)}$ the constant sheaf on $\sigma(\Delta^q)$ with values in $\mathbb{Z}$. Then the (p,q)-th chain group is defined as

$$C_{p,q}(X) = \bigoplus_{\sigma:\Delta^q \to X \text{ allowable}} \mathrm{Hom}\left(\Omega_X^p, \mathbb{Z}_{\sigma(\Delta^q)}\right).$$

Elements in $C_{p,q}(X)$ are denoted as $\sum_\sigma \sigma \otimes \eta_\sigma$, where $\eta_\sigma \in \mathrm{Hom}\left(\Omega_X^p, \mathbb{Z}_{\sigma(\Delta^q)}\right)$. The boundary map of $C_{p,\bullet}(X)$ is given as

$$\begin{aligned} C_{p,q+1}(X) &\longrightarrow C_{p,q}(X) \\ \sigma \otimes \eta &\longmapsto \sum_{\tau \in \partial\sigma} \tau \otimes (r_\tau \circ \eta), \end{aligned}$$

where ∂ is the boundary map from singular homology and $r_\tau : \mathbb{Z}_{\sigma(\Delta^{q+1})} \to \mathbb{Z}_{\tau(\Delta^q)}$ is the restriction map. By abuse of notation we denote the boundary maps of $C_{p,\bullet}(X)$ again by ∂ and we define the (p,q)-th *tropical homology group* $H_{p,q}(X) = H_q(C_{p,\bullet}(X))$. The cochain complexes are $C^{p,\bullet} = \mathrm{Hom}(C_{p,\bullet}, \mathbb{Z})$ and the *tropical cohomology groups* are $H^{p,q}(X) = H^q(C^{p,\bullet}(X))$.

Recall that there is a *tropical cycle class map*

$$\mathrm{cyc} : Z_k(X) \longrightarrow H_{k,k}(X)$$

which assigns to any tropical k-cycle a class in tropical homology. We usually suppress cyc from the notation and identify a tropical cycle in $Z_k(X)$ with its image in $H_{k,k}(X)$. This map is defined for rational polyhedral spaces with boundary in [GS19a, Section 5], and is given a convenient description in the $k = 1$ case for boundaryless polyhedral spaces in [GS19b]. We recall the latter

formula for $A \in Z_1(X)$, generalized to the case when the support of $|A|$, which is a graph, is allowed to have boundary vertices, but no boundary edges.

For every edge e in $|A|$ choose a generator for $T_x^{\mathbb{Z}}|A|$ for some $x \in e$. By parallel transport this gives rise to a generator for any $T_y^{\mathbb{Z}}|A|$ with $y \in e$ and hence a morphism $\Omega^1_{|A|} \to \mathbb{Z}_e$. Precomposing with $\Omega^1_X \to \Omega^1_{|A|}$ induced by the inclusion $|A| \hookrightarrow X$ we obtain $\eta_e \in \mathrm{Hom}(\Omega^1_X, \mathbb{Z}_e)$. Let $\gamma_e : \Delta^1 \to X$ be a parametrization of e in the direction given by η_e. Then

$$\mathrm{cyc}(A) = \sum_e A(e)\gamma_e \otimes \eta_e \in C_{1,1}(X).$$

Let us check that $\partial\,\mathrm{cyc}(A) = 0$. To do so, let v be a finite vertex of $|A|$ and assume that all edges e incident to v have been oriented away from v. Then each e contributes $-A(e)\eta_e|_v$ to $\partial\,\mathrm{cyc}(A)$. But now the sum over these contributions is 0 because A was assumed to be a tropical cycle and in particular balanced. On the other hand, if v is an infinite vertex, then the stalk of $\Omega^1_{|A|}$ at v is trivial, hence the element η_e (corresponding to any incident edge e) is equal to 0.

There are natural pushforward maps of tropical homology classes and pullback maps of tropical cohomology classes along morphisms of rational polyhedral spaces. These maps are compatible with the cycle class map in the sense that proper pushforward of tropical cycles and pushforward of tropical homology classes commute with the cycle class map [GS19a, Proposition 5.6]. Finally, there is a cap product which, if X admits a fundamental class, gives rise to Poincaré duality

$$H^{p,q}(X) \simeq H_{n-p,n-q}(X), \quad c \longmapsto c \frown \mathrm{cyc}[X].$$

3.7. **Tropical Cartier divisors.** Let X be a rational polyhedral space. A *rational function* on X is a continuous function $f : X \to \mathbb{R}$ that is piecewise affine on every chart. Denote the sheaf of rational functions by $\mathcal{M}_X$. Clearly, every affine function is rational, so there is an inclusion $\mathrm{Aff}_X \to \mathcal{M}_X$. Denote the quotient $\mathcal{M}_X/\mathrm{Aff}_X$ by $\mathcal{D}\mathrm{iv}(X)$, so that there is a short exact sequence of sheaves

$$0 \longrightarrow \mathrm{Aff}_X \longrightarrow \mathcal{M}_X \longrightarrow \mathcal{D}\mathrm{iv}(X) \longrightarrow 0. \tag{13}$$

The group of global sections $\mathrm{Div}(X) = \Gamma\big(X, \mathcal{D}\mathrm{iv}(X)\big)$ is the group of *Cartier divisors* on X. If $f : X \to Y$ is a morphism of rational polyhedral spaces, then there is an induced pullback map on Cartier divisors $f^* : \mathrm{Div}(Y) \to \mathrm{Div}(X)$.

The group $H^1(X, \mathrm{Aff}_X)$ classifies *tropical line bundles* on X, and the short exact sequence (13) gives rise to a boundary homomorphism $\mathrm{Div}(X) = H^0\big(X, \mathcal{D}\mathrm{iv}(X)\big) \to H^1(X, \mathrm{Aff}_X)$ that associates to a Cartier divisor D a tropical line bundle $\mathcal{L}(D)$. Pullback of Cartier divisors commutes with this association [GS19a, Proposition 3.15]. Furthermore, the short exact sequence defining Ω^1_X

$$0 \longrightarrow \mathbb{R}_X \longrightarrow \mathrm{Aff}_X \longrightarrow \Omega^1_X \longrightarrow 0$$

gives rise to the *first Chern class map* $c_1 : H^1(X, \mathrm{Aff}_X) \to H^1(X, \Omega^1_X) = H^{1,1}(X)$.

Finally, there is a natural intersection pairing

$$\mathrm{Div}(X) \times Z_k(X) \longmapsto Z_{k-1}(X), \qquad (D, A) \longmapsto D \cdot A.$$

If X has a fundamental cycle, then for $k = \dim X$ this gives an isomorphism $\mathrm{Div}(X) \cong Z_{\dim X - 1}(X)$. We recall from [GS19a, Proposition 5.12] that $\mathrm{cyc}(D \cdot A) = c_1(\mathcal{L}(D)) \cap \mathrm{cyc}(A)$ for every Cartier divisor D and tropical cycle A.

4. Tropical abelian varieties

In this section, we recall the theory of tropical abelian varieties. The definitions that we use were introduced in [LZ22] and differ slighly from the standard definitions (see e.g. [FRSS18]), but are equivalent to them. We prove a number of elementary results concerning morphisms of tropical abelian varieties. We then recall the Jacobian variety of a tropical curve (already introduced in [MZ08]) and the Prym variety of a double cover (introduced in [JL18]), and show that they satisfy natural universal properties. Finally, we prove Theorem 4.27, which is a tropical version of the homological formula for the image of the Abel–Prym (which is classically a part of Welters' criterion characterizing Prym varieties).

4.1. **Integral tori.** Let Λ and Λ' be finitely generated free abelian groups of the same rank and let $[\cdot,\cdot] : \Lambda \times \Lambda' \to \mathbb{R}$ be a non-degenerate pairing. The triple $(\Lambda, \Lambda', |\cdot,\cdot])$ defines a *real torus with integral structure* $\Sigma = \mathrm{Hom}(\Lambda, \mathbb{R})/\Lambda'$, or simply an *integral torus*, where the inclusion $\Lambda' \subseteq \mathrm{Hom}(\Lambda, \mathbb{R})$ is given by $\lambda' \mapsto [\cdot, \lambda']$. The *dual torus* $\Sigma^\vee = \mathrm{Hom}(\Lambda', \mathbb{R})/\Lambda$ is defined by the transposed triple $(\Lambda', \Lambda, [\cdot,\cdot]^t)$. The *dimension* of an integral torus is $\dim_{\mathbb{R}} \Sigma = \mathrm{rk}\,\Lambda = \mathrm{rk}\,\Lambda'$. From now on we abuse notation and refer to the triples as integral tori as well. In particular, there is a group structure on integral tori.

Remark 4.1. Integral tori are rational polyhedral spaces as follows. Identifying $\mathrm{Hom}(\Lambda, \mathbb{Z})$ with $\mathbb{Z}^g$ endows the universal cover $\mathrm{Hom}(\Lambda, \mathbb{R})$ with the structure of a rational polyhedral space. The integral affine linear functions on $\mathrm{Hom}(\Lambda, \mathbb{R})$ are those affine linear functions that take integer values on the lattice $\mathrm{Hom}(\Lambda, \mathbb{Z})$. In other words, affine functions are precisely elements of $\mathrm{Hom}(\mathrm{Hom}(\Lambda, \mathbb{Z}), \mathbb{Z}) \oplus \mathbb{R} \cong \Lambda \oplus \mathbb{R}$. The torus inherits the rational polyhedral structure from the universal covering via the quotient map. Note that $\Omega^1_\Sigma = \Lambda$ and $H_1(\Sigma, \mathbb{Z}) = \Lambda'$ for any integral torus $\Sigma = (\Lambda, \Lambda', [\cdot,\cdot])$.

We first classify morphisms of integral tori (as rational polyhedral spaces). Recall that a holomorphic map of complex tori factors as a group homomorphism followed by a translation. An identical classification holds for integral tori. We first define the two types of maps.

Definition 4.2. Let Σ be an integral torus. For every $y \in \Sigma$ the *translation* $t_y : \Sigma \to \Sigma$ is given by $t_y(x) = x + y$.

It is clear that translations are morphisms of rational polyhedral spaces, inducing identity maps on $\Omega^1(\Sigma)$ and $H^{p,q}(\Sigma)$.

Definition 4.3. A *homomorphism of integral tori* $f = (f^\#, f_\#) : (\Lambda_1, \Lambda_1', [\cdot,\cdot]_1) \to (\Lambda_2, \Lambda_2', [\cdot,\cdot]_2)$ consists of a pair of homomorphisms $f^\# : \Lambda_2 \to \Lambda_1$ and $f_\# : \Lambda_1' \to \Lambda_2'$ satisfying the relation

$$\left[f^\#(\lambda_2), \lambda_1'\right]_1 = \left[\lambda_2, f_\#(\lambda_1')\right]_2 \tag{14}$$

for all $\lambda_1' \in \Lambda_1'$ and $\lambda_2 \in \Lambda_2$. The maps $f^\#$ and $f_\#$ necessarily have the same rank, denoted $\mathrm{rk}\, f$.

Note that for a homomorphism $f = (f^\#, f_\#)$, the Hom-dual $\mathrm{Hom}(\Lambda_1, \mathbb{R}) \to \mathrm{Hom}(\Lambda_2, \mathbb{R})$ of $f^\#$ restricts to $f_\#$ on Λ_1' and hence descends to a group homomorphism on the underlying tori

$$f : \Sigma_1 = \mathrm{Hom}(\Lambda_1, \mathbb{R})/\Lambda_1' \longrightarrow \Sigma_2 = \mathrm{Hom}(\Lambda_2, \mathbb{R})/\Lambda_2',$$

which we also denote f again by abuse of notation. The *dual homomorphism* of f is given by the transposed pair $f^\vee = (f_\#, f^\#) : \Sigma_2^\vee \to \Sigma_1^\vee$.

Lemma 4.4. *Let $\Sigma_i = (\Lambda_i, \Lambda_i', [\cdot,\cdot]_i)$ for $i = 1, 2$ be integral tori and let $f : \Sigma_1 \to \Sigma_2$ be a map of rational polyhedral spaces. Then f factors uniquely as a homomorphism $g : \Sigma_1 \to \Sigma_2$ followed by a translation $t : \Sigma_2 \to \Sigma_2$.*

Proof. Define $g = t_{-f(0)} \circ f$, then clearly g is a map of rational polyhedral spaces with $g(0) = 0$. In particular, g pulls back affine linear functions defined in a neighborhood of $0 \in \Sigma_2$ to affine linear functions on a neighborhood of $0 \in \Sigma_1$. Since $g(0) = 0$, the pullback of a linear function is in fact linear. But for any integral torus $\Sigma = (\Lambda, \Lambda', [\cdot,\cdot])$, linear functions in a neighborhood of 0 are simply given by lattice points in Λ. Hence pullback defines a group homomorphism $g^\# : \Lambda_2 \to \Lambda_1$ whose Hom-dual induces the map g on the tori, which is therefore a homomorphism of integral tori. □

Lemma 4.5. *Let $\Sigma_i = (\Lambda_i, \Lambda_i', [\cdot,\cdot]_i)$ for $i = 1, 2$ be integral tori and let $f = (f^\#, f_\#)$ be a homomorphism $\Sigma_1 \to \Sigma_2$. Then for any $y \in \Sigma_1$ the diagram*

$$\begin{tikzcd} \Sigma_1 \arrow[r, "f"] \arrow[d, "t_y"] & \Sigma_2 \arrow[d, "t_{f(y)}"] \\ \Sigma_1 \arrow[r, "f"] & \Sigma_2 \end{tikzcd}$$

commutes.

Proof. This is clear because f is a homomorphism with respect to the group structures of the integral tori Σ_1 and Σ_2, in other words $f(x + y) = f(x) + f(y)$. □

We may classify homomorphisms of integral tori according to two properties: the structure of the induced map on the underlying groups, and the dilation properties of the map of rational polyhedral spaces.

Definition 4.6. Let $\Sigma_i = (\Lambda_i, \Lambda_i', [\cdot,\cdot]_i)$ for $i = 1, 2$ be integral tori of dimensions g_i. A homomorphism $f = (f^\#, f_\#) : \Sigma_1 \to \Sigma_2$ of integral tori is said to be

(1) *surjective* if $\operatorname{rk} f = g_2$ (equivalently, if $f^\#$ is injective),
(2) *finite* if $\operatorname{rk} f = g_1$ (equivalently, if $f^\#(\Lambda_2)$ has finite index in Λ_1),
(3) *injective* if it is finite and $f_\#(\Lambda_1')$ is saturated in Λ_2',
(4) an *isogeny* if it is surjective and finite (equivalently, if $\operatorname{rk} f = g_1 = g_2$),
(5) a *free isogeny* if it is an isogeny and $f^\#(\Lambda_2) = \Lambda_1$ (equivalently, if $f^\#$ is an isomorphism),
(6) a *dilation* if is an isogeny and injective, (equivalently, if $f_\#$ is an isomorphism), and
(7) an *isomorphism* if $f_\#$ and $f^\#$ are isomorphisms.

We now unwind the definitions. First, we see that f is surjective, finite, injective, or a dilation if and only if the induced map $f : \Sigma_1 \to \Sigma_2$ of real tori, viewed as a group homomorphism, is respectively surjective, has finite kernel (which is then identified with the quotient of the saturation of $f_\#(\Lambda_1')$ in Λ_2' by $f_\#(\Lambda_1')$), injective, or is an isomorphism. If f is an isogeny, then $f_\#$ controls the kernel, while $f^\#$ determines the dilation. The dual of a surjective homomorphism is finite (and vice versa), the dual of an isogeny is an isogeny, and the dual of a free isogeny is a dilation (and vice versa). Finally, we may canonically factor any isogeny as follows:

Lemma 4.7. *Let $f = (f^\#, f_\#) : (\Lambda_1, \Lambda_1', [\cdot,\cdot]_1) \to (\Lambda_2, \Lambda_2', [\cdot,\cdot]_2)$ be an isogeny of integral tori. Then there exists an integral torus Σ_3 together with a free isogeny $h : \Sigma_1 \to \Sigma_3$ and a dilation $g : \Sigma_3 \to \Sigma_2$ such that f factors as $f = g \circ h$. Moreover, this factorization is unique in the sense that for any other factorization*

$\Sigma_1 \to \Pi \to \Sigma_2$ *into a free isogeny followed by a dilation, there is an isomorphism* ϕ *of integral tori such that*

$$\begin{tikzcd} & \Sigma_3 \arrow[dr, "g"] & \\ \Sigma_1 \arrow[ur, "h"] \arrow[dr] & & \Sigma_3 \\ & \Pi \arrow[uu, "\phi"] \arrow[ur] & \end{tikzcd}$$

commutes.

We note that we can also uniquely factor an isogeny as a dilation followed by a free isogeny.

Proof. In order to factor f, we define a third real torus with integral structure $\Sigma_3 = (\Lambda_1, \Lambda_2', [\cdot,\cdot]_3)$. To define the pairing $[\cdot,\cdot]_3$ note the following. By the existence of a Smith normal form for $f_\#$, we may choose $\mathbb{Z}$-bases $e_1, \dots, e_g$ of Λ_1' and $f_1, \dots, f_g$ of Λ_2' such that $f_\#(e_i) = a_i f_i$ for nonzero integers $a_i \in \mathbb{Z}$. We now define

$$[\lambda, f_i]_3 = \frac{1}{a_i}[\lambda, e_i]_1$$

for any $\lambda \in \Lambda_1$. We now factor f as

$$\begin{tikzcd} \Sigma_1 \arrow[r, "h"] & \Sigma_3 \arrow[r, "g"] & \Sigma_2 \\ \Lambda_1 & \Lambda_1 \arrow[l, "\mathrm{Id}"'] & \Lambda_2 \arrow[l, "f^\#"'] \\ \Lambda_1' \arrow[r, "f_\#"] & \Lambda_2' \arrow[r, "\mathrm{Id}"] & \Lambda_2'. \end{tikzcd}$$

It is easy to check Condition (14) for h and g, hence they are homomorphisms. Moreover, h is a free isogeny because $h^\# = \mathrm{Id}$ is an isomorphism, and g is a dilation because $g_\# = \mathrm{Id}$ is surjective.

In order to show uniqueness, let $\Pi = (\Delta, \Delta', [\cdot,\cdot]_\Pi)$ be another integral torus and let $\Sigma_1 \xrightarrow{\tilde{h}} \Pi \xrightarrow{\tilde{g}} \Sigma_2$ be another factorization of f as a free isogeny $\tilde{h}$ followed by a dilation $\tilde{g}$. Since $\tilde{h}^\#$ and $\tilde{g}_\#$ are isomorphisms, it suffices to show that $\phi = ((\tilde{h}^\#)^{-1}, \tilde{g}_\#)$ is a homomorphism, i.e. we need to check Condition (14). Let $\delta_1, \dots, \delta_g$ be the basis of Δ' such that $\tilde{g}_\#(\delta_i) = f_i$. Then $\tilde{h}_\#(e_i) = a_i\delta_i$ because $\mathrm{Id} \circ h_\# = \tilde{g}_\# \circ \tilde{h}_\#$ and it becomes clear that

$$\left[(\tilde{h}^\#)^{-1}(\lambda), \delta_i\right]_\Pi = \frac{1}{a_i}[\lambda, e_i]_1 = [\lambda, f_i]_3 = \left[\lambda, \tilde{g}_\#(\delta_i)\right]_3$$

for all $\lambda \in \Lambda_1$ and $i = 1, \dots, g$. □

In some sense the key observation to prove Lemma 4.7 was simply that an isogeny f is free if and only if $f^\#$ is an isomorphism and it is a dilation if and only if $f_\#$ is an isomorphism.

4.2. **Kernels and cokernels.** Let $f : \Sigma_1 \to \Sigma_2$ be a homomorphism of integral tori. We define the integral torus

$$(\operatorname{Ker} f)_0 = \left((\operatorname{Coker} f^\#)^{\mathrm{tf}}, \operatorname{Ker} f_\#, [\cdot,\cdot]_K\right),$$

where $(\cdot)^{\mathrm{tf}}$ denotes the quotient of an abelian group by its torsion subgroup, and $[\cdot,\cdot]_K$ is the pairing induced by $[\cdot,\cdot]_1$. On the level of groups, $(\operatorname{Ker} f)_0$ is the connected component of the identity of $f^{-1}(0)$. There is a natural injective homomorphism $i : (\operatorname{Ker} f)_0 \to \Sigma_1$ given by the inclusion $i_\# : \operatorname{Ker} f_\# \hookrightarrow \Lambda_1'$ and the quotient map $i^\# : \Lambda_1 \twoheadrightarrow (\operatorname{Coker} f^\#)^{\mathrm{tf}}$.

Dually, we define the cokernel of f as the integral torus

$$\operatorname{Coker} f = \left(\operatorname{Ker} f^\#, (\operatorname{Coker} f_\#)^{\mathrm{tf}}, [\cdot,\cdot]_C\right),$$

where $[\cdot,\cdot]_C$ is the pairing induced by $[\cdot,\cdot]_2$. Similarly to the kernel, there is a natural surjective map $q : \Sigma_2 \to \operatorname{Coker} f$ given by the inclusion $q^\# : \operatorname{Ker} f^\# \hookrightarrow \Lambda_2$ and the quotient map $q_\# : \Lambda_2' \twoheadrightarrow (\operatorname{Coker} f_\#)^{\mathrm{tf}}$.

It is clear that $(\operatorname{Ker} f)_0$ is trivial if and only if f is finite, and that $\operatorname{Coker} f$ is trivial if and only if f is surjective. Moreover, unwinding the definitions it is easy to see that the following lemma holds.

Lemma 4.8. *Let $f : \Sigma_1 \to \Sigma_2$ be a homomorphism of integral tori. Then $(\operatorname{Ker} f)_0^\vee \cong \operatorname{Coker}(f^\vee)$.*

There is a universal property for the kernel and cokernel of a homomorphism of integral tori.

Proposition 4.9. *Let $f : \Sigma_1 \to \Sigma_2$ be a homomorphism of integral tori. Given an integral torus $\Pi = (\Delta, \Delta', [\cdot,\cdot]_\Delta)$ and a homomorphism $g : \Pi \to \Sigma_1$ such that $f \circ g = 0$, there exists a unique homomorphism $u : \Pi \to (\operatorname{Ker} f)_0$ such that*

$$\begin{tikzcd} (\operatorname{Ker} f)_0 \arrow[r, "i"] & \Sigma_1 \\ \Pi \arrow[u, "u"] \arrow[ur, "g"'] & \end{tikzcd}$$

commutes.

Proof. The assumption $f \circ g = 0$ means that $g^\# \circ f^\# = 0$ and $f_\# \circ g_\# = 0$. By the universal properties of kernels and cokernels of abelian groups, and using the fact that Δ is torsion free, we obtain unique morphisms $u^\#$ and $u_\#$ such that the diagrams

$$\begin{tikzcd} (\operatorname{Coker} f^\#)^{\mathrm{tf}} \arrow[d, "u^\#"'] & \Lambda_1 \arrow[l, "i^\#"'] \arrow[dl, "g^\#"] \\ \Delta & \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} \operatorname{Ker} f_\# \arrow[r, "i_\#"] & \Lambda_1' \\ \Delta' \arrow[u, "u_\#"] \arrow[ur, "g_\#"'] & \end{tikzcd}$$

commute. We need to verify that $u = (u^\#, u_\#)$ is a homomorphism of integral tori. Let $\lambda \in \Lambda_1$ and $\delta' \in \Delta'$. Denote the class of λ in $(\operatorname{Coker} f^\#)^{\mathrm{tf}}$ by $\overline{\lambda}$. Then

$$\left[\overline{\lambda}, u_\#(\delta')\right]_K = \left[\lambda, \underbrace{u_\#(\delta')}_{=g_\#(\delta')}\right]_1 = \left[\underbrace{g^\#(\lambda)}_{=u^\#(\overline{\lambda})}, \delta'\right]_\Delta,$$

as required. □

The universal property for $\operatorname{Coker} f$ is stated and proved in complete analogy.

4.3. **Polarizations and tropical homology.** Let $\Sigma = (\Lambda, \Lambda', [\cdot,\cdot])$ be an integral torus. A *polarization* on Σ is a group homomorphism $\xi : \Lambda' \to \Lambda$ such that $(\cdot,\cdot) = [\xi(\cdot),\cdot] : \Lambda_\mathbb{R} \times \Lambda_\mathbb{R} \to \mathbb{R}$ is a symmetric and positive definite bilinear form. A polarization is necessarily injective, and is called *principal* if it is bijective. The invariant factors $(a_1, \ldots, a_g)$ of the Smith normal form (where $a_i \geqslant 1$ and $a_i | a_{i+1}$ for $i = 1, \ldots, g-1$) define the *type* of a polarization ξ, and a polarization is principal if and only if all $a_i = 1$. A polarization defines a homomorphism $(\xi, \xi) : \Sigma \to \Sigma^\vee$ to the dual, which is always an isogeny, and is an isomorphism if and only if the polarization is principal.

Let $f : \Sigma_1 \to \Sigma_2$ be a finite homomorphism of integral tori. Given a polarization ξ_2 on Σ_2, we define a polarization $f^*(\xi_2) = f^\# \circ \xi_2 \circ f_\#$ on Σ_1, called the *induced polarization*. A polarization induced from a principal polarization need not itself be principal. However, a non-principal

polarization can be rescaled into a principal polarization by means of a dilation, a result that has no analogue in the algebraic setting.

Lemma 4.10. *Let $\Sigma = (\Lambda, \Lambda', [\cdot,\cdot])$ be an integral torus with polarization ξ of type $(a_1, \dots, a_g)$. Let $A = a_g$ be the least common multiple of the a_i. Then there exists an integral torus $\Sigma^{pp} = (\Lambda, \Lambda', [\cdot,\cdot]^{pp})$ with a principal polarization ζ together with a dilation $f : \Sigma^{pp} \to \Sigma$ such that $f^*\xi = A\zeta$.*

Proof. Consider $\mathbb{Z}$-bases $e_1, \dots, e_g$ of Λ and $e'_1, \dots, e'_g$ of Λ' such that $\xi(e'_i) = a_i e_i$ for all $1 \leqslant i \leqslant g$. Define

$$[e_i, \lambda']^{pp} = \frac{a_i}{A}[e_i, \lambda'] \qquad \text{for all } \lambda' \in \Lambda'$$

and define ζ by $e'_i \mapsto e_i$. Then ζ is clearly a principal polarization on $\Sigma^{pp} = (\Lambda, \Lambda', [\cdot,\cdot]^{pp})$. Define a homomorphism $f = (f^\#, f_\#)$ by

$$f^\# : \Lambda \longrightarrow \Lambda, \quad e_i \longmapsto \frac{A}{a_i} e_i$$

and $f_\# = \mathrm{Id}$. We have that

$$\left[f^\#(e_i), \lambda'\right]^{pp} = \frac{A}{a_i}[e_i, \lambda']^{pp} = [e_i, \lambda'] = \left[e_i, f_\#(\lambda')\right]$$

for $i = 1, \dots, g$ and $\lambda' \in \Lambda'$, hence f is a homomorphism of integral tori, and indeed a dilation because $f_\#$ is surjective and $f^\#$ is injective and has full rank. Finally, one easily checks that $f^*\xi = A\zeta$. □

We note that the map $f : \Sigma^{pp} \to \Sigma$ is a bijection on the underlying groups, only the integral structure changes.

Definition 4.11. An integral torus together with a principal polarization is called *principally polarized tropical abelian variety* or *pptav* for short. A morphism $f : \Sigma_1 \to \Sigma_2$ of pptavs is an *isomorphism of pptavs* if is is an isomorphism of integral tori and if the principal polarization on Σ_1 is the polarization induced from Σ_2.

We define the dual of a (not necessarily principal) polarization on an integral torus.

Definition 4.12. Let $\Sigma = (\Lambda, \Lambda', [\cdot,\cdot])$ be a tropical abelian variety with polarization $\xi : \Lambda' \to \Lambda$ of type $(a_1, \dots, a_g)$. The *dual polarization* $\xi^\vee$ on the dual torus $\Sigma^\vee = (\Lambda', \Lambda, [\cdot,\cdot]^t)$ is defined in the following way. Let $e_1, \dots, e_g$ and $e'_1, \dots, e'_g$ be bases of Λ and Λ', respectively, such that $\xi(e'_i) = a_i e_i$. We define $\xi^\vee : \Lambda \to \Lambda'$ by

$$\xi^\vee(e_i) = \frac{a_1 a_g}{a_i} e'_i.$$

The dual polarization has type $(a_1, a_1 a_g/a_{g-1}, \dots, a_1 a_g/a_2, a_g)$ and is principal if and only if ξ is principal. Furthermore, the compositions $\xi \circ \xi^\vee$ and $\xi^\vee \circ \xi$ are multiplication by $a_1 a_g$ maps on $\Sigma^\vee$ and Σ, respectively.

Finally, we recall the tropical homology and cohomology groups of an integral torus $\Sigma = (\Lambda, \Lambda', [\cdot,\cdot])$, and the relationship with polarizations. The groups can be computed explicitly (see [GS19b, Section 6]):

$$H_{p,p}(\Sigma) \cong \bigwedge^p \Lambda' \otimes_{\mathbb{Z}} \bigwedge^p \Lambda^*,$$
$$H^{p,p}(\Sigma) \cong \bigwedge^p (\Lambda')^* \otimes_{\mathbb{Z}} \bigwedge^p \Lambda.$$

Note that in particular $H^{1,1}(\Sigma) = (\Lambda')^* \otimes_{\mathbb{Z}} \Lambda = \mathrm{Hom}_{\mathbb{Z}}(\Lambda', \Lambda)$. Via this identification, we may view a polarization ξ as an element in $H^{1,1}(\Sigma)$. This element can also be identified with the tropical first Chern class $[\Theta]$ of a tropical line bundle on Σ corresponding to the *theta divisor* (see Section 3.7, and see [MZ08] for the definition of the theta divisor in terms of tropical theta functions). The following proposition then allows us to check homologically whether two principally polarized tropical abelian varieties are isomorphic.

Proposition 4.13. *Let $\Sigma_i = (\Lambda_i, \Lambda_i', [\cdot,\cdot]_i)$ for $i = 1, 2$ be tropical abelian varieties with principal polarizations ξ_i repectively. Let $f : \Sigma_1 \to \Sigma_2$ be an isomorphism of integral tori. Then the following are equivalent.*

(1) f is an isomorphism of pptavs.
(2) The following diagram commutes

$$\begin{tikzcd} \Lambda_1 & \Lambda_2 \arrow[l, "f^{\#}"'] \\ \Lambda_1' \arrow[u, "\xi_1"] \arrow[r, "f_{\#}"] & \Lambda_2' \arrow[u, "\xi_2"'] \end{tikzcd}$$

(3) The bilinear forms $[\xi_i(\cdot), \cdot]_i$ commute with the isomorphism $f_{\#} : \Lambda_1' \to \Lambda_2'$.
(4) ξ_1 and ξ_2 are identified via the induced isomorphism $f^* : H^{1,1}(\Sigma_2) \to H^{1,1}(\Sigma_1)$.

4.4. **Tropical Jacobians.** Let Γ be a tropical curve. Given an oriented model G of Γ, the simplicial chain group $C_1(G, \mathbb{Z})$ is the free abelian group on the edges of G, containing the simplicial homology group $H_1(G, \mathbb{Z})$. These groups fit into a directed system with respect to refinements of models (see [BF11] for details), and we denote the direct limit by $C_1(\Gamma, \mathbb{Z})$. The images of the $H_1(G, \mathbb{Z})$ are all equal and are denoted $H_1(\Gamma, \mathbb{Z})$.

There is a natural isomorphism $H_1(\Gamma, \mathbb{Z}) \to \Omega^1_\Gamma$ sending a cycle $\sum a_e\, e$ to the 1-form $\sum a_e\, de$. In addition, the *integration pairing*

$$\begin{aligned} [\cdot,\cdot] : \Omega^1_\Gamma \times C_1(\Gamma, \mathbb{Z}) &\longrightarrow \mathbb{R} \\ (\omega, \gamma) = \Big(\sum a_e de, \sum b_e e\Big) &\longmapsto \int_\gamma \omega = \sum a_e b_e \ell(e) \end{aligned}$$

restricts to a perfect pairing $\Omega^1_\Gamma \times H_1(\Gamma, \mathbb{Z}) \to \mathbb{R}$. Hence we have a pptav

$$\mathrm{Jac}(\Gamma) = \big(\Omega^1_\Gamma, H_1(\Gamma, \mathbb{Z}), [\cdot,\cdot]\big) = \mathrm{Hom}(\Omega^1_\Gamma, \mathbb{R})/H_1(\Gamma, \mathbb{Z})$$

of dimension equal to the genus $g(\Gamma)$, the *tropical Jacobian variety* of Γ.

Definition 4.14. Fix a base point $q \in \Gamma$. The *Abel–Jacobi map* relative to q is given by

$$\begin{aligned} \phi_q : \Gamma &\longrightarrow \mathrm{Jac}(\Gamma) \\ p &\longmapsto \Big(\omega \mapsto \int_{\gamma_p} \omega\Big), \end{aligned}$$

where γ_p denotes any path from q to p.

The Abel–Jacobi map naturally extends to symmetric powers of Γ and hence to divisors. This map respects linear equivalence, and the tropical Abel–Jacobi theorem (see Theorem 6.3 in [MZ08]) states that the induced map $\mathrm{Pic}_0(\Gamma) \to \mathrm{Jac}(\Gamma)$ (which does not depend on the choice

of a base point) is an isomorphism. Under this identification, the Abel–Jacobi map can also be described as $p \mapsto p - q$.

We now prove that the Abel–Jacobi map enjoys a universal property among morphisms of rational polyhedral spaces to integral tori, which is an exact analogue of the algebraic property (see Proposition 11.4.1 in [BL04], and see Section 1.4 in [BN07] for the corresponding property of the Jacobian of a finite graph). We first make the following elementary observation, which does not appear to have a proof in the literature.

Proposition 4.15. *The Abel–Jacobi map is a map of rational polyhedral spaces.*

Proof. Let $\eta \in \Omega^1_{\mathrm{Jac}(\Gamma)}$ be a 1-form. We need to show that its pullback along ϕ_q is a 1-form on Γ. Recall that $\Omega^1_{\mathrm{Jac}(\Gamma)} \cong \Omega^1_\Gamma$ and denote the 1-form on Γ that corresponds to η under this identification by $\omega = \sum_{e \in E(\Gamma)} a_e de$ with $a_e \in \mathbb{Z}$. We show that the pullback of η along ϕ_q is ω. Thinking of η as a linear function on $\mathrm{Hom}(\Omega^1_\Gamma, \mathbb{R})$, we easily see that

$$\eta\big(\phi_q(p)\big) = \eta\Big(\int_{\gamma_p} -\Big) = \int_{\gamma_p} \omega$$

for any $p \in \Gamma$ and γ_p a path from q to p. The second equality is simply the identification of $\eta \in \mathrm{Hom}\big(\mathrm{Hom}(\Omega^1_\Gamma, \mathbb{R}), \mathbb{R}\big)$ with $\omega \in \Omega^1_\Gamma$. But now we are already done because the coefficients a_e of ω are precisely $\frac{1}{\ell(e)} \int_e \omega$, with e parametrized with the orientation indicated by de. □

Proposition 4.16. *Let $\chi : \Gamma \to X$ be a morphism of rational polyhedral spaces from a tropical curve Γ to an integral torus $X = (\Lambda, \Lambda', [\cdot,\cdot])$. Then there exists a unique homomorphism $\mu : \mathrm{Jac}(\Gamma) \to X$ of integral tori such that the diagram*

$$\begin{tikzcd} \Gamma \arrow[r, "\chi"] \arrow[d, "\phi_q"] & X \arrow[d, "t_{-\chi(q)}"] \\ \mathrm{Jac}(\Gamma) \arrow[r, "\mu"] & X \end{tikzcd} \tag{15}$$

commutes for all $q \in \Gamma$.

Proof. Fix $q \in \Gamma$. Clearly, the composition $\chi_0 = t_{-\phi(q)} \circ \chi$ maps q to 0. We show that χ_0 factors via the Abel–Jacobi map. To do so, we need to describe maps $\mu^\# : \Lambda \to \Omega^1_\Gamma$ and $\mu_\# : H_1(\Gamma, \mathbb{Z}) \to \Lambda'$ which are compatible with the pairings on the integral tori.

The morphism χ_0 of rational polyhedral spaces induces a morphism of cotangent sheaves

$$(\chi_0)^{-1}\Omega^1_X \to \Omega^1_\Gamma.$$

Since $\Omega^1_X \cong \Lambda$, this is in fact the desired map $\mu^\#$. Passing to singular homology, the continuous map χ_0 induces a pushforward map $H_1(\Gamma, \mathbb{Z}) \to H_1(X, \mathbb{Z})$. Identifying $H_1(X, \mathbb{Z}) = \Lambda'$, we let $\mu_\#$ be the pushforward map. To show that the pair $\mu = (\mu^\#, \mu_\#)$ defines a homomorphism of integral tori we need to verify compatibility with pairings, in other words we need to show that

$$\big[\lambda, \mu_\#(\gamma)\big] = \int_\gamma \mu^\# \lambda \tag{16}$$

for all $\lambda \in \Lambda$ and $\gamma \in H_1(\Gamma, \mathbb{Z})$.

We now show that the pairing $[\cdot,\cdot]$ on X can be interpreted as integration as well. Indeed, let $\lambda \in \Lambda = \Omega^1_X$, which we view as the differential of an affine linear function f on X. Choose the

integration constant so that $f(0) = 0$, then f is linear and can be extended to a linear function on the universal cover $\mathrm{Hom}(\Lambda, \mathbb{R})$, namely

$$\begin{aligned} F : \mathrm{Hom}(\Lambda, \mathbb{R}) &\longrightarrow \mathbb{R} \\ u &\longmapsto u(\lambda). \end{aligned}$$

Let $\gamma \in H_1(X, \mathbb{Z}) = \Lambda'$. Choose a piecewise smooth representative and lift γ to a path $\gamma' : [0, 1] \to \mathrm{Hom}(\Lambda, \mathbb{R})$ on the universal cover of X going from 0 to some point $\lambda' \in \Lambda' \subseteq \mathrm{Hom}(\Lambda, \mathbb{R})$. Then $\int_\gamma \lambda = \int_{\gamma'} \lambda = F(\gamma'(1)) - F(\gamma'(0)) = F(\lambda') = \lambda'(\lambda)$. But now recall that Λ' is embedded in $\mathrm{Hom}(\Lambda, \mathbb{R})$ as $[\cdot, \lambda']$. This shows that $[\cdot, \cdot]$ is just integration of 1-forms along closed paths, and Equation (16) is simply the change-of-variables formula for line integrals. Hence μ is a homomorphism of integral tori.

Finally, we show that μ makes the diagram (15) commute for any $q' \in \Gamma$ (and not just the q that we fixed at the beginning of the proof). Indeed, it is clear that $\phi_q = t_{-\phi_{q'}(q)} \circ \phi_{q'}$. By Lemma 4.5 we obtain

$$\begin{tikzcd} \mathrm{Jac}(\Gamma) \arrow[r, "\mu"] \arrow[d, "t_{-\phi_{q'}(q)}"'] & X \arrow[d, "t_{\mu(-\phi_{q'}(q))}"] \\ \mathrm{Jac}(\Gamma) \arrow[r, "\mu"] & X \end{tikzcd}$$

But now $\mu(-\phi_{q'}(q)) = \mu(\phi_q(q')) = \chi(q') - \chi(q)$, and we are done. □

Remark 4.17. For the proof of Theorem 1.2 it is essential that all tropical curves be smooth and hence carry a fundamental cycle. Given a metric graph with finite univalent vertices, this is achieved by adding compact infinite rays to such vertices (see Example 3.8). We emphasize that the universal property in Proposition 4.16 is still valid in this context: any morphism of polyhedral spaces $\Gamma \to X$ maps each infinite ray to a single point in X, because any affine linear function on the ray is eventually constant.

4.5. **Tropical Prym variety.** Let $\pi : \widetilde{\Gamma} \to \Gamma$ be a harmonic double cover of tropical curves. We recall the construction of the tropical Prym variety (see [JL18], [LU21], [LZ22]), which we adopt slightly for our purposes. We assume a choice of graph model $p : \widetilde{G} \to G$ for π. Recall that an edge or vertex of Γ is called *free* if it has two preimages in $\widetilde{\Gamma}$ each of which has dilation factor equal to 1, and *dilated* if it has a unique preimage with dilation factor equal to 2. The set of dilated edges and vertices form the *dilation subgraph* of Γ. We say that π is *free* if the dilation subgraph is empty and *dilated* otherwise.

We note that the tropicalization of an algebraic étale double cover has the additional property of being *unramified*. This condition involves vertex weights, which we do not use, but also imposes a restriction on the dilation subgraph: each vertex must have even valency (see [JL18, Corollary 5.5]). This restriction does not naturally arise in the tropical setting, and we do not impose it.

A free edge e of Γ has two distinct preimages that we arbitrarily label $\widetilde{e}^+$ and $\widetilde{e}^-$, while a dilated edge e has a unique preimage that we denote $\widetilde{e}^+ = \widetilde{e}^-$ by abuse of notation. The maps

$$\begin{aligned} \mathrm{Nm}^\# = \pi^* : \Omega^1_\Gamma &\longrightarrow \Omega^1_{\widetilde{\Gamma}} \\ de &\longmapsto d\widetilde{e}^+ + d\widetilde{e}^- \end{aligned} \qquad \text{and} \qquad \mathrm{Nm}_\# = \pi_* : H_1(\widetilde{\Gamma}, \mathbb{Z}) \longrightarrow H_1(\Gamma, \mathbb{Z})$$

define the surjective *tropical norm homomorphism* $\mathrm{Nm} = (\mathrm{Nm}^{\#}, \mathrm{Nm}_{\#}) : \mathrm{Jac}(\widetilde{\Gamma}) \to \mathrm{Jac}(\Gamma)$. The kernel of Nm has two connected components if π is free, and a single connected component if π is dilated (see [JL18, Proposition 6.1], which we reprove below using a different method). The integral torus $(\mathrm{Ker}\,\mathrm{Nm})_0$ (see Section 4.2) comes equipped with the (generally non-principal) polarization ξ induced from $\mathrm{Jac}(\widetilde{\Gamma})$ by pullback along the injective homomorphism $i : (\mathrm{Ker}\,\mathrm{Nm})_0 \to \mathrm{Jac}(\widetilde{\Gamma})$. By Lemma 4.10, there exists a tropical abelian variety $(\mathrm{Ker}\,\mathrm{Nm})_0^{\mathrm{pp}}$ with a principal polarization ζ and a dilation $f : (\mathrm{Ker}\,\mathrm{Nm})_0^{\mathrm{pp}} \to (\mathrm{Ker}\,\mathrm{Nm})_0$ (which is an isomorphism of the underlying groups) such that the pullback $f^*\xi$ is a multiple of ζ. By abuse of notation, we denote the composition $i \circ f$ (which is still an injective homomorphism of integral tori) again by $i : (\mathrm{Ker}\,\mathrm{Nm})_0^{\mathrm{pp}} \to \mathrm{Jac}(\widetilde{\Gamma})$.

Definition 4.18. The *tropical Prym variety* of the double cover $\pi : \widetilde{\Gamma} \to \Gamma$ is the principally polarized tropical abelian variety $\mathrm{Prym}(\widetilde{\Gamma}/\Gamma) = (\mathrm{Ker}\,\mathrm{Nm})_0^{\mathrm{pp}}$. The dimension of $\mathrm{Prym}(\widetilde{\Gamma}/\Gamma)$ is equal to $g_0 = g(\widetilde{\Gamma}) - g(\Gamma)$.

Our definition of the tropical Prym variety differs from the existing literature by the introduction of the rescaling $(\mathrm{Ker}\,\mathrm{Nm})_0^{\mathrm{pp}} \to (\mathrm{Ker}\,\mathrm{Nm})_0$ (which is in fact implicitly present in [LZ22], but not spelled out). In this way, our Prym varieties are always principally polarized, which is convenient for intersection theory calculations. Furthermore, as we will soon show, the pullback of the principal polarization ξ of $\mathrm{Jac}(\widetilde{\Gamma})$ along i is always twice the principal polarization ζ on $\mathrm{Prym}(\widetilde{\Gamma}/\Gamma)$. This unifies the two cases of [LU21, Theorem 1.5.7] in the sense that we may always write

$$2\zeta = i^*\xi$$

for the inclusion $i : \mathrm{Prym}(\widetilde{\Gamma}/\Gamma) \to \mathrm{Jac}(\widetilde{\Gamma})$ without a distinction between the dilated and the free case.

In order to describe the polarization type of $(\mathrm{Ker}\,\mathrm{Nm})_0$ associated to a double cover $\pi : \widetilde{\Gamma} \to \Gamma$, and for other explicit calculations, we construct explicit bases for the homology groups $H_1(\widetilde{\Gamma}, \mathbb{Z})$ and $H_1(\Gamma, \mathbb{Z})$. Propositions 4.19 and 4.20 sharpen and improve on Lemma 1.5.4 in [LU21][1].

We first introduce some conventions and notation. A double cover $\pi : \widetilde{\Gamma} \to \Gamma$ induces an involution $\iota : \widetilde{\Gamma} \to \widetilde{\Gamma}$, which is fixed-point-free if and only if π is free. Let $\iota_* : H_1(\widetilde{\Gamma}, \mathbb{Z}) \to H_1(\widetilde{\Gamma}, \mathbb{Z})$ be the induced map. We have canonical isomorphisms $H_1(\widetilde{\Gamma}, \mathbb{Z}) = \Omega^1_{\widetilde{\Gamma}}$ and $H_1(\Gamma, \mathbb{Z}) = \Omega^1_{\Gamma}$ (the principal polarizations on $\mathrm{Jac}(\widetilde{\Gamma})$ and $\mathrm{Jac}(\Gamma)$), so we henceforth identify 1-forms and 1-cycles wherever this simplifies notation. Under this identification, we have a pullback map $\pi^* : H_1(\Gamma, \mathbb{Z}) \to H_1(\widetilde{\Gamma}, \mathbb{Z})$ on homology. It is easy to verify that

$$\pi^* \circ \pi_* = \mathrm{Id} + \iota_*$$

on $H_1(\widetilde{\Gamma}, \mathbb{Z})$.

For a free double cover $\pi : \widetilde{\Gamma} \to \Gamma$ we have $g(\widetilde{\Gamma}) = 2g(\Gamma) - 1$ and therefore $g_0 = g(\Gamma) - 1$. For a dilated double cover, choose a graph model $\pi : \widetilde{G} \to G$ and let $G_{\mathrm{dil}} \subset G$ and $\widetilde{G}_{\mathrm{dil}} \subset \widetilde{G}$ denote respectively the dilated subgraph of G and its (isomorphic) preimage in $\widetilde{G}$. Denote by $m_d = |E(G_{\mathrm{dil}})|$ and $n_d = |V(G_{\mathrm{dil}})|$ the number of dilated edges and dilated vertices, respectively, and denote by d the number of connected components of the dilation subgraph. The numbers m_d and n_d depend on the choice of model but their difference does not, and we introduce the

[1] In our opinion, there are certain gaps in the proof of Lemma 1.5.4 in [LU21].

invariants

$$\begin{aligned} A &= g(G) - m_d + n_d - d, \\ B &= d - 1, \\ C &= m_d - n_d + d. \end{aligned} \tag{17}$$

Furthermore, it is easy to see that $|E(\widetilde{G})| = 2|E(G)| - m_d$ and $|V(\widetilde{G})| = 2|V(G)| - n_d$, therefore

$$A + B = g(G) - m_d + n_d - 1 = |E(G)| - |V(G)| - m_d + n_d = g(\widetilde{G}) - g(G) = g_0$$

is the dimension of the Prym variety.

We now find explicit bases for the homology groups, starting with the free case.

Proposition 4.19. *Let $\pi : \widetilde{\Gamma} \to \Gamma$ be a connected free double cover and let $g(\Gamma) = g$ and $g(\widetilde{\Gamma}) = 2g - 1$. Then there exists a basis $\alpha_1, \dots, \alpha_{g-1}, \gamma_1$ of $H_1(\Gamma, \mathbb{Z})$ and a basis $\widetilde{\alpha}_1^{\pm}, \dots, \widetilde{\alpha}_{g-1}^{\pm}, \widetilde{\gamma}_1$ of $H_1(\widetilde{\Gamma}, \mathbb{Z})$ such that*

$$\begin{aligned} &\iota_*(\widetilde{\alpha}_i^{\pm}) = \widetilde{\alpha}_i^{\mp}, && \pi_*(\widetilde{\alpha}_i^{\pm}) = \alpha_i, && \pi^*(\alpha_i) = \widetilde{\alpha}_i^{+} + \widetilde{\alpha}_i^{-}, && i = 1, \dots, g-1, \\ &\iota_*(\widetilde{\gamma}_1) = \widetilde{\gamma}_1, && \pi_*(\widetilde{\gamma}_1) = 2\gamma_1, && \pi^*(\gamma_1) = \widetilde{\gamma}_1. \end{aligned}$$

Proof. The proof of this statement is included as part of Constructions A and B in [LZ22], and we briefly summarize it. Choose a graph model $\pi : \widetilde{G} \to G$ for the double cover. Choose an orientation on G and a spanning tree $T \subset G$, and denote the complementary edges by $E(G) \backslash E(T) = \{e_0, \dots, e_{g-1}\}$. Let $\widetilde{e}_i^{\pm}$ and $\widetilde{T}^{\pm} \subset \widetilde{G}$ denote the preimages of e_i and T, respectively. Let $S \subset \{e_0, \dots, e_{g-1}\}$ denote the set of those complementary edges whose lifts connect the two trees $\widetilde{T}^{\pm}$. The set S is nonempty since $\widetilde{G}$ is connected, so we assume without loss of generality that $e_0 \in S$. It follows that $\widetilde{T} = \widetilde{T}^{+} \cup \widetilde{T}^{-} \cup \{\widetilde{e}_0^{+}\}$ is a spanning tree for $\widetilde{G}$.

For a cycle $\gamma \in H_1(G, \mathbb{Z})$ and an oriented edge $e \in E(G)$, denote by $\langle \gamma, e \rangle$ the coefficient with which e appears in γ, and similarly for $H_1(\widetilde{G}, \mathbb{Z})$. We denote $\varepsilon_i \in H_1(G, \mathbb{Z})$ for $i = 0, \dots, g-1$ the unique cycle on the graph $T \cup \{e_i\}$ such that $\langle \varepsilon_i, e_i \rangle = 1$ Similarly, let $\widetilde{\varepsilon}_0 \in H_1(\widetilde{G}, \mathbb{Z})$ and $\widetilde{\varepsilon}_i^{\pm} \in H_1(\widetilde{G}, \mathbb{Z})$ for $i = 1, \dots, g-1$ denote the unique cycles on $\widetilde{T} \cup \{\widetilde{e}_0^{-}\}$ and $\widetilde{T} \cup \{\widetilde{e}_i^{\pm}\}$ such that $\langle \widetilde{\varepsilon}_0, \widetilde{e}_0^{-} \rangle = 1$ and $\langle \widetilde{\varepsilon}_i^{\pm}, \widetilde{e}_i^{\pm} \rangle = 1$, respectively. The cycles $\varepsilon_0, \dots, \varepsilon_{g-1}$ form a basis for $H_1(G, \mathbb{Z})$, and furthermore the coordinates of any $\gamma \in H_1(G, \mathbb{Z})$ with respect to this basis are given by

$$\gamma = \langle \gamma, e_0 \rangle \varepsilon_0 + \cdots + \langle \gamma, e_{g-1} \rangle \varepsilon_{g-1},$$

and a similar statement holds for $\widetilde{\varepsilon}_0, \widetilde{\varepsilon}_1^{\pm}, \dots, \widetilde{\varepsilon}_{g-1}^{\pm}$.

The action of ι_*, π_*, and π^* on these bases is computed by looking at the coefficients of the edges e_i and $\widetilde{e}_i^{\pm}$. Since $\langle \widetilde{\varepsilon}_0, \widetilde{e}_0^{-} \rangle = 1$, we see that $\iota_*(\widetilde{\varepsilon}_0) = \widetilde{\varepsilon}_0$, $\pi_*(\widetilde{\varepsilon}_0) = 2\varepsilon_0$, and $\pi^*(\varepsilon_0) = \widetilde{\varepsilon}_0$. Now denote $c_i = \langle \widetilde{\varepsilon}_i^{+}, \widetilde{e}_0^{+} \rangle$ for $i = 1, \dots, g-1$ (this number is equal to 0 or ± 1 since $\widetilde{\varepsilon}_i^{+}$ is a simple cycle). Comparing the coefficients of $\widetilde{e}_0^{-}$ and $\widetilde{e}_i^{-}$, we see that $\iota_*(\widetilde{\varepsilon}_i^{+}) = \widetilde{\varepsilon}_i^{-} + c_i \widetilde{\varepsilon}_0$ and $\iota_*(\widetilde{\varepsilon}_i^{-}) = \widetilde{\varepsilon}_i^{+} - c_i \widetilde{\varepsilon}_0$. Similarly, comparing the coefficients of e_0 and e_i, we see that $\pi_*(\widetilde{\varepsilon}_i^{\pm}) = \varepsilon_i \pm c_i \varepsilon_0$. Finally, using the relation $\pi^* \pi_* = \mathrm{Id} + \iota_*$ we find that

$$\pi^*(2\varepsilon_0) = \pi^*(\pi_*(\widetilde{\varepsilon}_0)) = (\mathrm{Id} + \iota_*)(\widetilde{\varepsilon}_0) = 2\widetilde{\varepsilon}_0$$

and

$$\pi^*(2\varepsilon_i) = \pi^* \pi_*(2\widetilde{\varepsilon}_i^{+} - c_i \widetilde{\varepsilon}_0) = (\mathrm{Id} + \iota_*)(2\widetilde{\varepsilon}_i^{+} - c_i \widetilde{\varepsilon}_0) = 2(\widetilde{\varepsilon}_i^{+} + \widetilde{\varepsilon}_i^{-}),$$

for $i = 1, \dots, g-1$. To complete the proof, we now set for $i = 1, \dots, g-1$

$$\begin{aligned} \widetilde{\alpha}_i^+ &= \widetilde{\varepsilon}_i^+, \\ \widetilde{\alpha}_i^- &= \widetilde{\varepsilon}_i^- + c_i \widetilde{\varepsilon}_0, \\ \alpha_i &= \varepsilon_i + c_i \varepsilon_0, \end{aligned}$$

and $\widetilde{\gamma}_1 = \widetilde{\varepsilon}_0$, $\gamma_1 = \varepsilon_0$. □

We now consider the dilated case.

Proposition 4.20. *Let $\pi : \widetilde{\Gamma} \to \Gamma$ be a dilated double cover. Then there exists a basis $\alpha_1, \dots, \alpha_A$, $\gamma_1, \dots, \gamma_C$ of $H_1(\Gamma, \mathbb{Z})$ and a basis $\widetilde{\alpha}_1^\pm, \dots, \widetilde{\alpha}_A^\pm$, $\widetilde{\beta}_1, \dots, \widetilde{\beta}_B$, $\widetilde{\gamma}_1, \dots, \widetilde{\gamma}_C$ of $H_1(\widetilde{\Gamma}, \mathbb{Z})$ with A, B, and C as defined in Equation (17), such that*

$$\begin{aligned} &\iota_*(\widetilde{\alpha}_i^\pm) = \widetilde{\alpha}_i^\mp, && \pi_*(\widetilde{\alpha}_i^\pm) = \alpha_i, && \pi^*(\alpha_i) = \widetilde{\alpha}_i^+ + \widetilde{\alpha}_i^-, && i = 1, \dots, A, \\ &\iota_*(\widetilde{\beta}_j) = -\widetilde{\beta}_j, && \pi_*(\widetilde{\beta}_j) = 0, && && j = 1, \dots, B, \\ &\iota_*(\widetilde{\gamma}_k) = \widetilde{\gamma}_k, && \pi_*(\widetilde{\gamma}_k) = \gamma_k, && \pi^*(\gamma_k) = 2\widetilde{\gamma}_k, && k = 1, \dots, C. \end{aligned}$$

Proof. We begin by contracting each dilated edge of G and the corresponding edge of $\widetilde{G}$. The result is a double cover $\pi' : \widetilde{G}' \to G'$ with associated involution $\iota' : \widetilde{G}' \to \widetilde{G}'$, whose dilated vertices correspond to the connected components of G_{dil}. Denote these vertices by $v'_0, \dots, v'_{d-1} \in V(G')$ and their preimages by $\widetilde{v}'_0, \dots, \widetilde{v}'_{d-1} \in V(\widetilde{G}')$. We now consider the free cover $p'' : \widetilde{G}'' \to G''$ obtained from $p' : \widetilde{G}' \to G'$ in the following way. For each $i = 0, \dots, d-1$, we replace v'_i with an undilated vertex v''_i with an attached loop e_i, and replace $\widetilde{v}'_i$ with a pair of vertices $\widetilde{v}''^\pm_i$ connected by a pair of edges $\widetilde{e}''^\pm_i$. For each half-edge $h \in H(G')$ rooted at v'_i, we attach its preimages $\widetilde{h}^\pm$ to the vertices $\widetilde{v}''^\pm_i$ in any manner. The result is a free cover $\pi'' : \widetilde{G}'' \to G''$ whose contraction along the loops $e_0, \dots, e_{d-1}$ is the edge-free cover $\pi' : \widetilde{G}' \to G'$. We denote $g(G'') = g$, so that $g(G) = g + m_d - n_d$.

We now pick a spanning tree $T \subset G''$ and let $E(G'') \backslash E(T) = \{e_0, \dots, e_{g-1}\}$ be the complementary edges, where the first d of the e_i are the loops at the vertices v_i, as defined above. Let $\varepsilon''_0, \dots, \varepsilon''_{g-1}$ and $\widetilde{\varepsilon}''_0, \widetilde{\varepsilon}''^\pm_1, \dots, \widetilde{\varepsilon}''^\pm_{g-1}$ be the bases of $H_1(G'', \mathbb{Z})$ and $H_1(\widetilde{G}'', \mathbb{Z})$ defined in Proposition 4.19. The edges $e_0, \dots, e_{d-1}$ are loops, so they form closed cycles and hence in fact $\varepsilon_i = e_i$ for $i = 0, \dots, d-1$. Furthermore, the edges $\widetilde{e}''^+_i$ and $\widetilde{e}''^-_i$ have the same root vertices $\widetilde{v}''^\pm_i$ for $i = 0, \dots, d-1$. This implies that $\widetilde{\varepsilon}''_0 = \widetilde{e}''^+_0 + \widetilde{e}''^-_0$, since the edge $\widetilde{e}''^+_0$ is contained in the spanning tree $\widetilde{T}$. Also, for $i = 1, \dots, d-1$ the cycle $\widetilde{\varepsilon}''^-_i$ is obtained from $\widetilde{\varepsilon}''^+_i$ by replacing $\widetilde{e}''^+_i$ with $\widetilde{e}''^-_i$ and reversing the direction of the remaining path.

We now let $\varepsilon'_0, \dots, \varepsilon'_{g-1}$ and $\widetilde{\varepsilon}'_0, \widetilde{\varepsilon}'^\pm_1, \dots, \widetilde{\varepsilon}'^\pm_{g-1}$ denote the cycles in respectively $H_1(G', \mathbb{Z})$ and $H_1(\widetilde{G}', \mathbb{Z})$ obtained by contracting the cycles $\varepsilon''_0, \dots, \varepsilon''_{g-1}$ and $\widetilde{\varepsilon}''_0, \widetilde{\varepsilon}''^\pm_1, \dots, \widetilde{\varepsilon}''^\pm_{g-1}$ defined above, in other words by setting e''_i and $\widetilde{e}''^\pm_i$ to zero for $i = 0, \dots, d-1$. We see that $\widetilde{\varepsilon}'_0 = 0$ and $\varepsilon'_i = 0$ for $i = 0, \dots, d-1$. The remaining cycles $\varepsilon'_d, \dots, \varepsilon'_{g-1}$ form a basis for $H_1(G', \mathbb{Z})$. Furthermore, we see that $\widetilde{\varepsilon}'^-_i = -\widetilde{\varepsilon}'^+_i$ for $i = 1, \dots, d-1$, and the cycles $\widetilde{\varepsilon}'^+_1, \dots, \widetilde{\varepsilon}'^+_{d-1}$ and $\widetilde{\varepsilon}'^\pm_d, \dots, \widetilde{\varepsilon}'^\pm_{g-1}$ form a basis for $H_1(\widetilde{G}', \mathbb{Z})$. These bases satisfy the relations

$$\begin{aligned} &\iota'_*(\widetilde{\varepsilon}'^+_i) = -\widetilde{\varepsilon}'^+_i, && \pi'_*(\widetilde{\varepsilon}'^+_i) = 0, && && i = 1, \dots, d-1, \\ &\iota'_*(\widetilde{\varepsilon}'^\pm_i) = \widetilde{\varepsilon}'^\mp_i, && \pi'_*(\widetilde{\varepsilon}'^\pm_i) = \varepsilon_i, && \pi'^*(\varepsilon'_i) = \widetilde{\varepsilon}'^+_i + \widetilde{\varepsilon}'^-_i, && i = d, \dots, g-1. \end{aligned}$$

The edge set $E(G')$ is identified with the set of non-dilated edges of $E(G)$, hence we can view each cycle $\widetilde{\varepsilon}_i'^{+}$ as a simplicial chain $\widetilde{\varepsilon}_i^{+}$ in G, with boundary $\delta(\widetilde{\varepsilon}_i^{+})$ supported on the set of dilated vertices. We claim that $\widetilde{\varepsilon}_i^{+}$ is in fact closed. Indeed, since $\iota_*(\widetilde{\varepsilon}_i^{+}) = -\widetilde{\varepsilon}_i^{+}$, it consists of a linear combination of expressions of the form $\widetilde{e}^{+} - \widetilde{e}^{-}$ for certain non-dilated pairs of edges, and if a root vertex of $\widetilde{e}^{+}$ is dilated, then it is also a root vertex of $\widetilde{e}^{-}$. Hence $\delta(\widetilde{\varepsilon}_i^{+}) = 0$ and $\widetilde{\varepsilon}_i^{+}$ is a cycle, and we relabel $\widetilde{\beta}_i = \widetilde{\varepsilon}_i^{+}$ for $i = 1, \ldots, B = d - 1$.

Similarly, for each $i = d, \ldots, g-1$, let $\widetilde{\varepsilon}_i^{\pm}$ be the cycle $\widetilde{\varepsilon}_i'^{\pm}$, but viewed as a chain on G. The boundaries $\delta(\widetilde{\varepsilon}_i^{+})$ and $\delta(\widetilde{\varepsilon}_i^{-})$ are equal and supported on the set of dilated vertices, so we can find a chain ζ_i supported on $\widetilde{G}_{dil}$ such that $\widetilde{\alpha}_i^{\pm} = \widetilde{\varepsilon}_{i+d-1}^{\pm} + \zeta_{i+d-1}$ for $i = 1, \ldots, A = g - d$ is a closed cycle on $\widetilde{G}$. Denoting $\alpha_i = \pi_*(\widetilde{\alpha}_i^{\pm})$, we see that the $\widetilde{\alpha}_i^{\pm}$ and the α_i satisfy the required relations.

Finally, we let $\gamma_1, \ldots, \gamma_C$ be a basis for $H_1(G_{dil}, \mathbb{Z})$, and let $\widetilde{\gamma}_1, \ldots, \widetilde{\gamma}_C$ be the preimages of these cycles on $\widetilde{G}_{dil}$. This completes the required basis. □

We now recall that $(\operatorname{Ker} \mathrm{Nm})_0$ is the integral torus $(K, K', [\cdot,\cdot]_P)$, where K is the torsion-free part of the cokernel of the pullback map $\pi^* : H_1(\Gamma, \mathbb{Z}) \to H_1(\widetilde{\Gamma}, \mathbb{Z})$ (recall that we have identified Ω^1 with H_1), K' is the kernel of the pushforward map $\pi_* : H_1(\widetilde{\Gamma}, \mathbb{Z}) \to H_1(\Gamma, \mathbb{Z})$, and $[\cdot,\cdot]_P$ is the induced pairing. The principal polarization on $\mathrm{Jac}(\widetilde{\Gamma})$ (which is now simply the identity map on $H_1(\widetilde{\Gamma}, \mathbb{Z})$) induces a polarization on $(\operatorname{Ker} \mathrm{Nm})_0$, and we compute its type. The following result sharpens Theorem 1.5.7 in [LU21].

Proposition 4.21. *Let $\pi : \widetilde{\Gamma} \to \Gamma$ be a double cover. The polarization $K' \to K$ on $(\operatorname{Ker} \mathrm{Nm})_0$ induced from the principal polarization on $\mathrm{Jac}\,\widetilde{\Gamma}$ has type $(2, \ldots, 2)$ when π is free, and type $(1, \ldots, 1, 2, \ldots, 2)$ when π is dilated, where the number of 1's and 2's is equal to B and A, respectively.*

Proof. We give explicit bases for K and K' using the bases of $H_1(\widetilde{\Gamma}, \mathbb{Z})$ and $H_1(\Gamma, \mathbb{Z})$ constructed in Propositions 4.19 and 4.20. The proof for the free and dilated cases is formally identical, provided that we set $A = g_0$ and $B = 0$ in the free case.

The kernel $K' = \operatorname{Ker} \pi_*$ has a basis $\widetilde{\alpha}_i = \widetilde{\alpha}_i^{+} - \widetilde{\alpha}_i^{-}$ for $i = 1, \ldots, A$ and $\widetilde{\beta}_j$ for $j = 1, \ldots, B$. Similarly, $K = (\operatorname{Coker} \pi)^{tf}$ has a basis $\widetilde{\alpha}^i = [\widetilde{\alpha}_i^{+}] = -[\widetilde{\alpha}_i^{-}]$ for $i = 1, \ldots, A$ and $\widetilde{\beta}^j = [\widetilde{\beta}_j]$ for $j = 1, \ldots, B$, where $[\cdot]$ denotes equivalence classes in K. The polarization $\xi : K' \to K$ induced from the principal polarization on $\mathrm{Jac}(\widetilde{\Gamma})$ (which, according to our convention, is the identity map on $H_1(\widetilde{\Gamma}, \mathbb{Z})$) is

$$\xi(\widetilde{\alpha}_i) = [\widetilde{\alpha}_i^{+}] - [\widetilde{\alpha}_i^{-}] = 2[\widetilde{\alpha}_i^{+}] = 2\widetilde{\alpha}^i,$$
$$\xi(\widetilde{\beta}_j) = [\widetilde{\beta}_j] = \widetilde{\beta}^j.$$

Hence the polarization has type $(1^B, 2^A)$, and the principal polarization $\zeta : K \to K'$ on $\mathrm{Prym}(\widetilde{\Gamma}/\Gamma)$ (constructed canonically using Lemma 4.10) is

$$\zeta(\widetilde{\alpha}_i) = \widetilde{\alpha}^i, \quad \zeta(\widetilde{\beta}_j) = \widetilde{\beta}^j.$$

□

Alternatively, the Prym variety may be described as follows. Recall that the involution $\iota : \widetilde{\Gamma} \to \widetilde{\Gamma}$ induces a pushforward map $\iota_* : H_1(\widetilde{\Gamma}, \mathbb{Z}) \to H_1(\widetilde{\Gamma}, \mathbb{Z})$ (given explicitly in Propositions 4.19 and 4.20). Identifying $H_1(\widetilde{\Gamma}, \mathbb{Z}) = \Omega^1_{\widetilde{\Gamma}}$ as before, the pullback map ι^* is equal to ι_*. The two maps (ι^*, ι_*) induce a homomorphism $\iota : \mathrm{Jac}(\widetilde{\Gamma}) \to \mathrm{Jac}(\widetilde{\Gamma})$ on the Jacobian.

Lemma 4.22. *The Prym variety* $\mathrm{Prym}(\widetilde{\Gamma}/\Gamma)$ *is isomorphic (as a principally polarized tropical abelian variety) to the kernel of* $\mathrm{Id}+\iota : \mathrm{Jac}(\widetilde{\Gamma}) \to \mathrm{Jac}(\widetilde{\Gamma})$.

Proof. Considering the basis for $H_1(\widetilde{\Gamma}, \mathbb{Z})$ from Proposition 4.20 (in case of a dilated double cover) or Proposition 4.19 (in case of a free double cover) is is easy to see that $\mathrm{Ker}(\mathrm{Id}+\iota)_0$ has the same underlying lattices as $(\mathrm{Ker}\,\mathrm{Nm})_0$. Furthermore, for both kernels the pairing and the induced (non-principal) polarization come from $\mathrm{Jac}(\widetilde{\Gamma})$. Hence, the two kernels are isomorphic as (non-principally) polarized tropical abelian varieties. The construction of the principal polarization from Lemma 4.10 is intrinsic to $(\mathrm{Ker}\,\mathrm{Nm})_0$ (resp. $\mathrm{Ker}(\mathrm{Id}+\iota)_0$) and in particular it does not depend on the maps Nm or $\mathrm{Id}+\iota$. Therefore, the induced principally polarized tropical abelian varieties are isomorphic as well. □

We now define the tropical Abel–Prym map. Recall that $\mathrm{Ker}\,\mathrm{Nm}$ has two connected components when π is free and one when π is dilated. In the former case, given a point $x \in \widetilde{\Gamma}$, the divisor $x - \iota(x)$ lies in the odd component, and the sum of two such divisors lies in the connected component of the identity. With this in mind, we make the following definition:

Definition 4.23. The *tropical Abel–Prym map* is defined as

$$\begin{aligned} \psi_q : \widetilde{\Gamma} &\longrightarrow \mathrm{Prym}(\widetilde{\Gamma}/\Gamma) \\ p &\longmapsto \left(\omega \mapsto \int_{\gamma_p} \omega - \int_{\iota_* \gamma_p} \omega \right), \end{aligned}$$

where γ_p is a path in $\widetilde{\Gamma}$ from a fixed point $q \in \widetilde{\Gamma}$ to p.

We note that the divisor corresponding to $\psi_q(p)$ is $p - q - \iota(p - q)$, which lies in the connected component of the identity.

It is convenient to give the following alternative definition of the Abel–Prym map. Let $i : \mathrm{Prym}(\widetilde{\Gamma}/\Gamma) \to \mathrm{Jac}(\widetilde{\Gamma})$ be the inclusion (more accurately, the composition of the inclusion $(\mathrm{Ker}\,\pi)_0 \to \mathrm{Jac}(\widetilde{\Gamma})$ with the dilation $\mathrm{Prym}(\widetilde{\Gamma}/\Gamma) \to (\mathrm{Ker}\,\pi)_0$ defined by Lemma 4.10). The tropical abelian varieties $\mathrm{Prym}(\widetilde{\Gamma}/\Gamma)$ and $\mathrm{Jac}(\widetilde{\Gamma})$ are principally polarized, hence we can identify them with their duals. We can then factor the Abel–Prym map as

$$\widetilde{\Gamma} \xrightarrow{\phi_q} \mathrm{Jac}(\widetilde{\Gamma}) \xrightarrow{i^\vee} \mathrm{Prym}(\widetilde{\Gamma}/\Gamma) \tag{18}$$

where ϕ_q is the Abel–Jacobi map with base point q and $i^\vee$ is the dual of i.

The Abel–Prym map posses a universal property as well.

Proposition 4.24. *Let* $\pi : \widetilde{\Gamma} \to \Gamma$ *be a double cover with associated involution* $\iota : \widetilde{\Gamma} \to \widetilde{\Gamma}$. *Let* $\chi : \widetilde{\Gamma} \to X$ *be a morphism of rational polyhedral spaces to an integral torus* X *such that* $\chi\iota = -\chi$. *Then there exists a unique homomorphism* $\mu : \mathrm{Prym}(\widetilde{\Gamma}/\Gamma) \to X$ *of integral tori such that the diagram*

$$\begin{tikzcd} \widetilde{\Gamma} \arrow[r, "\chi"] \arrow[d, "\psi_q"] & X \arrow[d, "t_{-\chi(q)}"] \\ \mathrm{Prym}(\widetilde{\Gamma}/\Gamma) \arrow[r, "\mu"] & X \end{tikzcd}$$

commutes for all $q \in \widetilde{\Gamma}$.

Proof. By the universal property of the tropical Jacobian we obtain a commutative square

$$\begin{tikzcd} \widetilde{\Gamma} \arrow[r, "\chi"] \arrow[d, "\phi_q"] & X \arrow[d, "t_{-\chi(q)}"] \\ \mathrm{Jac}(\widetilde{\Gamma}) \arrow[r, "\mu"] & X. \end{tikzcd}$$

Now note that $\mathrm{Id}+\iota : \mathrm{Jac}(\widetilde{\Gamma}) \to \mathrm{Jac}(\widetilde{\Gamma})$ is self-dual in the sense that after identifying $\mathrm{Jac}(\widetilde{\Gamma})$ and $\mathrm{Jac}(\widetilde{\Gamma})^\vee$ via the principal polarization, the dual homomorphism $(\mathrm{Id}+\iota)^\vee : \mathrm{Jac}(\widetilde{\Gamma}) \to \mathrm{Jac}(\widetilde{\Gamma})$ is equal to $\mathrm{Id}+\iota$ again. By Lemma 4.22 we have $\mathrm{Prym}(\widetilde{\Gamma}/\Gamma) = \mathrm{Ker}(\mathrm{Id}+\iota)_0^{\mathrm{pp}}$ and hence by Lemma 4.8 we think of $\mathrm{Prym}(\widetilde{\Gamma}/\Gamma)$ as the cokernel of $\mathrm{Id}+\iota$ as well. The assumption $\chi\iota = -\chi$ implies $\mu\iota = -\mu$, or equivalently, $\mu \circ (\mathrm{Id}+\iota) = 0$. By the universal property of the cokernel μ now factors through $\mathrm{Prym}(\widetilde{\Gamma}/\Gamma)$

$$\begin{tikzcd} & \widetilde{\Gamma} \arrow[r, "\chi"] \arrow[d, "\phi_q"] \arrow[dl, "\psi_q"'] & X \arrow[d, "t_{-\chi(q)}"] \\ & \mathrm{Jac}(\widetilde{\Gamma}) \arrow[r, "\mu"] \arrow[d, "j^\vee"] & X \\ \mathrm{Prym}(\widetilde{\Gamma}/\Gamma) \arrow[r, "\text{dilation}"] & \mathrm{Coker}(\mathrm{Id}+\iota), \arrow[ur, dashed] & \end{tikzcd}$$

where $j : \mathrm{Ker}(\mathrm{Id}+\iota)_0 \to \mathrm{Jac}(\widetilde{\Gamma})$ is the inclusion and we have identified $\mathrm{Jac}(\widetilde{\Gamma}) \cong \mathrm{Jac}(\widetilde{\Gamma})^\vee$ via its principal polarization. The desired map is now the composition of the dilation $\mathrm{Prym}(\widetilde{\Gamma}/\Gamma) \to \mathrm{Coker}(\mathrm{Id}+\iota)$ with the dashed arrow. □

Remark 4.25. The d-fold product of the tropical Abel-Prym map $\psi_q^d : \widetilde{\Gamma}^d \to \mathrm{Prym}(\widetilde{\Gamma}/\Gamma)$ and the composition of dilation and inclusion $i : \mathrm{Prym}(\widetilde{\Gamma}/\Gamma) \to \mathrm{Jac}(\widetilde{\Gamma})$ are both proper morphisms of rational polyhedral spaces.

4.6. **The tropical Poincaré–Prym formula.** Let Γ be a metric graph of genus g, let $1 \leqslant d \leqslant g$, and let $\phi_q^d : \Gamma^d \to \mathrm{Jac}(\Gamma)$ be the d-fold product of the Abel–Jacobi map with an arbitrary base point q, and let $\widetilde{W}_d$ denote the image of ϕ_q^d. The tropical Poincaré formula [GS19b] states that

$$[\widetilde{W}_d] = \frac{1}{(g-d)!}[\Theta]^{g-d} \in H_{d,d}(\mathrm{Jac}(\Gamma)),$$

where $[\Theta] \in H^{1,1}\big(\mathrm{Jac}(\Gamma)\big)$ is the class of the theta divisor of $\mathrm{Jac}(\Gamma)$, which lives in homology via Poincaré duality. The algebraic Poincaré formula has an analogue for Prym varieties, which is part of Welters' criterion and which we call the Poincaré–Prym formula. We conjecture that this formula holds in the tropical setting as well:

Conjecture 4.26. *(The tropical Poincaré–Prym formula) Let $\widetilde{\Gamma} \to \Gamma$ be a (possibly dilated) double cover of tropical curves and fix a base point $q \in \widetilde{\Gamma}$. Let $g_0 = \dim \mathrm{Prym}(\widetilde{\Gamma}/\Gamma)$, let $1 \leqslant d \leqslant g_0$, and denote by $\widetilde{Y}_d$ the image of the d-fold Abel–Prym map $\psi_q^d : \widetilde{\Gamma}^d \to \mathrm{Prym}(\widetilde{\Gamma}/\Gamma)$. Then*

$$[\widetilde{Y}_d] = \frac{2^d}{(g_0-d)!}[\Xi]^{g_0-d} \in H_{d,d}\big(\mathrm{Prym}(\widetilde{\Gamma}/\Gamma)\big),$$

where $[\Xi]$ is the class of the principal polarization of $\mathrm{Prym}(\widetilde{\Gamma}/\Gamma)$.

We only prove this result for $d = 1$, which is all that we require to prove our main Theorems 1.1 and 1.2:

Theorem 4.27. *Conjecture 4.26 holds for* $d = 1$.

We give a proof of Conjecture 4.26 assuming the following conjecture, which clearly holds for $d = 1$ and thus establishes Theorem 4.27:

Conjecture 4.28. *For all* $1 \leqslant d \leqslant g_0$ *the locus* $\widetilde{Y}_d \subseteq \mathrm{Prym}(\widetilde{\Gamma}/\Gamma)$ *is purely* d*-dimensional.*

We will make use of the factorization (18) of the tropical Abel–Prym map through the tropical Abel–Jacobi map. We start by describing the map $i^\vee : \mathrm{Jac}(\widetilde{\Gamma}) \to \mathrm{Prym}(\widetilde{\Gamma}/\Gamma)$. This map is obtained by composing the dual $j^\vee : \mathrm{Jac}(\widetilde{\Gamma})^\vee \to \mathrm{Prym}(\widetilde{\Gamma}/\Gamma)^\vee$ of the inclusion map $j : \mathrm{Prym}(\widetilde{\Gamma}/\Gamma) \to \mathrm{Jac}(\widetilde{\Gamma})$ with the isomorphism $\zeta^{-1} : \mathrm{Prym}(\widetilde{\Gamma}/\Gamma)^\vee \to \mathrm{Prym}(\widetilde{\Gamma}/\Gamma)$ defined by the principal polarization ζ on $\mathrm{Prym}(\widetilde{\Gamma}/\Gamma)$ (since we have identified $H_1(\widetilde{\Gamma}, \mathbb{Z})$ with $\Omega^1_{\widetilde{\Gamma}}$, we may also identify $\mathrm{Jac}(\widetilde{\Gamma}) = \mathrm{Jac}(\widetilde{\Gamma})^\vee$). In terms of the bases of $H_1(\widetilde{\Gamma}, \mathbb{Z})$, K, and K' defined in Propositions 4.19, 4.20, and 4.21 (where we set $C = 1$ for a free cover) the map $i^\vee = ((i^\vee)^\#, (i^\vee)_\#)$ is given by

$$\begin{aligned} (i^\vee)^\# : K &\longrightarrow H_1(\widetilde{\Gamma}, \mathbb{Z}) \\ \widetilde{\alpha}^i &\longmapsto \widetilde{\alpha}_i^+ - \widetilde{\alpha}_i^- \\ \widetilde{\beta}^j &\longmapsto \widetilde{\beta}_j \end{aligned} \qquad \begin{aligned} (i^\vee)_\# : H_1(\widetilde{\Gamma}, \mathbb{Z}) &\longrightarrow K' \\ \widetilde{\alpha}_i^\pm &\longmapsto \pm\widetilde{\alpha}_i \\ \widetilde{\beta}_j &\longmapsto 2\widetilde{\beta}_j \\ \widetilde{\gamma}_k &\longmapsto 0. \end{aligned} \tag{19}$$

On the level of real tori, the homomorphism $i^\vee$ is given by

$$\begin{aligned} \mathrm{Hom}\left(H_1(\widetilde{\Gamma}, \mathbb{Z}), \mathbb{R}\right) &\longrightarrow \mathrm{Hom}\left(K, \mathbb{R}\right) \\ (\widetilde{\alpha}_i^\pm)^* &\longmapsto \pm(\widetilde{\alpha}^i)^* \\ (\widetilde{\beta}_j)^* &\longmapsto (\widetilde{\beta}^j)^* \\ (\widetilde{\gamma}_k)^* &\longmapsto 0 \end{aligned} \tag{20}$$

where Hom-dual bases are denoted with $(\cdot)^*$.

From [GST22, Theorem 8.3] and [GS19b, Proposition 8.3] we know that $\widetilde{W}_d$ is the support of $(\phi_q^d)_*[\widetilde{\Gamma}^d]$, is purely d-dimensional, and, crucially, has a fundamental cycle. At this point it is clear that $i^\vee(\widetilde{W}_d) = \widetilde{Y}_d$ is of dimension at most d. By Conjecture 4.28 we know that in fact the support of $(\psi_q^d)_*[\widetilde{\Gamma}^d]$ (or equivalently $i_*^\vee[\widetilde{W}_d]$) is equal to $\widetilde{Y}_d$. Note that these pushforwards are well-defined because ψ_q^d and $i^\vee$ are proper.

Proposition 4.29. *Let* $1 \leqslant d \leqslant h$. *The locus* $\widetilde{Y}_d$ *has a fundamental class and*

$$(\psi_q^d)_*[\widetilde{\Gamma}^d] = d!2[\widetilde{Y}_d] \in Z_d\big(\mathrm{Prym}(\widetilde{\Gamma}/\Gamma)\big).$$

Proof. We know by [GS19b, Proposition 8.3] that $\widetilde{W}_d$ has a fundamental cycle and that $(\phi_q^d)_*[\widetilde{\Gamma}^d] = d![\widetilde{W}_d]$ in $Z_d(\mathrm{Jac}(\widetilde{\Gamma}))$. It suffices to show that the tropical cycle $i_*^\vee[\widetilde{W}_d]$ has weight 2 on every component of $\widetilde{Y}_d^{\mathrm{reg}}$ because in this case $\frac{1}{2}i_*^\vee[\widetilde{W}_d]$ is a tropical cycle with support $\widetilde{Y}_d$ and weight 1 everywhere.

Recall the definition of proper push-forward in tropical homology: for a point $x \in \mathrm{Prym}(\widetilde{\Gamma}/\Gamma)^{\mathrm{reg}}$, each preimage in $\mathrm{Jac}(\widetilde{\Gamma})$ contributes with multiplicity given by the index of integral tangent spaces

$$\left[T_0^{\mathbb{Z}}(\mathrm{Prym}(\widetilde{\Gamma}/\Gamma)) \colon i_*^\vee T_0^{\mathbb{Z}}(\mathrm{Jac}(\widetilde{\Gamma}))\right]. \tag{21}$$

The integral tangent space of an integral torus $\Sigma = (\Lambda, \Lambda', [\cdot,\cdot])$ is $\mathrm{Hom}(\Lambda, \mathbb{Z})$ and thus we see that the induced map $d(i^\vee)$ on the tangent spaces is the map given in (20) restricted to integer valued functionals. In particular, the index in Equation (21) is 1 and $i_*^\vee[\widetilde{W}_d] = 2[\widetilde{Y}_d]$ because there are two preimages. □

We now compute the class of $\mathrm{cyc}[\widetilde{Y}_1]$ in $H_{1,1}\big(\mathrm{Prym}(\widetilde{\Gamma}/\Gamma)\big) = K' \otimes K^*$.

Proposition 4.30. *In terms of the bases of* K *and* K' *defined in Proposition 4.21, the class of* $\widetilde{Y}_1$ *is*

$$\mathrm{cyc}[\widetilde{Y}_1] = 2\left(\sum_{i=1}^{A} \tilde{\alpha}_i \otimes (\tilde{\alpha}^i)^* + \sum_{j=1}^{B} \tilde{\beta}_j \otimes (\tilde{\beta}^j)^*\right) \in H_{1,1}\big(\mathrm{Prym}(\widetilde{\Gamma}/\Gamma)\big).$$

Proof. The tropical cycle class map commutes with proper push-forward [GS19a, Proposition 5.6], i.e. the diagram

$$\begin{array}{ccc} Z_1\big(\mathrm{Jac}(\widetilde{\Gamma})\big) & \xrightarrow{i_*^\vee} & Z_1\big(\mathrm{Prym}(\widetilde{\Gamma}/\Gamma)\big) \\ \downarrow{\scriptstyle \mathrm{cyc}} & & \downarrow{\scriptstyle \mathrm{cyc}} \\ H_{1,1}\big(\mathrm{Jac}(\widetilde{\Gamma})\big) & \xrightarrow{i_*^\vee} & H_{1,1}\big(\mathrm{Prym}(\widetilde{\Gamma}/\Gamma)\big) \end{array}$$

commutes. From [GS19b, Proposition 9.2] we know that

$$\mathrm{cyc}[\widetilde{W}_1] = \sum_{i=1}^{A} \big(\tilde{\alpha}_i^+ \otimes (\tilde{\alpha}_i^+)^* + \tilde{\alpha}_i^- \otimes (\tilde{\alpha}_i^-)^*\big) + \sum_{j=1}^{B} \tilde{\beta}_j \otimes (\tilde{\beta}_j)^* + \sum_{k=1}^{C} \tilde{\gamma}_k \otimes (\tilde{\gamma}_k)^*.$$

The push forward in homology is given by the maps described in (19) and (20). Hence we obtain

$$\begin{aligned} i_*^\vee : \tilde{\alpha}_i^+ \otimes (\tilde{\alpha}_i^+)^* + \tilde{\alpha}_i^- \otimes (\tilde{\alpha}_i^-)^* &\longmapsto 2\tilde{\alpha}_i \otimes (\tilde{\alpha}^i)^* \\ \tilde{\beta}_j \otimes (\tilde{\beta}_j)^* &\longmapsto 2\tilde{\beta}_j \otimes (\tilde{\beta}^j)^* \\ \tilde{\gamma}_k \otimes (\tilde{\gamma}_k)^* &\longmapsto 0 \end{aligned}$$

and the claim follows. □

Proof of Theorem 4.27. We use the simplified notation

$$\{\epsilon_i \mid 1 \leqslant i \leqslant g_0\} = \big\{\tilde{\alpha}_i, \tilde{\beta}_j \;\big|\; i = 1, \dots, A, j = 1, \dots, B\big\}$$

for the basis of K'. With this notation, Proposition 4.30 can be rephrased as

$$\mathrm{cyc}[\widetilde{Y}_1] = 2\Big(\sum_{i=1}^{g_0} \epsilon_i \otimes \zeta(\epsilon_i)^*\Big).$$

It is now purely formal to repeat the proof of [GS19b, Lemma 9.4] to obtain

$$\mathrm{cyc}[\widetilde{Y}_d] = 2^d \sum_{\substack{I \subseteq \{1,\dots,g_0\} \\ |I| = d}} \bigwedge_{k \in I} \epsilon_i \otimes \bigwedge_{k \in I} \zeta(\epsilon_i)^*. \tag{22}$$

Similarly, the proof of [GS19b, Lemma 9.7] can be repeated verbatim for the divisor Ξ defining the principal polarization of $\mathrm{Prym}(\widetilde{\Gamma}/\Gamma)$ to obtain

$$\mathrm{cyc}\left([\Xi]^{g_0-d}\right) = (g_0 - d)! \sum_{\substack{I \subseteq \{1,\dots,g_0\} \\ |I| = d}} \bigwedge_{k \in I} \epsilon_i \otimes \bigwedge_{k \in I} \zeta(\epsilon_i)^*. \tag{23}$$

The only noteworthy detail is that the bases $\{\epsilon_i\}_{i=1,\dots,g_0}$ and $\{\zeta(\epsilon_i)^*\}_{i=1,\dots,g_0}$ are dual to each other with respect to the inner product on $\mathrm{Hom}\left(\mathrm{Coker}(\pi^*)^{tf}, \mathbb{R}\right)$ induced by the principal polarization of $\mathrm{Prym}(\widetilde{\Gamma}/\Gamma)$.

Now comparing expressions (22) and (23) proves the claim. □

5. Compatibility of the n-gonal construction and tropical abelian varieties

We are now ready to prove the main theorems stated in the introduction. We restate the theorems for convenience, and begin with the trigonal construction.

Theorem 5.1 (Theorem 1.2)**.** *Let K be a metric tree. The tropical trigonal and Recillas constructions establish a one-to-one correspondence*

$$\left\{\begin{array}{c}\textit{Tropical curves } \Pi \textit{ with a}\\ \textit{harmonic map of degree 4 to } K\\ \textit{with dilation profiles nowhere}\\ (4) \textit{ or } (2,2).\end{array}\right\} \underset{\textit{trigonal construction}}{\overset{\textit{Recillas construction}}{\rightleftarrows}} \left\{\begin{array}{c}\textit{Free double covers } \widetilde{\Gamma} \to \Gamma \textit{ with}\\ \textit{a harmonic map of degree 3}\\ \textit{from } \Gamma \textit{ to } K.\end{array}\right\}$$

and under this correspondence, the Prym variety of a double cover $\mathrm{Prym}(\widetilde{\Gamma}/\Gamma)$ *and the Jacobian* $\mathrm{Jac}(\Pi)$ *of the tetragonal curve are isomorphic as principally polarized tropical abelian varieties.*

The techniques of tropical homology allows us to closely model the proof of the algebraic version of the theorem (see Theorem 12.7.2 in [BL04]).

Proof. Here we present the outline of the proof, and postpone the necessary calculations and checks to a series of lemmas that are given later in this chapter. Recall that we have already established the bijection in Proposition 2.15, and it only remains to show that $\mathrm{Prym}(\widetilde{\Gamma}/\Gamma) \simeq \mathrm{Jac}(\Pi)$.

We recall the setup and notation. Let $k : \Pi \to K$ be a generic tetragonal curve, so that K does not have any points over which the degree profile of k is $(2,2)$ or (4). By the tropical Recillas construction (Definition 2.13) we obtain a tower $\widetilde{\Gamma} \xrightarrow{\pi} \Gamma \xrightarrow{f} K$, where $f : \Gamma \to K$ is a trigonal curve and $\pi : \widetilde{\Gamma} \to \Gamma$ is a free double cover. We denote $\iota : \widetilde{\Gamma} \to \widetilde{\Gamma}$ the associated involution. We choose graph models for our tropical curves, and by abuse of notation refer to edges and vertices of $\widetilde{\Gamma}$, Γ, Π, and K.

First, we **define a map** of rational polyhedral spaces $\chi : \widetilde{\Gamma} \to \mathrm{Jac}(\Pi)$ such that $\chi\iota = -\chi$. To this end, we choose an Abel–Jacobi map $\Pi^2 \to \mathrm{Jac}(\Pi)$ suited to our purposes. Fix a point $x \in K$ and let

$$D = \sum_{y \in k^{-1}(x)} d_k(y) \cdot y \in \mathrm{Div}_4^+(\Pi)$$

be the fiber of k above x. The group $\mathrm{Pic}_0(\Pi) \simeq \mathrm{Jac}(\Pi)$ is divisible because it is a real torus, hence we can find $M \in \mathrm{Div}_1(\Pi)$ such that $4M \sim D$ in $\mathrm{Pic}_4(\Pi)$. Let $\phi_M : \Pi \to \mathrm{Jac}(\Pi)$ be the Abel–Jacobi map associated to M (note that M may fail to be effective, in which case ϕ_M is actually a translation of an Abel–Jacobi map by a fixed divisor class) and denote

$$\phi_L = \phi_M + \phi_M : \Pi^2 \to \mathrm{Jac}(\Pi).$$

The map ϕ_L is symmetric, hence it descents to a map $\mathrm{Div}_2^+(\Pi) \to \mathrm{Jac}(\Pi)$. Define χ to be the composition of the inclusion $\widetilde{\Gamma} \subseteq \mathrm{Div}_2^+(\Pi)$ from Equation (10) with this descent of ϕ_L. Since we are not using the polyhedral structure on $\mathrm{Div}_2^+(\Pi)$, we need to verify by hand that χ is a map

of rational polyhedral spaces, and we do this in Lemma 5.3 (where we also check that χ has the desired property $\chi\iota = -\chi$).

We may now apply the universal property of the Prym variety from Proposition 4.24 to obtain a commutative square

$$\begin{tikzcd} \widetilde{\Gamma} \arrow[r, "\chi"] \arrow[d, "\psi_q"] & \mathrm{Jac}(\Pi) \arrow[d, "t_{-\chi(q)}"] \\ \mathrm{Prym}(\widetilde{\Gamma}/\Gamma) \arrow[r, "\mu"] & \mathrm{Jac}(\Pi). \end{tikzcd}$$

Our goal is to show that ϕ' is an isomorphism of principally polarized tropical abelian varieties. We saw already in Proposition 2.15 that $\dim \mathrm{Jac}(\Pi) = g(\Pi) = g(\widetilde{\Gamma}) - 1 = \dim \mathrm{Prym}(\widetilde{\Gamma}/\Gamma)$. By Theorem 4.27 we know that the pushforward of the fundamental class of $\widetilde{\Gamma}$ along ψ_q is equal to

$$(\psi_q)_*[\widetilde{\Gamma}] = \frac{2}{(g_0-1)!}[\Xi]^{g_0-1} \in H_{1,1}\big(\mathrm{Prym}(\widetilde{\Gamma}/\Gamma)\big),$$

where $g_0 = \dim \mathrm{Prym}(\widetilde{\Gamma}/\Gamma)$, Ξ is the principal polarization, and we identify 1-cycles with their homology classes. If we can show that

$$\chi_*[\widetilde{\Gamma}] = \frac{2}{(g_0-1)!}[\Theta]^{g_0-1} \in H_{1,1}\big(\mathrm{Jac}(\Pi)\big), \tag{24}$$

where Θ is principal polarization of $\mathrm{Jac}(\Pi)$, then $\mu_*[\Xi] = [\Theta]$ and $\mathrm{Prym}(\widetilde{\Gamma}/\Gamma)$ is isomorphic to $\mathrm{Jac}(\Pi)$ by the homological criterion of Proposition 4.13.

We now **define a tropical 1-cycle** $A \in Z_1(\Pi^2)$ as follows. Recall that we can view $\widetilde{\Gamma}$ as a subset of $\mathrm{Sym}^2(\Pi)$ (see Equation 10), however, this does not induce the correct edge lengths on $\widetilde{\Gamma}$. Instead, we manually construct a cycle A that represents the lift of $\widetilde{\Gamma}$ to Π^2 via the natural projection map $\Pi^2 \to \mathrm{Sym}^2(\Pi)$.

A tropical 1-cycle on Π^2 is a map $\Pi^2 \to \mathbb{Z}$, and for $x, y \in \Pi$ we set

$$A(x,y) = \begin{cases} 0 & \text{if } k(x) \neq k(y), \\ 1 & \text{if } k(x) = k(y) \text{ and } x \neq y, \\ d-1 & \text{if } x = y, \end{cases}$$

where $d = d_k(x) = d_k(y)$ is the dilation factor of k at $x = y$. The support $|A|$ is a purely 1-dimensional rational polyhedral space, contained in the preimage of the diagonal Δ_K. The proof that A is indeed a tropical 1-cycle is given in Lemma 5.4 below.

The key calculation is the following formula

$$[\Delta_\Pi] + A = 4[\Pi \times p'] + 4[p' \times \Pi] \in H_{1,1}(\Pi^2), \tag{25}$$

where Δ_Π is the diagonal, $p' \in \Pi$ is an arbitrary point, $[\cdot]$ denotes the fundamental class, and we identify cycles in $Z_1(\Pi^2)$ with their classes in $H_{1,1}(\Pi^2)$ via the cycle class map.

We prove this formula in several steps. First, we note that the spaces K^2 and Π^2 have fundamental cycles, so by Poincaré duality we can identify $H^{1,1}(K^2) \simeq H_{1,1}(K^2)$ and $H^{1,1}(\Pi^2) \simeq H_{1,1}(\Pi^2)$. Under this identification, pullback on cohomology induces a map $(k \times k)^* : H_{1,1}(K^2) \to H_{1,1}(\Pi^2)$, and we show in Lemma 5.5 that

$$[\Delta_\Pi] + A = (k \times k)^*[\Delta_K] \in H_{1,1}(\Pi^2). \tag{26}$$

We then use Lemma 5.6 to rewrite the right hand side of Equation (26) as

$$(k \times k)^*[\Delta_K] = (k \times k)^*\big([K \times p] + [p \times K]\big) \in H_{1,1}(\Pi^2), \tag{27}$$

where $p \in K$ is an arbitrary point. Finally, it is easy to show that $k^*[K] = [\Pi]$, which implies that the right hand sides of Equations (25) and (27) are equal.

To complete the proof, we compute the pushforward of Equation (25) to $H_{1,1}(\mathrm{Jac}(\Pi))$ along the map $\phi_L : \Pi^2 \to \mathrm{Jac}(\Pi)$.

(1) **Claim:** $(\phi_L)_*[\Delta_\Pi] = 4(\phi_M)_*[\Pi]$. Indeed, by definition $\phi_L(x, y) = \phi_M(x) + \phi_M(y)$, so the restriction of ϕ_L to the diagonal Δ_Π is ϕ_M applied to the first coordinate followed by multiplication by 2. But multiplication by 2 on an integral torus induces multiplication by 4 on $H_{1,1}$, so the claim follows.

(2) **Claim:** $(\phi_L)_*A = 2\chi_*[\widetilde{\Gamma}]$. Consider the involution $\Pi^2 \to \Pi^2$ given by $(x, y) \mapsto (y, x)$. Obviously, it restricts to an involution ι of $|A|$. It suffices to show that $|A|/\iota \cong \widetilde{\Gamma}$ and that the quotient map is a harmonic double cover $\tau : |A| \to \widetilde{\Gamma}$. Once we have that, then by definition $\phi_L = \chi \circ \tau$ and therefore

$$(\phi_L)_*A = \chi_*(\tau_*A) = \chi_*(2[\widetilde{\Gamma}]) = 2\chi_*[\widetilde{\Gamma}].$$

(3) **Claim:** $(\phi_L)_*[\Pi \times p] = (\phi_L)_*[p \times \Pi] = (\phi_M)_*[\Pi]$. As above, $\phi_L(x, p) = \phi_M(x) + \phi_M(p)$ for all $x \in \Pi$. But this means that the restriction of ϕ_L to $\Pi \times p$ is the composition of ϕ_M with a translation. Since translations induce the identity on homology, the claim follows.

Summing up we have just shown $4(\phi_M)_*[\Pi] + 2\chi_*[\widetilde{\Gamma}] = 8(\phi_M)_*[\Pi]$. Solving for $\chi_*[\widetilde{\Gamma}]$ and plugging in the Poincaré formula [GS19b, Theorem A]

$$(\phi_M)_*[\Pi] = \frac{1}{(g_0 - 1)!}[\Theta]^{g_0 - 1}$$

we obtain Equation 24, and the proof of Theorem 1.2 is complete. □

Example 5.2. We consider the $(3, 2)$-graph tower and the tetragonal graph shown on Figure 6. We promote these to a tower $\widetilde{\Gamma} \to \Gamma \to K$ of tropical curves and a generic tetragonal curve $\Pi \to K$ by assigning edge lengths to K, which we denote, going left to right, by a, b, c, d, and e.

Let us compute the Prym and Jacobian varieties in this example explicitly to see that they are the same. We begin by computing $\mathrm{Prym}(\widetilde{\Gamma}/\Gamma)$. It is possible to construct a basis for $\mathrm{Ker}\,\pi_*$ using Proposition 4.19, but in this case it is easier to choose a basis by hand. It is clear that $\mathrm{Ker}\,\pi_*$ is spanned by the elements $\widetilde{\eta}_1^+ - \widetilde{\eta}_1^-$ and $\widetilde{\eta}_2^+ - \widetilde{\eta}_2^-$, where $\widetilde{\eta}_i^\pm$ are the elements of $H_1(\widetilde{\Gamma}, \mathbb{Z})$ shown on Figure 9. Furthermore, the module $(\mathrm{Coker}\,\pi^*)^{\mathrm{tf}}$ (viewed as a quotient of $H_1(\widetilde{\Gamma}, \mathbb{Z})$, which has been canonically identified with $\Omega^1_{\widetilde{\Gamma}}$) is spanned by the classes $[\widetilde{\eta}_1^+]$ and $[\widetilde{\eta}_2^+]$. Keeping in mind that the dilation factors of $\Gamma \to K$ also affect the lengths of edges in $\widetilde{\Gamma}$, we obtain that the intersection matrix is

$[\cdot, \cdot]^{\mathrm{pp}}$	$[\widetilde{\eta}_1^+]$	$[\widetilde{\eta}_2^+]$
$\widetilde{\eta}_1^+ - \widetilde{\eta}_1^-$	$2(b + c + d)$	$b + c + d$
$\widetilde{\eta}_2^+ - \widetilde{\eta}_2^-$	$b + c + d$	$\frac{3}{2}a + 2b + \frac{3}{2}c + 2d + \frac{3}{2}e$.

To compute the intersection matrix for $\mathrm{Jac}(\Pi)$, we choose the basis ϵ, δ for $H_1(\Pi, \mathbb{Z})$ depicted in Figure 10. The edge length pairing $[\cdot, \cdot] : H_1(\Pi, \mathbb{Z}) \times H_1(\Pi, \mathbb{Z}) \to \mathbb{R}$ (where we have identified $\Omega^1_\Pi = H_1(\Pi, \mathbb{Z})$ via the principal polarization) evaluated on the basis yields

$[\cdot,\cdot]$	ϵ	δ
ϵ	$2(b+c+d)$	$b+c+d$
δ	$b+c+d$	$\frac{3}{2}a+2b+\frac{3}{2}c+2d+\frac{3}{2}e$,

which is evidently the same table as before. Hence $\mathrm{Prym}(\widetilde{\Gamma}/\Gamma) \cong \mathrm{Jac}(\Pi)$ as principally polarized tropical abelian varieties.

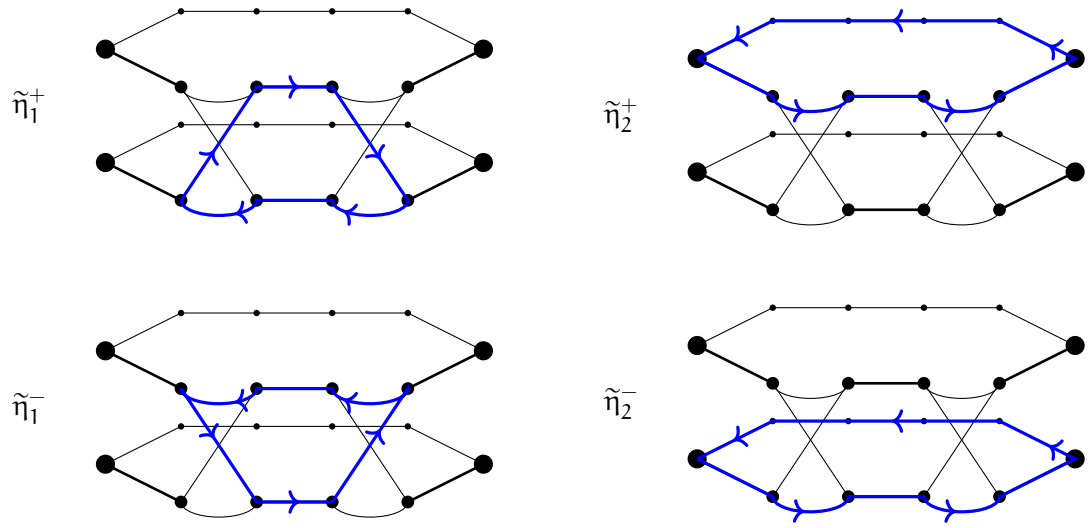

FIGURE 9. The four generators of $H_1(\widetilde{\Gamma},\mathbb{Z})$ of Type I that we use for our computation.

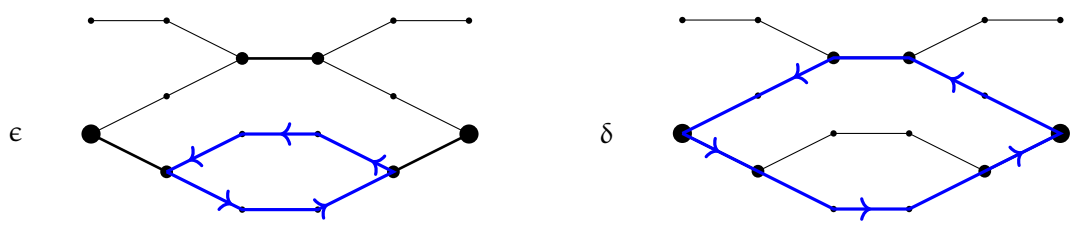

FIGURE 10. Basis for $H_1(\Pi,\mathbb{Z})$.

We now present a series of lemmas which fill in the missing details in the proof of Theorem 1.2 above.

Lemma 5.3. *The map $\chi : \widetilde{\Gamma} \to \mathrm{Jac}(\Pi)$ defined in the proof of Theorem 5.1 is a map of rational polyhedral spaces. Furthermore, it satisfies $\chi\iota = -\chi$.*

Proof. We start by verifying that χ is a map of rational polyhedral spaces. We have to check that the pullback of a 1-form $\omega \in \Omega^1_{\mathrm{Jac}(\Pi)} \cong \Omega^1_{\Pi}$ along χ is a 1-form on $\widetilde{\Gamma}$. For this we need check that $\chi^*\omega$ has integer slopes on every edge of $\widetilde{\Gamma}$, and that the sum of slopes around every vertex of $\widetilde{\Gamma}$ is 0.

Recall that an edge $\widetilde{e} \in E(\widetilde{\Gamma})$ corresponds to a pair of edges of $f_1, f_2 \in E(\Pi)$ (which may be the same), with all three mapping to the same edge $e \in E(K)$ (see table in Definition 2.13). The slope

of $\chi^*\omega$ on $\widetilde{e}$ is equal to

$$\operatorname{slope}_{\widetilde{e}}(\chi^*\omega) = \frac{1}{\ell(\widetilde{e})}\Big(s_1\ell(f_1) + s_2\ell(f_2)\Big),$$

where $s_1 = \operatorname{slope}_{f_1}(\omega)$ and $s_2 = \operatorname{slope}_{f_2}(\omega)$. If we denote the composed map $\widetilde{f} = f \circ \pi$, then by harmonicity, we have

$$\frac{\ell(f_i)}{\ell(\widetilde{e})} = \frac{\ell(e)/d_k(f_i)}{\ell(e)/d_{\widetilde{f}}(\widetilde{e})} = \frac{d_{\widetilde{f}}(\widetilde{e})}{d_k(f_i)}$$

for $i = 1, 2$. We can now verify case-by-case that $\operatorname{slope}_{\widetilde{e}}(\chi^*\omega)$ is an integer

$d_{\widetilde{f}}(\widetilde{e})$	$d_k(f_1)$	$d_k(f_2)$	$\operatorname{slope}_{\widetilde{e}}(\chi^*\omega)$
1	1	1	$s_1 + s_2$
1	2	2	$s_1 = s_2$
2	1	2	$2s_1 + s_2$
3	1	3	$3s_1 + s_2$
3	3	3	$2s_1 = 2s_2$

where in the second and fifth row the edges $f_1 = f_2$ are the same and hence $s_1 = s_2$.

Now let $v \in V(\widetilde{\Gamma})$ be a vertex of $\widetilde{\Gamma}$, corresponding to two (not necessarily distinct) vertices $w_1, w_2 \in V(\Pi)$. We want to show that the sum of outgoing slopes of $\chi^*\omega$ over the half-edges rooted at v is 0, i.e. that

$$\sum_{\widetilde{h} \in T_v\widetilde{\Gamma}} \operatorname{slope}_{\widetilde{h}}(\chi^*\omega) = \sum_{h \in T_{\widetilde{f}(v)}K} \left(\sum_{\widetilde{h} \in \widetilde{f}^{-1}(h) \cap T_v\widetilde{\Gamma}} \operatorname{slope}_{\widetilde{h}}(\chi^*\omega) \right) = 0,$$

where we have split the sum according to the image half-edge $h = \widetilde{f}(\widetilde{h})$. We do this by showing that for every vertex v there are integers $a, b \in \mathbb{Z}$ such that for every $g \in T_{\widetilde{f}(v)}K$ we have

$$\sum_{\widetilde{h} \in \widetilde{f}^{-1}(h) \cap T_v\widetilde{\Gamma}} \operatorname{slope}_{\widetilde{h}}(\chi^*\omega) = a\left(\sum_{\widetilde{h}_1 \in k^{-1}(h) \cap T_{w_1}\Pi} \operatorname{slope}_{\widetilde{h}_1}(\omega) \right) + b\left(\sum_{\widetilde{h}_2 \in k^{-1}(h) \cap T_{w_2}\Pi} \operatorname{slope}_{\widetilde{h}_2}(\omega) \right). \quad (28)$$

Summing Equation (28) over all $h \in T_{\widetilde{f}(v)}K$, the right hand side is 0 by harmonicity of ω. The numbers a and b are determined by case distinction and direct computation in Figure 11.

Now let us check that $\chi\iota = -\chi$. Let $p' \in \widetilde{\Gamma}$ correspond to $p_1 + p_2$ with $k(p_1) = k(p_2) = p$. Then $\iota(p')$ corresponds to $p_3 + p_4$, where $p_1 + p_2 + p_3 + p_4$ is the fiber of k over $p \in K$. Therefore,

$$\chi(\iota(p')) + \chi(p') = p_1 + p_2 + p_3 + p_4 - 2L \sim k^{-1}(p) - D$$

by construction of L. The right hand side is now linearly equivalent to 0 because D is a fiber of k and all fibers of k are linearly equivalent (here we use that K is a tree). □

Lemma 5.4. *Keeping notation from the proof of Theorem 1.2 above, A is a tropical 1-cycle.*

Proof. We first give an explicit description of the graph $|A|$. Pick an edge $e \in E(K)$. By assumption, Π does not have dilation profiles (4) or (2, 2), hence $k^{-1}(e) = \{e_1, \ldots, e_m\}$ with $m \in \{2, 3, 4\}$, and we can further assume that e_1 is the only possibly dilated edge, i.e. that $d_k(e_1) = 5 - m$ and $d_k(e_i) = 1$ for $i \geqslant 2$. This fiber gives rise to edges of $|A|$ corresponding to

$$(e_1, e_2), \ldots, (e_1, e_m), (e_2, e_1), (e_2, e_3), \ldots, (e_2, e_m), \ldots, (e_m, e_{m-1})$$

$d_{\tilde{f}}(v)$	w_1 and w_2	Local picture of $\tilde{\Gamma}$ at v over h together with outgoing slopes of $\chi^*\omega$.	Local picture of Π at w_1 and w_2 over h, and outgoing slopes of ω	(a, b)
1	$d_k(w_1) = d_k(w_2) = 1$ and $w_1 \neq w_2$	$s_1 + s_2$	s_1 s_2	$a = b = 1$
	$d_k(w_1) = 2$ and $w_1 = w_2$	$s_1 + s_2$ s	s_1 s_2 s	$a = 1$
2	$d_k(w_1) = 1$, $d_k(w_2) = 2$	$s_1 + s_2$ $s_1 + s_3$ $2s_1 + s_2$	s_1 s_2 s_3 s_1 s_2	$a = 2, b = 1$
3	$d_k(w_1) = 1$, $d_k(w_2) = 3$	$3s_1 + s_2$ $2s_1 + s_2$ $s_1 + s_3$ $s_1 + s_2$ $s_1 + s_3$ $s_1 + s_4$	s_1 s_2 s_1 s_2 s_3 s_1 s_2 s_3 s_4	$a = 3, b = 1$
	$d_k(w_1) = 3$ and $w_1 = w_2$	$2s$ $s_1 + 2s_2$ s_1 $s_1 + s_2$ $s_1 + s_3$ $s_2 + s_3$	s s_1 s_2 s_1 s_2 s_3	$a = 2$

FIGURE 11. Case by case proof that at every vertex v the sum of outgoing slopes of $\chi^*\omega$ on edges in $\tilde{\Gamma}$ over $h \in E(K)$ is an integer linear combination of the corresponding sums at the vertices w_1 and w_2.

and additionally (e_1, e_1) if $d_k(e_1) > 1$ (i.e. if $m \leqslant 3$). The vertices of $|A|$ corresponding to $v \in V(K)$ and $k^{-1}(v) = \{v_1, \ldots, v_l\}$ admit a similar description as

$$(v_1, v_2), \ldots, (v_1, v_l), (v_2, v_1), (v_2, v_3), \ldots, (v_2, v_l), \ldots, (v_l, v_{l-1})$$

with the possible addition of (v_1, v_1) if $l \leqslant 3$. It is clear that there is a subdivision of the natural polyhedral structure of Π^2 which induces this graph structure on $|A|$, hence A is constructible.

We need to verify the balancing condition (11) at each vertex of $|A|$. The support of A consists of diagonals (e, e') of cells $e \times e' \subseteq \Pi^2$ such that e and e' map to the same edge of K. We first

observe the following. Let $(e, e') \subseteq e \times e' \subseteq \Pi^2$ be an edge of $|A|$. Let η, η' be generators of $\Omega^1_\Pi|_e$ and $\Omega^1_\Pi|_{e'}$, respectively. By abuse of notation we denote the induced generators of $\Omega^1_{\Pi^2}|_{e\times e'}$ by η and η' as well. Then

$$\frac{1}{\gcd\left(d_k(e), d_k(e')\right)}\left(d_k(e')\eta + d_k(e)\eta'\right)$$

generates $\Omega^1_{\Pi^2}|_{(e,e')}$. If $e \neq e'$ then we can assume without loss of generality that $d_k(e') = 1$, hence the generator is $\eta + d_k(e)\eta'$. On the other hand, if $e = e'$ then the generator is $\eta + \eta' = 2\eta$, where we can now identify η and η' on the diagonal.

We now check balancing at a vertex (u, v) of $|A|$, where $u, v \in V(\Pi)$ are vertices mapping to the same vertex $w = k(u) = k(v)$ of K. To avoid overly cumbersome notation, we separately consider the cases $u \neq v$ and $u = v$.

If $u \neq v$, then we can further assume that $d_k(v) = 1$. For each $e_i \in E(K)$ incident to w let $e_{i,1}, \ldots, e_{i,m_i}$ be the edges in Π over e_i which are incident to u. Since $d_k(v) = 1$, each edge e_i has a unique preimage f_i in Π incident to v, and $d_k(f_i) = 1$. The edges of $|A|$ which are incident to (u, v) are $(e_{i,j}, f_i)$ for $i = 1, \ldots, n$ and $j = 1, \ldots, m_i$.

Now orient all edges e_i away from w and denote the by $\eta_{i,j}$ and ρ_i the primitive generators compatible with the outwards orientation of $\Omega^1_\Pi|_{e_{i,j}}$ and $\Omega^1_\Pi|_{f_i}$, respectively. Since $d_k(f_i) = 1$, the generator of $\Omega^1_{\Pi^2}|_{(e_{i,j},f_i)}$ is simply $\eta_{i,j} + d_k(e_{i,j})\rho_i$. Then sum of outgoing primitive tangent vectors in $|A|$ at (u, v) is then

$$\begin{aligned}\sum_i\sum_j\left(\eta_{i,j} + d_k(e_{i,j})\rho_i\right) &= \sum_i\Big(\sum_j \eta_{i,j}\Big) + \sum_i\Big(\underbrace{\sum_j d_k(e_{i,j})}_{=d_k(u)}\Big)\rho_i \\ &= \underbrace{\sum_i\sum_j \eta_{i,j}}_{=0} + d_k(u)\underbrace{\sum_i \rho_i}_{=0} = 0,\end{aligned}$$

where the last two sums are zero because Π is smooth and hence its fundamental cycle is balanced at respectively u and v. Therefore A is balanced around (u, v).

Now assume that $u = v$, and denote by $e_{i,1}, \ldots, e_{i,m_i}$ the edges of Π which are incident to u and lie over $e_i \in E(K)$. Without loss of generality we assume that $d_k(e_{i,j}) = 1$ for all $j \geqslant 2$. The set of edges in $|A|$ which are incident to (u, u) are

$$\left\{(e_{i,j}, e_{i,l}) \;\middle|\; j \neq l \text{ or } j = l = 1 \text{ and } d_k(e_{i,1}) \geqslant 2\right\},$$

and the values of A are

$$A(e_{i,j}, e_{i,l}) = \begin{cases} 1, & j \neq l, \\ d_k(e_{i,1}) - 1, & j = l = 1. \end{cases}$$

Let $\eta_{i,j}$ be a generator for $\Omega^1_\Pi|_{e_{i,j}}$ oriented away from u. As discussed above, a generator of $\Omega^1_\Pi|_{(e_{i,j},e_{i,l})}$ is $d_k(e_{i,l})\eta_{i,j} + d_k(e_{i,j})\eta_{i,l}$ if $j \neq l$ and $2\eta_{i,1}$ if $j = l = 1$. The balancing condition at

(u,u) now requires us to check that

$$\sum_i \Big(\sum_{j\neq l} \underbrace{A(e_{i,j},e_{i,l})}_{=1} \big(d_k(e_{i,l})\eta_{i,j} + d_k(e_{i,j})\eta_{i,l}\big) + A(e_{i,1},e_{i,1})2\eta_{i,1}\Big)$$
$$= 2\sum_i \Big(\sum_{j\neq l} d_k(e_{i,l})\eta_{i,j} + (d_k(e_{i,1})-1)\eta_{i,1}\Big).$$

is equal to 0. We compute

$$\sum_i\sum_{j\neq l} d_k(e_{i,l})\eta_{i,j} = \sum_i\sum_j \Big(\eta_{i,j} \underbrace{\sum_{l\neq j} d_k(e_{i,l})}_{=d_k(u)-d_k(e_{i,j})}\Big) = d_k(v)\underbrace{\sum_i\sum_j \eta_{i,j}}_{=0} - \sum_i\sum_j d_k(e_{i,j})\eta_{i,j}, \tag{29}$$

where the sum of the $\eta_{i,j}$ vanishes because Π is balanced at u. Since $d_k(e_{i,j}) = 1$ unless $j = 1$, we can write the right hand side of Equation (29) as

$$-\sum_i\sum_j d_k(e_{i,j})\eta_{i,j} = -\underbrace{\sum_i\sum_j \eta_{i,j}}_{=0} - \sum_i (d_k(e_{i,1})-1)\eta_{i,1}.$$

Putting these together, we see that A satisfies the balancing condition at (u,u), and hence $A \in Z_1(\Pi^2)$ is a tropical 1-cycle. □

Lemma 5.5. *With the notation as in the proof of Theorem 1.2 above, we have*

$$[\Delta_\Pi] + A = (k\times k)^*[\Delta_K] \in H_{1,1}(\Pi^2).$$

Proof. Above we explained that the pullback $(k\times k)^*[\Delta_k]$ is really a pullback of cohomology cycles. However, in order to compute $(k\times k)^*[\Delta_k]$ explicitly, we give a different interpretation. Because $[\Delta_K]$ is a cycle in codimension 1 and $K\times K$ has a fundamental class, we can identify $[\Delta_K]$ with a Cartier divisor $D \in \mathrm{Div}(K^2)$, which admits a pullback to $\mathrm{Div}(\Pi^2)$. This is a tropical 1-cycle in $Z_1(\Pi^2)$ after intersection with $[\Pi^2]$. Let us quickly verify, that these are in fact the same thing.

Starting with $D \in \mathrm{Div}(K^2)$, we have

$$\mathrm{cyc}[\Delta_K] = \mathrm{cyc}\big(D\cdot[K\times K]\big) = c_1(\mathcal{L}(D)) \cap \mathrm{cyc}[K\times K],$$

in other words, the Poincaré dual of $\mathrm{cyc}[\Delta_K]$ is $c_1(\mathcal{L}(D))$. Similarly, we treat the pullback $(k\times k)^*D$

$$\mathrm{cyc}\Big(\big((k\times k)^*D\big)\cdot[\Pi\times\Pi]\Big) = c_1\Big(\mathcal{L}\big((k\times k)^*D\big)\Big) \cap \mathrm{cyc}[\Pi\times\Pi].$$

But now pullback of Cartier divisors commutes with taking the line bundle [GS19a, Proposition 3.15] and that in turn commutes with the Chern class map by naturality. Hence the two interpretations of the pullback coincide.

Let us now prove the formula that we claimed. Let e be an edge of K and consider the cell $e\times e \subseteq K\times K$. The diagonal of this cell is defined as a Cartier divisor by the piecewise affine (i.e. rational) function

$$f : e\times e \longrightarrow \mathbb{R},$$
$$(x,y) \longmapsto \min\{y-x, 0\}.$$

Let $k^{-1}(e) = \{e_1, \dots, e_m\}$ be the fiber over e with $m \in \{2,3,4\}$. Without loss of generality we assume that the dilation factors along $e_2, \dots, e_m$ are 1. Consider the cell $e_i \times e_j \subseteq \Pi \times \Pi$ and the pull-back of f to restricted to this cell

$$g = \big((k \times k) \circ f\big)\big|_{e_i \times e_j} : e_i \times e_j \longrightarrow \mathbb{R}$$
$$(x,y) \longmapsto \min\left\{ \frac{1}{\gcd\big(d_k(e_i), d_k(e_j)\big)} \big(d_k(e_j)y - d_k(e_i)x\big), 0 \right\}.$$

In order to determine the multiplicity of $\operatorname{div}(g)$ along the support of g, we have to evaluate g on a lattice normal vector of the support. The support of $\operatorname{div}(g)$ has primitive integer tangent vector $\frac{1}{\gcd\left(d_k(e_i), d_k(e_j)\right)} \begin{pmatrix} d_k(e_j) \\ d_k(e_i) \end{pmatrix}$ and a lattice normal vector is determined by completing this vector into a $\mathbb{Z}$-basis for $\mathbb{Z}^2$. This is symmetric in i and j, so we may assume without loss of generality $i \leqslant j$ and find the lattice normal vector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ with function value $g(0,1) = d_k(e_j)$. This means that the value of $(k \times k)^*[\Delta_K]$ on $\big\{(x,y) \in e_i \times e_j \;\big|\; k(x) = k(y)\big\}$ is $d_k(e_1)$ for $i = j = 1$ and 1 otherwise. This is precisely the same as the value of $[\Delta_\Pi] + A$ on this locus. □

Lemma 5.6. *Let K be a smooth metric tree (so that each extremal edge is infinite), and let $p \in K$ be any point. Then*

$$[\Delta_K] = [K \times p] + [p \times K] \in H_{1,1}(K^2),$$

where Δ_K is the diagonal and $[\cdot]$ denotes the fundamental class.

Proof. We note that the terms in the above equation are tropical homology classes associated to 1-cycles (see Example 3.9 for the diagonal cycle), and that the equation certainly does not hold in in $Z_1(K^2)$.

We proceed in two steps. As a base case, we first prove the claim for the compactified real line $L = \mathbb{R} \cup \{-\infty, \infty\}$. We orient L from $-\infty$ to ∞. Let η be a generator for Ω^1_L in the direction of this orientation, more precisely

$$\Omega^1_{L,x} = \begin{cases} \langle \eta \rangle & \text{if } x \in \mathbb{R} \\ 0 & \text{otherwise .} \end{cases}$$

Then $\eta_1 = \begin{pmatrix} \eta \\ 0 \end{pmatrix}$ and $\eta_2 = \begin{pmatrix} 0 \\ \eta \end{pmatrix}$ are generators for $\Omega^1_{L^2}$, or again more precisely

$$\Omega^1_{L^2,(x,y)} = \begin{cases} \langle \eta_1, \eta_2 \rangle & \text{if } (x,y) \in \mathbb{R}^2 \\ \langle \eta_1 \rangle & \text{if } x \in \mathbb{R} \text{ and } y = \pm\infty \\ \langle \eta_2 \rangle & \text{if } y \in \mathbb{R} \text{ and } x = \pm\infty \\ 0 & \text{otherwise .} \end{cases}$$

Let $\sigma : \Delta^2 \to \big\{(x,y) \in L^2 \;\big|\; x \geqslant y\big\}$ be a singular 2-simplex parametrizing the area below the diagonal of L^2 with orientation compatible with that on L. Write

$$\tau_1 : \Delta^1 \longrightarrow \big\{(x, -\infty) \in L^2\big\}$$
$$\tau_2 : \Delta^1 \longrightarrow \big\{(\infty, y) \in L^2\big\}$$
$$\delta : \Delta^1 \longrightarrow \big\{(x,x) \in L^2\big\}$$

for the restriction of σ to the faces of Δ^2, so that the boundary of σ as a singular chain is $\partial\sigma = \tau_1 - \delta + \tau_2$. Then $B = \sigma \otimes (\eta_1^* + \eta_2^*)$ is a $(1,2)$-chain and

$$\begin{aligned}\partial B &= \tau_1 \otimes (\eta_1^* + \underbrace{\eta_2^*}_{=0 \text{ on } \operatorname{Im}\tau_1}) - \delta \otimes (\eta_1^* + \eta_2^*) + \tau_2 \otimes (\underbrace{\eta_1^*}_{=0 \text{ on } \operatorname{Im}\tau_2} + \eta_2^*) \\ &= \operatorname{cyc}[L \times -\infty] - \operatorname{cyc}[\Delta_L] + \operatorname{cyc}[\infty \times L].\end{aligned}$$

Note that the second equality only holds because the images of τ_1 and τ_2 are contained in the boundary of the rational polyhedral space L^2. If we did the same computation with a finite interval $L = [a, b]$, we would not see $\operatorname{cyc}[L \times a]$ and $\operatorname{cyc}[b \times L]$ in $\partial\big(\sigma \otimes (\eta_1^* + \eta_2^*)\big)$ because η_1^* and η_2^* do not vanish on the topological boundary.

To finish the proof for L, we want to argue that $\operatorname{cyc}[L \times \infty] = \operatorname{cyc}[L \times p]$ for any $p \in \mathbb{R}$. Let σ' be a singular 2-complex (consisting of two simplices) parametrizing $\big\{(x, y) \in L^2 \;\big|\; -\infty \leqslant y \leqslant p\big\}$. Then

$$\partial(\sigma' \otimes \eta_1^*) = \operatorname{cyc}[L \times -\infty] - \operatorname{cyc}[L \times p].$$

The key here is that η_1^* vanishes on $[-\infty, p] \times \{\pm\infty\}$ so that there is no contribution to $\partial(\sigma' \otimes \eta_1^*)$ from the remaining two edges of $\partial\sigma'$. Similarly we see $[\infty \times L] = [p \times L]$ which completes the proof for the base case.

Now let K be any smooth tree. Fix two distinct points $-\infty, \infty \in K$ on the boundary and let $L \subseteq K$ be the path from $-\infty$ to ∞. As a rational polyhedral space, L is isomorphic to the compactified real line from the base case. Furthermore, L is a deformation retract of K. Denote the retraction map $\rho : K \to L$. This is a proper map of rational polyhedral spaces, so there is a pushforward map ρ_* in homology.

We claim that the induced map $\rho_*^2 : H_{1,1}(K^2) \to H_{1,1}(L^2)$ is an isomorphism. We first show that ρ_* is an isomorphism. Indeed, $H_{1,0}$ and $H_{0,1}$ of K and L are trivial because K and L are trees. On the other hand, the pushforward map on $H_{0,0}$ is an isomorphism sending the class of a point to the class of a point, and similarly on $H_{1,1}$ we have an isomorphism given by sending the fundamental class of K to the fundamental class of L. By the Künneth formula, $H_{1,1}(K^2)$ decomposes as

$$H_{0,0}(K) \otimes H_{1,1}(K) \quad \oplus \quad \underbrace{H_{0,1}(K) \otimes H_{1,0}(K)}_{=0} \quad \oplus \quad \underbrace{H_{1,0}(K) \otimes H_{0,1}(K)}_{=0} \quad \oplus \quad H_{1,1}(K) \otimes H_{0,0}(K),$$

and similarly for L. Hence ρ_* being an isomorphism implies that $\rho_*^2 : H_{1,1}(K^2) \to H_{1,1}(L^2)$ is an isomorphism as well.

We have already proved our claim in $H_{1,1}(L^2)$, so it suffices to show that

$$\begin{aligned}\rho_*^2 \operatorname{cyc}[\Delta_K] &= \operatorname{cyc}[\Delta_L], \\ \rho_*^2 \operatorname{cyc}[K \times p] &= \operatorname{cyc}[L \times \rho(p)], \quad \text{and} \\ \rho_*^2 \operatorname{cyc}[p \times K] &= \operatorname{cyc}[\rho(p) \times L].\end{aligned}$$

But the cycle class map and pushforward commute, so we may verify this on the level of tropical cycles, where it is easy to see. On every edge e of K, the map ρ is either the identity (if e is part of L) or constant. Hence the index of tangent spaces in Equation (12) is either 1 or 0, i.e. the part of the cycles $[\Delta_K]$, $[K \times p]$, and $[p \times K]$ that is already in L survives the-push forward while the rest is weighted with 0. This finishes the proof. □

Finally, we consider the bigonal construction. Recall that the Prym variety $\mathrm{Prym}(\widetilde{\Gamma}/\Gamma)$ of a double cover $\widetilde{\Gamma} \to \Gamma$ is defined as the pptav associated (by Lemma 4.10) to the connected component of the identity of the kernel $(\mathrm{Ker}\,\mathrm{Nm})_0$ of the norm map $\mathrm{Nm} : \mathrm{Jac}(\widetilde{\Gamma}) \to \mathrm{Jac}(\Gamma)$. The tropical abelian variety $(\mathrm{Ker}\,\mathrm{Nm})_0$ carries a polarization (induced from $\mathrm{Jac}(\widetilde{\Gamma})$), which is not necessarily principal. We now restate our main result:

Theorem 5.7 (Theorem 1.1)**.** *Let $\widetilde{\Gamma} \xrightarrow{\pi} \Gamma \xrightarrow{f} K$ be a tower of harmonic morphisms of metric graphs of degrees $\deg \pi = \deg f = 2$, where K is a metric tree. Assume that there is no point $x \in K$ with the property that $|f^{-1}(x)| = 2$ and $|(f \circ \pi)^{-1}(x)| = 2$. Then the output $\widetilde{\Pi} \xrightarrow{\pi'} \Pi \xrightarrow{f'} K$ of the bigonal construction has the same property, and applying the bigonal construction to it produces the original tower. If moreover $\widetilde{\Gamma}$ and $\widetilde{\Pi}$ are both connected, then*

$$\big(\mathrm{Ker}\,\mathrm{Nm}(\pi)\big)_0 \cong \big(\mathrm{Ker}\,\mathrm{Nm}(\pi')\big)_0^\vee,$$

where $(\cdot)^\vee$ denotes the dual tropical abelian variety.

The algebraic version of Theorem 5.7 [Pan86] was proved by reduction to the tetragonal construction. Since this is not an option for us, we give the following proof which is an adaptation of the proof of Theorem 5.1 that we just discussed.

Proof of Theorem 5.7. Recall that we have already established the properties of the bigonal construction in Propositions 2.7 and 2.8, and it only remains to compare the Pryms. Specifically, we show that if $\big(\mathrm{Ker}\,\mathrm{Nm}(\pi)\big)_0$ has polarization type $(1^B, 2^A)$, then the type of $\big(\mathrm{Ker}\,\mathrm{Nm}(\pi')\big)_0$ is $(1^A, 2^B)$. We emphasize that throughout the proof $\mathrm{Prym}(\cdot)$ will always refer to the pptav induced by Lemma 4.10 from $\big(\mathrm{Ker}\,\mathrm{Nm}(\cdot)\big)_0$.

By construction, the curve $\widetilde{\Pi}$ comes with an embedding in $\mathrm{Sym}^2\,\widetilde{\Gamma}$, compare Equation (9). Fix a base point $q \in \widetilde{\Gamma}$ and let $\chi : \widetilde{\Pi} \to \mathrm{Prym}(\widetilde{\Gamma}/\Gamma)$ be the restriction of the second power of the Abel–Prym map $\psi_q : \widetilde{\Gamma} \to \mathrm{Prym}(\widetilde{\Gamma}/\Gamma)$ to $\widetilde{\Pi}$. Similarly to the above, one can check that χ is a morphism of rational polyhedral spaces and it satisfies $\chi \circ \iota = -\chi$. Fix a base point $q' \in \widetilde{\Pi}$. By the universal property of the Prym variety (Proposition 4.24) we obtain a commutative diagram

$$\begin{tikzcd}
\widetilde{\Pi} \arrow[r, "\chi"] \arrow[d, "\psi_{q'}"] & \mathrm{Prym}(\widetilde{\Gamma}/\Gamma) \arrow[d, "t_{-\chi(q')}"] \\
\mathrm{Prym}(\widetilde{\Pi}/\Pi) \arrow[r, "\mu"] & \mathrm{Prym}(\widetilde{\Gamma}/\Gamma).
\end{tikzcd}$$

We know from Theorem 4.27 that $(\psi_{q'})_*[\widetilde{\Pi}] = \frac{2}{(g_0-1)!}[\Psi]^{g_0-1}$, where $g_0 = \dim \mathrm{Prym}(\widetilde{\Pi}/\Pi)$. We emphasize that Ψ is the principal polarization on $\mathrm{Prym}(\widetilde{\Pi}/\Pi)$ and not the induced polarization. We claim that it thus suffices to show that

$$\chi_*[\widetilde{\Pi}] = 2\frac{2}{(g_0-1)!}[\Xi]^{g_0-1}, \tag{30}$$

where Ξ is the principal (not the induced) polarization on $\mathrm{Prym}(\widetilde{\Gamma}/\Gamma)$. In particular, this means that μ is an isomorphism of integral tori, but not an isomorphism of principally polarized tropical abelian varieties.

To see that Equation (30) suffices, let ξ and ζ denote the induced non-principal polarizations of $\big(\mathrm{Ker}\,\mathrm{Nm}(\pi)\big)_0$ and $\big(\mathrm{Ker}\,\mathrm{Nm}(\pi')\big)_0$, respectively. Let $(\alpha_1, \ldots, \alpha_A, \beta_1, \ldots, \beta_B)$ be a basis of $\mathrm{Ker}(\pi'_*)$

that exhibits the type of ζ, i.e. such that $\zeta(\alpha_i)$ is twice a generator of $\big(\mathrm{Coker}(\pi'^*)\big)^{\mathrm{tf}}$ and $\zeta(\beta_j)$ is a generator of $\big(\mathrm{Coker}(\pi'^*)\big)^{\mathrm{tf}}$. If Equation (30) holds, then $\mu^*\Xi = 2\Psi$, which means more specifically that

$$\mu^\# \circ \Xi \circ \mu_\#(\alpha_i) = 2\Psi(\alpha_i) = \zeta(\alpha_i),$$
$$\mu^\# \circ \Xi \circ \mu_\#(\beta_j) = 2\Psi(\beta_j) = 2\zeta(\beta_j).$$

Since $\mu_\#$ and $\mu^\#$ are isomorphisms, this can be rearranged as

$$(\mu^\#)^{-1} \circ \zeta \circ (\mu_\#)^{-1}(\alpha_i) = \Xi\big((\mu_\#)^{-1}(\alpha_i)\big),$$
$$(\mu^\#)^{-1} \circ \zeta \circ (\mu_\#)^{-1}(\beta_j) = 2\Xi\big((\mu_\#)^{-1}(\beta_j)\big),$$

and we see that ξ necessarily has the complementary type of ζ. But this is precisely how the dual (non-principal) polarization was defined and hence we are done.

We will now outline the proof of Equation (30). Let $k = f' \circ \pi'$ be the map from $\widetilde{\Pi} \to K$. Just like in the proof of Theorem 5.1 above, we define a tropical 1-cycle $B \in Z_1(\widetilde{\Gamma}^2)$ which represents the lift of $\widetilde{\Pi} \subseteq \mathrm{Sym}^2(\widetilde{\Gamma})$ to $\widetilde{\Gamma}^2$. Again, one has to carefully verify that B is in fact a tropical cycle. Once this is established, we may prove that the key formula

$$[\Delta_{\widetilde{\Gamma}}] + (\mathrm{Id} \times \iota)_*[\Delta_{\widetilde{\Gamma}}] + B = (k \times k)^*[\Delta_K] = 4[\widetilde{\Gamma} \times p'] + 4[p' \times \widetilde{\Gamma}] \tag{31}$$

holds in $H_{1,1}(\widetilde{\Gamma}^2)$ for an arbitrary point $p' \in \widetilde{\Gamma}$. Here the first equality is proved similarly to Lemma 5.5 and the second equality is the same as before. We then apply $(\psi_q^2)_*$ to Equation (31) to obtain an expression in $H_{1,1}\big(\mathrm{Prym}(\widetilde{\Gamma}/\Gamma)\big)$ consisting of the following terms.

(1) $(\psi_q^2)_*[\Delta_{\widetilde{\Gamma}}] = 4(\psi_q)_*[\widetilde{\Gamma}]$ because again the pushforward along ψ_q^2 acts as ψ_q followed by multiplication by 2 on the Prym variety which then induces multiplication by 4 on homology.

(2) $(\psi_q^2)_*(\mathrm{Id} \times \iota)_*[\Delta_{\widetilde{\Gamma}}] = 0$. This is because

$$\psi_q^2\big(p + \iota(p)\big) = \big(p - q - \iota(p) + \iota(q)\big) + \big(\iota(p) - q - p + \iota(q)\big) = -2q + 2\iota(q)$$

is constant.

(3) $(\psi_q^2)_* B = 2\chi_*[\widetilde{\Pi}]$, because ψ_q^2 factors as the degree 2 quotient map $\widetilde{\Gamma}^2 \to \mathrm{Sym}^2\,\widetilde{\Gamma}$ followed by χ.

(4) $(\psi_q^2)_*[\widetilde{\Gamma} \times p'] = (\psi_q^2)_*[p' \times \widetilde{\Gamma}] = (\psi_q)_*[\widetilde{\Gamma}]$.

Solving the (ψ_q^2)-pushforward of Equation (31) for $\chi_*[\widetilde{\Pi}]$ gives Equation 30 and we are done. □

Example 5.8. Let us return to the towers of graphs depicted in Figure 4. Again, we assign real edge lengths a, b, and c to the three edges of K and thus obtain two $(2,2)$-towers $\widetilde{\Gamma}_1 \to \Gamma_1 \to K$ and $\widetilde{\Gamma}_2 \to \Gamma_2 \to K$ of tropical curves. To compute the (non-principally polarized) tropical Prym varieties $(\mathrm{Ker}\,\mathrm{Nm}(\pi_1))_0$ and $(\mathrm{Ker}\,\mathrm{Nm}(\pi_2))_0$, we work with the bases

$$\mathrm{Ker}(\widetilde{\Gamma}_1 \to \Gamma_1) = \langle \widetilde{\eta}_1^+ - \widetilde{\eta}_1^-, \widetilde{\eta}_2 \rangle$$
$$\mathrm{Ker}(\widetilde{\Gamma}_2 \to \Gamma_2) = \langle \widetilde{\epsilon}_1, \widetilde{\epsilon}_2^+ - \widetilde{\epsilon}_2^- \rangle$$

depicted in Figure 12. For $i = 1, 2$ we identify $H_1(\widetilde{\Gamma}_i, \mathbb{Z})$ with $\Omega^1(\widetilde{\Gamma}_i)$ using the principal polarization ζ_i of $\mathrm{Jac}(\widetilde{\Gamma}_i)$. With this, the pairings induced from the integration pairings on the $\widetilde{\Gamma}_i$ are

$[\cdot,\cdot]_1$	$\zeta_1(\widetilde{\eta}_1^+ - \widetilde{\eta}_1^-)$	$\zeta_1(\widetilde{\eta}_2)$
$\widetilde{\eta}_1^+ - \widetilde{\eta}_1^-$	$2b$	b
$\widetilde{\eta}_2$	$2b$	$a+2b+c$

and

$[\cdot,\cdot]_2$	$\zeta_2(\widetilde{\varepsilon}_1)$	$\zeta_2(\widetilde{\varepsilon}_2^+ - \widetilde{\varepsilon}_2^-)$
$\widetilde{\varepsilon}_1$	$2b$	$2b$
$\widetilde{\varepsilon}_2^+ - \widetilde{\varepsilon}_2^-$	b	$a+2b+c$

and we clearly see that transposing one of these matrices gives the other. This shows that $\big(\operatorname{Ker}\operatorname{Nm}(\pi_1)\big)_0 \cong \big(\operatorname{Ker}\operatorname{Nm}(\pi_2)\big)_0^\vee$ on the level of integral tori. But even more is true: by Definition 4.12 the polarization of $\big(\operatorname{Ker}\operatorname{Nm}(\pi_2)\big)_0^\vee$ is given by

$$\begin{aligned} \xi_2^\vee : \Omega^1(\widetilde{\Gamma}_2) &\longrightarrow H_1(\widetilde{\Gamma}_2,\mathbb{Z}) \\ \zeta_2(\widetilde{\varepsilon}_1) &\longmapsto 2\widetilde{\varepsilon}_1 \\ \zeta_2(\widetilde{\varepsilon}_2^+ - \widetilde{\varepsilon}_2^-) &\longmapsto \widetilde{\varepsilon}_2^+ - \widetilde{\varepsilon}_2^- \end{aligned}$$

and hence the pull-back of $\xi_2^\vee$ along the isomorphism $\operatorname{Prym}(\widetilde{\Gamma}_1/\Gamma_1) \to \operatorname{Prym}(\widetilde{\Gamma}_2/\Gamma_2)^\vee$ yields the induced polarization ξ_1 of $\big(\operatorname{Ker}\operatorname{Nm}(\pi_1)\big)_0$.

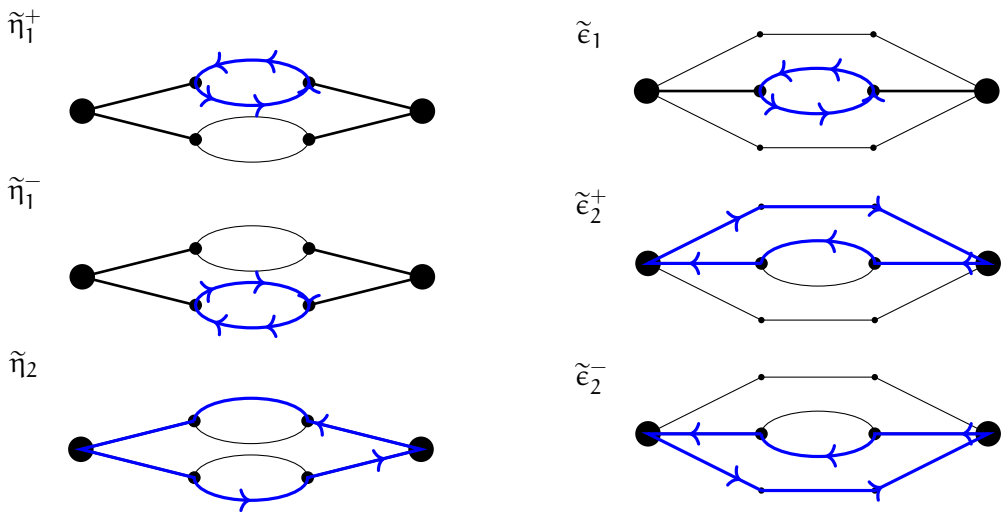

FIGURE 12. Bases of the first homology $H_1(\widetilde{\Gamma}_1,\mathbb{Z})$ and $H_1(\widetilde{\Gamma}_2,\mathbb{Z})$ used in the computation of the Prym varieties.

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