A discrete analogue of the modified Novikov-Veselov hierarchy

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Abstract

We construct a discrete analogue of the integrable two-dimensional Dirac operator and describe the spectral properties of its eigenfunctions. We construct an integrable discrete analogue of the modified Novikov-Veselov hierarchy. We derive the first two equations of the hierarchy and give explicit formulas for the eigenfunctions in terms of the theta-functions of the associated spectral curve.

1 Introduction

The purpose of this paper is to construct a discrete analogue of the modified Novikov-Veselov hierarchy and its algebro-geometric solutions, and to describe the spectral theory of the corresponding discrete Dirac operator.

The modified Novikov–Veselov (mNV) hierarchy is an integrable hierarchy of equations introduced by Bogdanov in [1], [2] as a special reduction of the Davey–Stewardson equation. The equations of the hierarchy have the form of Manakov L, A, B-triples

$$\frac{\partial L}{\partial t_n} = [L, A_n] - B_n L, \tag{1.1}$$

where $L = D$ is the two-dimensional Dirac operator

$$D\psi = \begin{pmatrix} u & \partial \\ -\bar{\partial} & u \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \tag{1.2}$$

and A_n and B_n are (2×2)-matrix differential operators. The mNV hierarchy describes deformations of the Dirac operator that preserve the zero energy level, i.e. isospectral deformations of the equation

$$D\psi = 0. \tag{1.3}$$

The first equation of the hierarchy has the form

$$u_t = \left(u_{zzz} + 3u_z v + \frac{3}{2} u v_z \right) + \left(u_{\bar{z}\bar{z}\bar{z}} + 3u_{\bar{z}} v + \frac{3}{2} u v_{\bar{z}} \right), \quad v_{\bar{z}} = (u^2)_z. \tag{1.4}$$

In [3], [4] Taimanov constructed algebro-geometric solutions of the mNV hierarchy and described the spectral theory of the Dirac operator (1.2). In recent times, the mNV hierarchy and its algebro-geometric solutions have attracted significant attention due to their applications to the classical theory

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of two-dimensional surfaces in three-dimensional Euclidean space, and in particular to the Willmore conjecture (see the survey [5] for an extensive bibliography).

It is possible to consider a more general two-dimensional Dirac operator of the form

$$D = \begin{pmatrix} u & \partial \\ -\bar\partial & v \end{pmatrix}. \tag{1.5}$$

The spectral theory of the two-dimensional Dirac operator (1.5) is equivalent to that of the two-dimensional scalar Schrödinger operator in a magnetic field

$$H = \partial\bar\partial + V\bar\partial + U. \tag{1.6}$$

The reduction of the Dirac operator (1.5) to the form (1.2) corresponds to a reduction on the Schrödinger operator in which the functions U and V satisty the relation

$$V = -\partial \ln U. \tag{1.7}$$

The analytic properties of Baker–Akhiezer functions which describe general Schrödinger operators of the form (1.6) that are integrable on the zero energy level were formulated in [10]. The reductions on the algebro-geometric data that describe the potential Schrödinger operator ($V = 0$), which is the auxiliary operator for the Novikov–Veselov hierarchy, were found in [8], [9].

The problem of constructing an integrable discretization of an integrable differential equation is not well-posed and does not have a universal solution. However, there are several methods in soliton theory that allow us to construct integrable discretizations. Most of them are based on constucting a discrete analogue of the auxiliary linear problems, which involves an appropriate deformation of the analytic properties of the solutions of these linear problems.

In the finite-gap case, the eigenfunction of the auxiliary linear differential operator, known as the Baker–Akhiezer function, is defined on an algebraic Riemann surface and is required to have exponential singularities controlled by the continuous variables at one or more marked points of the surface. To construct a discrete analogue of the operator, we replace each exponential singularity with a pair of meromorphic singularities consising of a pole and a zero of the same order, which we consider as the discrete variable. This deformed eigenfunction then satisfies a infinite system of linear difference and differential equations, whose compatibility conditions are the discretization of the original integrable hierarchy. This method was used for constructing algebro-geometric solutions of the Ablowitz–Ladik equation [11], [12], which is a discretization of the nonlinear Schrödinger equation, and for constructing Darboux–Egoroff lattices, which are the discrete analogue of Darboux–Egoroff metrics [6].

Using this approach, Grushevsky and Krichever have given an algebro-geometric construction of an integrable discretization of the two-dimensional Schrödinger operator (1.6). In the second paragraph, we describe a matrix variant of this construction, which leads to a two-dimensional matrix difference operator of the form

$$D\psi = \left[\begin{pmatrix} T_2 & 0 \\ 0 & T_1 \end{pmatrix} - \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\right]\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \tag{1.8}$$

where the functions ψ_i and the coefficients of the operator are functions of two discrete variables $n, m \in \mathbb{Z}$, and T_1, T_2 denote the translation operators in the discrete variables. The operator D, which we call the *discrete Dirac operator*, can be considered as a discrete analogue of the general Dirac operator of the form (1.5).

The coefficients of a discrete Dirac operator (1.8) depend, up to gauge transformation, on two arbitrary functions of the discrete variables. In the second paragraph, we show that a discretization of the algebro-geometric data corresponding to operators of the form (1.2) leads to operators whose coefficients depend on only one arbitrary function, namely operators of the form

$$D\psi = \left[\begin{pmatrix} T_2 & 0 \\ 0 & T_1 \end{pmatrix} - \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix}\right]\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \tag{1.9}$$

where the coefficients satisfy the relation

$$\alpha^2 - \beta^2 = 1 \tag{1.10}$$

In the third paragraph we introduce time dependence into the eigenfunctions and construct an integrable hierarchy of isospectral deformations of the zero energy level of the operator (1.9). We call this hierarchy, which has the form of Manakov L, A, B-triples, the *discrete modified Novikov-Veselov hierarchy.*

In the fourth paragraph we derive the explicit form of the first two equations of the hierarchy (equations (4.22), (4.24), (4.26)). The first equation has the following form:

$$\frac{\partial\psi(n,m)}{\partial\tau_1^1} = \sqrt{\left(e^{2\varphi(n-1,m+1)} - e^{2\varphi(n-1,m)}\right)\left(e^{-2\varphi(n,m)} - e^{-2\varphi(n,m+1)}\right)}, \tag{1.11}$$

where the two functions satisfy the non-local relation

$$\varphi(n,m+1) - \varphi(n,m) = \psi(n+1,m) - \psi(n,m). \tag{1.12}$$

In the final paragraph we give explicit formulas for the Baker–Akhiezer functions in terms of theta-functions associated to the spectral curve.

2 Reduction of general discrete Dirac operators

Consider the following discrete linear equation

$$D\psi = \left[\begin{pmatrix} T_2 & 0 \\ 0 & T_1 \end{pmatrix} - \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\right]\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = 0, \tag{2.1}$$

where $\psi = (\psi_1(n,m), \psi_2(n,m))^T$ is a vector function of two discrete variables $n, m \in \mathbb{Z}$, and

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha(n,m) & \beta(n,m) \\ \gamma(n,m) & \delta(n,m) \end{pmatrix} \tag{2.2}$$

is a (2×2)-matrix function of the discrete variables. We call D the *discrete Dirac operator*. We use T_1 and T_2 to denote the translation operators in the discrete variables

$$T_1 f(n,m) = f(n+1,m), \quad T_2 f(n,m) = f(n,m+1), \tag{2.3}$$

while t_1 and t_2 will be used to denote the translated functions, so that for example $T_1(fg) = (t_1 f)(t_1 g)$. In this chapter, we construct algebro-geometric solutions of equation (2.1) and some of its reductions.

The main method of constructing algebro-geometric solutions of linear differential or difference equations such as (2.1) is to consider functions ψ_i defined on an auxiliary Riemann surface, called the *spectral curve*, and having certain prescribed singularities on that curve. Generally, to construct solutions of difference equations, we consider functions that are meromorphic on the spectral curve with prescribed pole singularities, while constructing solutions of differential equations requires us to consider functions with prescribed essential singularities, called *Baker-Akhiezer functions*.

Let X be a smooth Riemann surface of genus g. We consider the following data on X:

Data A.

- Four distinct marked points $P_1^{\pm}, P_2^{\pm}$ on X.
- Local parameters $z_i^{\pm} = (k_i^{\pm})^{-1}$ defined in some neighborhoods of these points.
- An effective divisor $D = \gamma_1 + \cdots + \gamma_{g+1}$ of degree $g+1$ on X, supported away from the marked points, which satisfies the following condition of general position:

$$h^1(D + (n-1)P_1^+ - nP_1^- + (m-1)P_2^+ - mP_2^-) = 0 \text{ for all } n, m \in \mathbb{Z}. \tag{2.4}$$

To construct solutions of equation (2.1), we consider spaces of meromorphic functions on X with singularities controlled by the discrete variables:

$$\Psi_{n,m} = H^0(D + nP_1^+ - nP_1^- + mP_2^+ - mP_2^-) \subset \mathrm{Mer}(X), \quad n, m \in \mathbb{Z}.$$

The Riemann-Roch theorem implies the following

Proposition 1 *Suppose that X is an algebraic curve with data A defined above. Then each of the spaces $\Psi_{n,m}$ is two-dimensional:*

$$\dim \Psi_{n,m} = h^0(D + nP_1^+ - nP_1^- + mP_2^+ - mP_2^-) = 2 \textit{ for all } n, m \in \mathbb{Z},$$

the intersection of two of these spaces at adjacent lattice points is one-dimensional:

$$\dim \Psi_{n,m} \cap \Psi_{n,m-1} = h^0(D + nP_1^+ - nP_1^- + (m-1)P_2^+ - mP_2^-) = 1 \textit{ for all } n, m \in \mathbb{Z},$$

$$\dim \Psi_{n,m} \cap \Psi_{n-1,m} = h^0(D + (n-1)P_1^+ - nP_1^- + mP_2^+ - mP_2^-) = 1 \textit{ for all } n, m \in \mathbb{Z},$$

and these two one-dimensional subspaces of $\Psi_{n,m}$ span the entire space, i.e. their intersection is trivial:

$$\dim \Psi_{n,m} \cap \Psi_{n,m-1} \cap \Psi_{n-1,m} = h^0(D + (n-1)P_1^+ - nP_1^- + (m-1)P_2^+ - mP_2^-) = 0 \textit{ for all } n, m \in \mathbb{Z}.$$

Therefore, we can fix a basis $\psi_1(n, m, P), \psi_2(n, m, P)$ in each of the spaces $\Psi_{n,m}$ by letting $\psi_1(n, m, P)$ be any non-zero element of $\Psi_{n,m} \cap \Psi_{n,m-1}$, and letting $\psi_2(n, m, P)$ to be any non-zero element of $\Psi_{n,m} \cap \Psi_{n-1,m}$:

$$\psi_1(n, m, P) \in H^0(D + nP_1^+ - nP_1^- + (m-1)P_2^+ - mP_2^-) - \{0\}, \tag{2.5}$$

$$\psi_2(n, m, P) \in H^0(D + (n-1)P_1^+ - nP_1^- + mP_2^+ - mP_2^-) - \{0\}. \tag{2.6}$$

The principal observation concerning these functions can be summarized in the following statement:

Proposition 2 *Suppose that X is a Riemann surface with data A as defined above. Then there exist functions $\alpha(n,m)$, $\beta(n,m)$, $\gamma(n,m)$, $\delta(n,m)$ such that the functions $\psi_1(n,m,P)$ and $\psi_2(n,m,P)$ defined by (2.5)-(2.6) satisfy the Dirac equation:*

$$D\psi = \left[\begin{pmatrix} T_2 & 0 \\ 0 & T_1 \end{pmatrix} - \begin{pmatrix} \alpha(n,m) & \beta(n,m) \\ \gamma(n,m) & \delta(n,m) \end{pmatrix}\right]\begin{pmatrix} \psi_1(n,m,P) \\ \psi_2(n,m,P) \end{pmatrix} = 0. \tag{2.7}$$

Proof. Indeed, by construction, both $\psi_1(n,m+1,P)$ and $\psi_2(n+1,m,P)$ actually lie in the space $\Psi_{n,m}$, hence they can be expressed as linear combinations of the basis functions $\psi_1(n,m,P)$ and $\psi_2(n,m,P)$, which is equivalent to saying that the satisfy the Dirac equation (2.7).

Therefore, a Riemann surface X together with the additional data given above allows us to construct a family of solutions $(\psi_1(n,m,P),\psi_2(n,m,P))^T$ of the Dirac equation (2.1), parametrized by the points P of X.

In order to construct reductions on the Dirac equation (2.7), we first express the coefficients $\alpha(n,m)$, $\beta(n,m)$, $\gamma(n,m)$ and $\delta(n,m)$ in terms of the principal parts of the basis functions at the marked points. In terms of the chosen local coordinates, the basis functions $\psi_1(n,m,P)$ and $\psi_2(n,m,P)$ have the following expansions at the marked points, where k denotes the appropriate local parameter $k^i_\pm$:

$$\psi_1(n,m,P) = \begin{cases} a_1^+(n,m)k^n + O(k^{n-1}), & \text{as } P \to P_1^+ \\ a_1^-(n,m)k^{-n} + O(k^{-n-1}), & \text{as } P \to P_1^- \\ O(k^{m-1}), & \text{as } P \to P_2^+ \\ a_2^-(n,m)k^{-m} + O(k^{-m-1}), & \text{as } P \to P_2^- \end{cases} \tag{2.8}$$

$$\psi_2(n,m,P) = \begin{cases} O(k^{n-1}), & \text{as } P \to P_+^1 \\ b_1^-(n,m)k^{-n} + O(k^{-n-1}), & \text{as } P \to P_-^1 \\ b_2^+(n,m)k^{m} + O(k^{m-1}), & \text{as } P \to P_+^2 \\ b_2^-(n,m)k^{-m} + O(k^{-m-1}), & \text{as } P \to P_-^2 \end{cases} \tag{2.9}$$

where the $a_i^\pm(n,m)$ and $b_i^\pm(n,m)$ are functions of the discrete variables n and m. Considering the Dirac equation (2.7) near the marked points $P_1^\pm, P_2^\pm$ gives us the following system of equations (in what follows, we usually suppress the indices n and m and replace them with the shift operators t_1 and t_2):

$$\begin{array}{rclcrcl} t_2a_1^+ &=& \alpha a_1^+, & & 0 &=& \gamma a_1^- + \delta b_1^-, \\ t_2a_1^- &=& \alpha a_1^- + \beta b_1^-, & & t_1b_2^+ &=& \delta b_2^+, \\ 0 &=& \alpha a_2^- + \beta a_2^+, & & t_1b_2^- &=& \gamma a_2^- + \delta b_2^-. \end{array} \tag{2.10}$$

The functions ψ_1 and ψ_2 have so far been defined up to multiplication by a constant factor dependent on n and m. We impose the following additional condition on the functions ψ_1 and ψ_2:

$$a_1^+a_1^- = 1, \qquad b_2^+b_2^- = 1. \tag{2.11}$$

It is easy to show using (2.10) that these conditions imply the following relations on the coefficients α, β, γ, δ:

$$\alpha\delta - \beta\gamma = \frac{\alpha}{\delta} = \frac{\delta}{\alpha} = \frac{(t_2a_1^+)(t_1b_2^-)}{a_1^+b_2^-} = \pm 1. \tag{2.12}$$

Condition (2.11) defines the constants a_1^+ and b_2^-, and hence the functions ψ_1 and ψ_2, only up to a factor of ± 1 that depends on n and m. This allows us to impose the following additional condition on the functions ψ_1 and ψ_2:

$$(t_2a_1^+)(t_1b_2^-) = a_1^+b_2^-. \tag{2.13}$$

In other words, we can choose the sign for the function ψ_2 arbitrarily, and then choose the sign for the function ψ_1 using the above relation. With this condition, the sign in equation (2.12) is positive. Therefore, reductions (2.11) and (2.13) impose the following relations on the coefficients of the Dirac operator (2.7):

$$\alpha\delta - \beta\gamma = 1, \qquad \alpha = \delta \tag{2.14}$$

In other words, the coefficients of a general Dirac operator of the form (2.7) depend, up to gauge equivalence, on two arbitrary functions of the discrete variables.

We now introduce a reduction under which the coefficients of the Dirac operator (2.7) depend on only one function of the variables n, m. Suppose that, in addition to data A described above, the spectral curve X has the following

Data B.

- A holomorphic involution $\sigma : X \to X$ that interchanges the marked points and the local parameters at the marked points as follows:

$$\sigma(P_i^{\pm}) = P_i^{\mp}, \quad \sigma(k_i^{\pm}) = k_i^{\mp}. \tag{2.15}$$

- A meromorphic 1-form ω on X which has simple poles at the marked points $P_i^{\pm}$ with residues ± 1 and no other singularities, whose zero divisor is $D + \sigma(D)$, and which is odd with respect to the involution.

Consider the meromorphic 1-form $\psi_1(n, m, P)\psi_2(n, m, \sigma(P))\omega(P)$. Comparing the singularities of the three terms, we see that this 1-form has simple poles at P_1^+ and P_2^- with residues $a_1^+ b_1^-$ and $-a_2^- b_2^+$, respectively, and no other singularities. Hence, the existence of the additional data above implies that the coefficients of the functions ψ_1 and ψ_2 satisfy the following additional condition:

$$a_1^+ b_1^- = a_2^- b_2^+. \tag{2.16}$$

Using (2.10) and (2.11), it is easy to show that this condition implies the following additional relation on the coefficients of the Dirac operator:

$$\beta = \gamma. \tag{2.17}$$

Using the involution σ we can rewrite the normalization conditions (2.11) and (2.13) in the following equivalent form:

$$\psi_1(P)\psi_1(\sigma(P))|_{P=P_1^+} = 1, \tag{2.18}$$

$$\psi_2(P)\psi_2(\sigma(P))|_{P=P_2^+} = 1, \tag{2.19}$$

$$\left.\frac{t_2\psi_1(P)}{\psi_1(P)}\right|_{P=P_1^+} = \left.\frac{t_1\psi_2(P)}{\psi_2(P)}\right|_{P=P_2^-}. \tag{2.20}$$

Therefore, we can summarize the result of this reduction as follows.

Proposition 3 *Suppose that X is a Riemann surface with data A and data B as defined above, and suppose the functions $\psi_1(P)$ and $\psi_2(P)$ defined by (2.5) and (2.6) satisfy the normalization conditions (2.18)-(2.20). Then there exist functions of the discrete variables α and β that satisfy the relation*

$$\alpha^2 - \beta^2 = 1 \tag{2.21}$$

and such that the functions $\psi_1(P)$ and $\psi_2(P)$ satisfy the discrete Dirac equation:

$$D\psi = \left[\begin{pmatrix} T_2 & 0 \\ 0 & T_1 \end{pmatrix} - \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix}\right]\begin{pmatrix} \psi_1(P) \\ \psi_2(P) \end{pmatrix} = 0. \tag{2.22}$$

We now construct a further reduction of the discrete Dirac equation (2.22) which is the discrete analogue of the real-valued reduction in the differential case. Suppose that, in addition to data A and data B above, the spectral curve X has the following

Data C.

- An anti-holomorphic involution $\tau: X \to X$ that interchanges the marked points and acts on the local parameters at the marked points as follows:

$$\tau(P_1^\pm) = P_2^\pm, \quad \tau(P_2^\pm) = P_1^\pm, \quad \tau(k_1^\pm) = \bar{k}_2^\pm, \quad \tau(k_2^\pm) = \bar{k}_1^\pm. \tag{2.23}$$

- A meromorphic function $f(P)$ on X with divisor $(f) = D - \tau(D)$ satisfying the conditions

$$f(P)\bar{f}(\tau(P)) = -1 \text{ for all } P \in X, \quad f(P_1^+)f(P_1^-) = 1. \tag{2.24}$$

For a function $f(n,m)$ of the discrete variables, we introduce the notation $f^*(n,m) = \bar{f}(m,n)$. Consider the two functions $\psi_2^*(n,m,\tau(P))$ and $\psi_1(n,m,P)f(P)$. Both these functions are meromorphic and lie in the one-dimensional space $H^0(\tau(D) + (n-1)P_2^+ - nP_2^- + mP_1^+ - mP_1^-)$, hence there exists a function $C(n,m)$ of n and m such that

$$\bar{\psi}_2(m,n,\tau(P)) = \psi_1(n,m,P)f(P)C(n,m). \tag{2.25}$$

Considering this equation at $P = P_1^+$ and $P = P_1^-$ and using conditions (2.11) and (2.24), we see that

$$C(n,m)^2 = 1 \text{ for all } n,m \in \mathbb{Z}. \tag{2.26}$$

We recall that the function ψ_2 was normalized by condition (2.11), which specifies it up to multiplication by a factor ± 1 dependent on n and m. Therefore, we can choose this factor in such a way that $C(n,m) = 1$ for all n and m, in other words we may impose the additional following condition:

$$\bar{\psi}_2(m,n,\tau(P)) = \psi_1(n,m,P)f(P). \tag{2.27}$$

Equation (2.24) then implies that the functions ψ_1 and ψ_2 chosen in this way satisfy the following relations:

$$\bar{\psi}_2(m,n,\tau(P)) = \psi_1(n,m,P)f(P), \quad \bar{\psi}_1(m,n,\tau(P)) = -\psi_2(n,m,P)f(P). \tag{2.28}$$

Plugging these relations into the reduced Dirac equation (2.22) gives us the following relations on the coefficients of the operator:

$$\alpha^* = \alpha, \quad \beta^* = -\beta. \tag{2.29}$$

We summarize the results of this reduction in the following proposition:

Proposition 4 *Suppose that X is an algebraic curve with data A, B and C as defined above, and suppose the functions $\psi_1(P)$ and $\psi_2(P)$ defined by (2.5) and (2.6) satisfy the normalization conditions (2.18)-(2.20) and (2.27). Then there exist functions of the discrete variables α and β that satisfy the relations*

$$\alpha^2 - \beta^2 = 1, \quad \alpha^* = \alpha, \quad \beta^* = -\beta \tag{2.30}$$

that the functions $\psi_1(P)$ and $\psi_2(P)$ satisfy the discrete Dirac equation:

$$D\psi = \left[\begin{pmatrix} T_2 & 0 \\ 0 & T_1 \end{pmatrix} - \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix}\right]\begin{pmatrix} \psi_1(P) \\ \psi_2(P) \end{pmatrix} = 0. \tag{2.31}$$

3 The discrete modified Novikov-Veselov hierarchy

In the previous section, we constructed algebro-geometric solutions of the discrete Dirac operator (2.7) and its reductions (2.22) and (2.31) by considering spaces of meromorphic functions $\Psi_{n,m}$ on an algebraic curve X with poles and zeroes determined by the numbers n and m. In this section, we embed these meromorphic solutions into a family of transcendental functions, called *Baker-Akhiezer functions*, and construct a hierarchy of commuting flows on the space of these functions. The set of compatibility conditions of these flows is the discrete analogue of the modified Novikov-Veselov hierarchy.

Let $\tau = \{\tau_s^1, \tau_s^2, s = 1, 2, \ldots\} \in \mathbb{C}^\infty \oplus \mathbb{C}^\infty$ denote two sequences of complex numbers, only finitely many of which are non-zero, which we think of as continuous time variables. We construct deformations $\Psi_{n,m,\tau}$ of the function spaces $\Psi_{n,m}$ constructed in Section 2 by considering functions which in addition have essential singularities at the marked points controlled by the times τ.

Proposition 5 *Suppose that X is an algebraic curve with data A and data B given as in the previous section. Denote by $\tilde{X} = X - P_1^+ - P_1^- - P_2^+ - P_2^-$ the curve X with the marked points removed. Consider the space $\Psi_{n,m,\tau} \in Mer(\tilde{X})$ of functions on $\tilde{X}$ defined by the following conditions*

1. *For all $\psi(n, m, \tau; P) \in \Psi_{n,m,\tau}$ we have $(\psi) + D \geq 0$, where (f) denotes the divisor of f.*
2. *At the marked points $P_i^\pm$ the elements $\psi(n, m, \tau; P)$ of $\Psi_{n,m,\tau}$ have essential singularities of the following form, where by k we denote the appropriate local coordinate $k_i^\pm$:*

$$\begin{aligned} \psi(n, m, \tau; P) &= \exp\left(\pm \sum_{s=1}^{\infty} \tau_s^1 k^s\right) O(k^{\pm n}) \text{ as } P \to P_1^\pm, \\ \psi(n, m, \tau; P) &= \exp\left(\pm \sum_{s=1}^{\infty} \tau_s^2 k^s\right) O(k^{\pm m}) \text{ as } P \to P_2^\pm. \end{aligned} \tag{3.1}$$

Then each of the spaces $\Psi_{n,m,\tau}$ is two-dimensional:

$$\dim \Psi_{n,m,\tau} = 2 \text{ for all } n, m \in \mathbb{Z}, \tag{3.2}$$

the intersection of two of these spaces at adjacent lattice points is one-dimensional:

$$\dim \Psi_{n,m,\tau} \cap \Psi_{n,m-1,\tau} = 1 \text{ for all } n, m \in \mathbb{Z}, \tag{3.3}$$

$$\dim \Psi_{n,m,\tau} \cap \Psi_{n-1,m,\tau} = 1 \text{ for all } n, m \in \mathbb{Z}, \tag{3.4}$$

and these two one-dimensional subspaces of $\Psi_{n,m,\tau}$ span the entire space, i.e. intersection is trivial:

$$\dim \Psi_{n,m,\tau} \cap \Psi_{n,m-1,\tau} \cap \Psi_{n-1,m,\tau} = 0 \text{ for all } n, m \in \mathbb{Z}. \tag{3.5}$$

Proof. The proof of this proposition is a standard application of the Riemann–Roch theorem.

This proposition allows us to define functions $\psi_1(n, m, \tau; P)$ and $\psi_2(n, m, \tau; P)$ using the same relations as in Section 2. We observe the normalization conditions (2.18)-(2.20) can be applied to elements of $\Psi_{n,m,\tau}$, since the exponential singularities cancel out.

Proposition 6 *There exist unique functions* $\psi_1(n,m,\tau;P)$ *and* $\psi_2(n,m,\tau;P)$ *that form a basis for the vector space* $\Psi_{n,m,\tau}$ *such that*

$$\psi_1(n,m,\tau;P)\in\Psi_{n,m,\tau}\cap\Psi_{n,m-1,\tau}-\{0\}, \tag{3.6}$$

$$\psi_2(n,m,\tau;P)\in\Psi_{n,m,\tau}\cap\Psi_{n-1,m,\tau}-\{0\}. \tag{3.7}$$

and which satisfy the normalization conditions (2.18)-(2.20). These functions satisfy the discrete Dirac equation

$$D\psi=\left[\begin{pmatrix} T_2 & 0\\ 0 & T_1\end{pmatrix}-\begin{pmatrix}\alpha & \beta\\ \beta & \alpha\end{pmatrix}\right]\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}=0, \tag{3.8}$$

where α *and* β *are functions of the variables* n, m, *and* τ *that satisfy the condition*

$$\alpha^2-\beta^2=1. \tag{3.9}$$

In Section 6, we give explicit formulas for the functions ψ_i in terms of theta-functions.

We now show that these functions satisfy a system of commuting linear equations. Let $\mathfrak{R}$ denote the ring of functions in the variables n, m and τ. We consider the ring $\mathfrak{O}=\mathfrak{R}[T_1,T_1^{-1},T_2,T_2^{-1}]$ of finite difference operators with coefficients in $\mathfrak{R}$, and the ring $\mathfrak{M}$ of (2×2) matrix operators with coefficients in $\mathfrak{O}$. By ψ we denote the column vector $(\psi_1(n,m,\tau;P,\psi_2(n,m,\tau;P))^T$.

Proposition 7 *There exist unique matrix difference operators* A_s^i *in* $\mathfrak{M}$

$$A_s^i=\begin{pmatrix}A_{s,1}^i & 0\\ 0 & A_{s,2}^i\end{pmatrix},\quad i=1,2, \tag{3.10}$$

$$A_{s,j}^i=\sum_{\mu=-s}^{s} f_{s,j,\mu}^i(n,m,\tau)T_i^\mu, \tag{3.11}$$

such that the functions $\psi_1(n,m,\tau;P)$ *and* $\psi_2(n,m,\tau;P)$ *satisfy the following system of differential equations:*

$$\frac{\partial}{\partial\tau_s^i}\psi=A_s^i\psi. \tag{3.12}$$

Proof. The proof is standard. For a given s we show how to construct the operator $A_{s,1}^1$, the other cases being similar.

The derivative of the function $\psi_1(n,m,\tau;P)$ with respect to τ_s^1 has the following expansions at the marked points $P_i^\pm$, where by k we denote the appropriate local coordinate $k_i^\pm$:

$$\frac{\partial}{\partial\tau_s^1}\psi_1(n,m,\tau;P)=\exp\left(\pm\sum_{\sigma=1}^\infty\tau_\sigma^1k^\sigma\right)\cdot O(k^{\pm n+s})\text{ as }P\to P_1^\pm, \tag{3.13}$$

$$\frac{\partial}{\partial\tau_s^1}\psi_1(n,m,\tau;P)=\exp\left(\sum_{\sigma=1}^\infty\tau_\sigma^2k^\sigma\right)O(k^{m-1})\text{ as }P\to P_2^+, \tag{3.14}$$

$$\frac{\partial}{\partial\tau_s^1}\psi_1(n,m,\tau;P)=\exp\left(-\sum_{\sigma=1}^\infty\tau_\sigma^2k^\sigma\right)\cdot O(k^{-m})\text{ as }P\to P_2^-. \tag{3.15}$$

Therefore, for an appropriate choice of functions $f^1_{s,i,\mu}(n,m,\tau)$, the function

$$\tilde{\psi}(n,m,\tau;P) = \frac{\partial}{\partial \tau^1_s}\psi_1(n,m,\tau;P) - \sum_{\mu=-s}^{s} f^1_{s,1,\mu}(n,m,\tau)\psi_1(n+\mu,m,\tau;P) \tag{3.16}$$

has the following expansions at $P_1^{\pm}$:

$$\tilde{\psi}(n,m,\tau;P) = \exp\left(\pm\sum_{\sigma=1}^{\infty}\tau^1_\sigma k^\sigma\right)\cdot O(k^{n-1}) \text{ as } P \to P_1^+, \tag{3.17}$$

$$\tilde{\psi}(n,m,\tau;P) = \exp\left(\pm\sum_{\sigma=1}^{\infty}\tau^1_\sigma k^\sigma\right)\cdot O(k^{-n}) \text{ as } P \to P_1^-, \tag{3.18}$$

and the same expansions (3.14)-(3.15) at $P_2^{\pm}$ as $\frac{\partial}{\partial \tau^1_s}\psi_1(n,m,\tau;P)$. Therefore, by (3.5) this function is identically zero on X. Hence, the function $\psi_1(n,m,\tau;P)$ satisfies the system of equations (3.12).

Proposition 8 *The left ideal of matrix difference operators in $\mathfrak{M}$ that annihilate ψ is the principal left ideal generated by the operator D.*

Proof. Suppose that A and B are two operators in $\mathfrak{O}$ that satisfy the following equation:

$$A\psi_1 + B\psi_2 = 0. \tag{3.19}$$

We need to show that there exist elements $C, D \in \mathfrak{O}$ such that $A = C(T_2-\alpha) - D\beta$ and $B = -C\beta + D(T_1-\alpha)$.

First, we multiply equation (3.19) on the left by sufficiently high powers of T_1 and T_2 so that the operators A and B become polynomial in T_1 and T_2. Next, we show that we can eliminate all terms containing mixed powers of T_1 and T_2. Indeed, suppose

$$A = \sum_{i=1}^{n-1} a_i T_1^i T_2^{n-i} + (\text{terms with no } T_1T_2) + (\text{terms of order} < n),$$

$$B = \sum_{i=1}^{n-1} b_i T_1^i T_2^{n-i} + (\text{terms with no } T_1T_2) + (\text{terms of order} < n),$$

then we can write

$$A = \sum_{i=1}^{n-1}\left[a_i T_1^i T_2^{n-i-1}(T_2-\alpha) - b_i T_1^i T_2^{n-i-1}\beta\right] + (\text{terms with no } T_1T_2) + (\text{terms of order} < n),$$

$$B = \sum_{i=1}^{n-1}\left[b_i T_1^i T_2^{n-i-1}(T_1-\alpha) - a_i T_1^i T_2^{n-i-1}\alpha\right] + (\text{terms with no } T_1T_2) + (\text{terms of order} < n),$$

and proceeding in this way, we can eliminate all terms which are not powers of only T_1 or T_2. Therefore, we can assume that $A = A_1(T_1) + A_2(T_2)$, $B = B_1(T_1) + B_2(T_2)$, where the A_i, B_i are polynomials in only T_i.

Suppose that $A_1 = \sum_{i=0}^{n} a_i T_1^i$ and $B_1 = \sum_{j=0}^{m} b_j T_1^j$. Comparing the singularities in (3.19) at the point P_1^+, we see that $m = n+1$. Subtracting $b_{n+1}T_1^n\left[(T_1-\alpha)\psi_2 - \beta\psi_1\right]$ from (3.19), we reduce the degree of B_1, and hence of A_1. In this way we can eliminate A_1, and similarly B_2. Therefore, we are left with showing that if $A = A_2(T_2)$ and $B = B_1(T_1)$ are linear polynomials satisfying (3.19), then they can be expressed as $A = f(T_2-\alpha) - g\beta$ and $B = -f\beta + g(T_1-\alpha)$ for some functions f and g, which can be easily shown.

Proposition 9 *There exist matrix difference operators B_s^i in $\mathfrak{M}$ such that the following equations are satisfied:*

$$-\frac{\partial}{\partial t_s^i} D = D A_s^i + B_s^i D \tag{3.20}$$

Proof. Equations (3.8) and (3.12) imply that

$$\left[\frac{\partial}{\partial t_s^i} - A_s^i, D\right]\psi = 0. \tag{3.21}$$

Since the operator in the left hand side does not contain derivation in time, it is inside $\mathfrak{M}$, hence by the above proposition it is a left multiple of D, which proves the statement.

Proposition 10 *The equations*

$$\frac{\partial}{\partial t_s^i} D + D A_s^i \equiv 0 \ mod\ D \tag{3.22}$$

define a commuting hierarchy of differential-difference equations.

We call this system the *discrete modified Novikov-Veselov (dmNV) hierarchy.* In the next section, we give the explicit form of the first two pairs of equations of the dmNV hierarchy.

4 First and second equations: explicit forms

In this section, we write down the explicit form of the dmNV hierarchy corresponding to times τ_1^1, τ_1^2, τ_2^1 and τ_2^2. We give the explicit calculations for τ_1^1, the derivations for the other times being similar.

It is difficult to write down the dmNV as they are defined in (3.22), since this involves performing division with remainder in a matrix algebra over a non-commutative operator ring. To circumvent this difficulty, we notice that the discrete Dirac equation (3.8), which is a difference equation of degree one on the two functions ψ_1 and ψ_2, is equivalent to a degree two difference equation on one of the ψ_1 or ψ_2.

Proposition 11 *Suppose the functions ψ_1 and ψ_2 satisfy the discrete Dirac equation (3.8). Then the functions ψ_1 and ψ_2 satisfy the following discrete Schrödinger equations*

$$H_1\psi_1 = \left[T_1T_2 - (t_1\alpha)T_1 - \frac{\alpha(t_1\beta)}{\beta}T_2 + \frac{t_1\beta}{\beta}\right]\psi_1 = 0 \tag{4.1}$$

$$H_2\psi_2 = \left[T_1T_2 - (t_2\alpha)T_2 - \frac{\alpha(t_2\beta)}{\beta}T_1 + \frac{t_2\beta}{\beta}\right]\psi_2 = 0. \tag{4.2}$$

Proof. This follows from excluding ψ_1 or ψ_2 from the system (3.8).

Conversely, we have an analogue of Proposition 3.4 for the operators H_i:

Proposition 12 *The left ideal of difference operators in $\mathfrak{D}$ that annihilate ψ_i is the principal left ideal generated by the operator H_i.*

Proof. Suppose that $A \in \mathfrak{D}$ is an operator such that $A\psi_1 = 0$. Then Proposition 3.4 implies that there exist operators $C, D \in \mathfrak{D}$ such that

$$A = C(T_2 - \alpha) - D\beta, \quad -C\beta + D(T_1 - \alpha) = 0.$$

Expressing $C = D(T_1 - \alpha)(\beta)^{-1}$ from the second equation and plugging it in to the first, we get that $A = D(t_1\beta)^{-1}H_1$. The case of ψ_2 is similar.

These two propositions allow us to write our hierarchy as a system of rank one difference equations of degree two.

Proposition 13 *The discrete modified Novikov-Veselov hierarchy (3.22) is equivalent to either of the following two systems of equations*

$$\frac{\partial}{\partial \tau_s^i} H_1 + H_1 A_{s,1}^i \equiv 0 \ mod \ H_1, \tag{4.3}$$

$$\frac{\partial}{\partial \tau_s^i} H_2 + H_2 A_{s,2}^i \equiv 0 \ mod \ H_2. \tag{4.4}$$

We now use this approach to construct the equations corresponding to times τ_1^1, τ_1^2, τ_2^1 and τ_2^2.

The functions ψ_1 and ψ_2 have the following power series expansions at the marked points $P_i^\pm$, where by k we denote the appropriate local coordinate $k_i^\pm$:

$$\begin{aligned}
\psi_1(n,m,\tau;P) &= k^{\pm n} \exp\left(\pm \sum_{\sigma=1}^\infty \tau_\sigma^1 k^\sigma\right) \cdot \left(\sum_{\alpha=0}^\infty \xi_{1,\alpha}^\pm(n,m,\tau)k^{-\alpha}\right) \text{ as } P \to P_1^\pm, \\
\psi_1(n,m,\tau;P) &= k^{\pm m} \exp\left(\pm \sum_{\sigma=1}^\infty \tau_\sigma^2 k^\sigma\right) \cdot \left(\sum_{\alpha=0}^\infty \xi_{2,\alpha}^\pm(n,m,\tau)k^{-\alpha}\right) \text{ as } P \to P_2^\pm, \\
\psi_2(n,m,\tau;P) &= k^{\pm n} \exp\left(\pm \sum_{\sigma=1}^\infty \tau_\sigma^1 k^\sigma\right) \cdot \left(\sum_{\alpha=0}^\infty \chi_{1,\alpha}^\pm(n,m,\tau)k^{-\alpha}\right) \text{ as } P \to P_1^\pm, \\
\psi_2(n,m,\tau;P) &= k^{\pm m} \exp\left(\pm \sum_{\sigma=1}^\infty \tau_\sigma^2 k^\sigma\right) \cdot \left(\sum_{\alpha=0}^\infty \chi_{2,\alpha}^\pm(n,m,\tau)k^{-\alpha}\right) \text{ as } P \to P_2^\pm,
\end{aligned} \tag{4.5}$$

where the $\xi_{i,s}^\pm(n,m,\tau)$ and $\chi_{i,s}^\pm(n,m,\tau)$ are analytic functions in the variables τ, and $\xi_{2,0}^+ = 0$, $\chi_{1,0}^+ = 0$. To make our notation consistent with (2.8)-(2.9), we denote

$$a_i^\pm = \xi_{i,0}^\pm, \quad b_i^\pm = \chi_{i,0}^\pm \tag{4.6}$$

$$c_i^\pm = \xi_{i,1}^\pm, \quad d_i^\pm = \chi_{i,1}^\pm \tag{4.7}$$

Plugging these expressions into (3.8), we see that these coefficients satisfy the following system of equations:

$$t_2\xi_{1,\alpha}^\pm = \alpha\xi_{1,\alpha}^\pm + \beta\chi_{1,\alpha}^\pm \tag{4.8}$$

$$t_2\xi^{\pm}_{2,\alpha\pm 1} = \alpha\xi^{\pm}_{2,\alpha} + \beta\chi^{\pm}_{2,\alpha} \tag{4.9}$$
$$t_1\chi^{\pm}_{1,\alpha\pm 1} = \beta\xi^{\pm}_{1,\alpha} + \alpha\chi^{\pm}_{1,\alpha} \tag{4.10}$$
$$t_1\chi^{\pm}_{2,\alpha} = \beta\xi^{\pm}_{2,\alpha} + \alpha\chi^{\pm}_{2,\alpha} \tag{4.11}$$

Also, since the functions ψ_1 and ψ_2 satisfy the normalization conditions (2.18)-(2.20), we also have

$$a_1^+a_1^- = 1, \quad b_2^+b_2^- = 1. \tag{4.12}$$

We now derive the dmNV equation corresponding to time τ_1^1 using its equivalent form (4.3). Let $\dot{f}$ denote differentiation by τ_1^1. We denote $A^1_{1,1} = AT_1 + BT_1^{-1} + C$ and $H_1 = T_1T_2 + xT_1 + yT_2 + z$. The equation in time τ_1^1 has the form

$$-\dot{x}T_1 - \dot{y}T_2 - \dot{z} \equiv (T_1T_2 + xT_1 + yT_2 + z)(AT_1 + BT_1^{-1} + C) \bmod H_1. \tag{4.13}$$

First, we express all of the coefficients of the above equation in terms of the variables a_1^+, b_2^+, α and β. The coefficients x, y, z of H_1 were found above in Proposition 4.1:

$$x = -t_1\alpha, \quad y = -\frac{\alpha(t_1\beta)}{\beta}, \quad z = \frac{t_1\beta}{\beta}. \tag{4.14}$$

To calculate the coefficients of the operator $A^1_{1,1}$, we use the method of Proposition 3.3. Comparing singularities, we see that if

$$A = f^1_{1,1,1} = \frac{a_1^+}{t_1a_1^+}, \quad B = f^1_{1,1,-1} = -\frac{a_1^-}{t_1^{-1}a_1^-} = -\frac{t_1^{-1}a_1^+}{a_1^+}, \tag{4.15}$$

then the functions ψ_1 and $\dot{\psi}_1 - AT_1\psi_1 - BT_1^{-1}\psi$ are proportional. Hence we can determine the third coefficient $C = f^1_{1,1,0}$ by comparing these two functions at either P_2^+ or P_2^-, which gives us two alternative expressions:

$$C = f^1_{1,1,0} = \frac{1}{c_2^+}\left(\frac{\partial c_2^+}{\partial\tau_1^1} - \frac{a_1^+}{t_1a_1^+}t_1c_2^+ + \frac{a_1^-}{t_1^{-1}a_1^-}t_1^{-1}c_2^+\right) = \frac{1}{a_2^-}\left(\frac{\partial a_2^-}{\partial\tau_1^1} - \frac{a_1^+}{t_1a_1^+}t_1a_2^- + \frac{a_1^-}{t_1^{-1}a_1^-}t_1^{-1}a_2^-\right). \tag{4.16}$$

We first these expressions by removing the coefficients a_2^- and c_2^+. From the system (4.8-4.11) we get that $c_2^+ = (t_2^{-1}\beta)(t_2^{-1}b_2^+)$ and $a_2^- = -\beta/(\alpha b_2^+)$. Using $t_1b_2^+ = \alpha b_2^+$, the first expression becomes

$$C = \frac{t_2^{-1}\dot{\beta}}{t_2^{-1}\beta} + \frac{t_2^{-1}\dot{b}_2^+}{t_2^{-1}b_2^+} - \frac{a_1^+}{t_1a_1^+}\frac{(t_2^{-1}\alpha)(t_1t_2^{-1}\beta)}{t_2^{-1}\beta} + \frac{t_1^{-1}a_1^+}{a_1^+}\frac{t_1^{-1}t_2^{-1}\beta}{(t_2^{-1}\beta)(t_1^{-1}t_2^{-1}\alpha)}$$

and the second expression becomes

$$C = f^1_{1,1,0} = \frac{\dot{\beta}}{\beta} - \frac{\dot{\alpha}}{\alpha} - \frac{\dot{b}_2^+}{b_2^+} - \frac{a_1^+}{t_1a_1^+}\frac{t_1\beta}{\beta(t_1\alpha)} + \frac{t_1^{-1}a_1^+}{a_1^+}\frac{\alpha(t_1^{-1}\beta)}{\beta}.$$

Expanding the right hand side of (4.13), we get

$$H_1A^1_{1,1} = (t_1t_2A)T_1^2T_2 + x(t_1A)T_1^2 + [t_1t_2C + y(t_2A)]\,T_1T_2 + [x(t_1C) + zA]\,T_1 +$$
$$+ [t_1t_2B + y(t_2C)]\,T_2 + x(t_1B) + zC + y(t_2B)T_1^{-1}T_2 + zBT_1^{-1}.$$

This expression is a Laurent polynomial in T_1 and T_2 whose terms have degrees i and j in T_1 and T_2, respectively, where $i = -1, 0, 1, 2$ and $j = 0, 1$. We need to express it as a left multiple of H_1 plus an operator containing terms of degrees $(0, 0)$, $(0, 1)$ and $(1, 0)$. First, to cancel the term containing $T_1^2 T_2$, we subtract the following left multiple of H_1:

$$(t_1t_2A)T_1H_1 = (t_1t_2A)T_1^2T_2 + (t_1t_2A)(t_1x)T_1^2 + (t_1t_2A)(t_1y)T_1T_2 + (t_1t_2A)(t_1z)T_1.$$

Using (4.1), (4.15) and (4.8), we see that the coefficient in front of T_1^2 in this difference vanishes:

$$x(t_1A) - (t_1x)(t_1t_2A) = -(t_1\alpha)\frac{t_1a_1^+}{t_1^2a_1^+} + (t_1^2\alpha)\frac{t_1t_2a_1^+}{t_1^2t_2a_1^+} = 0.$$

Similarly, to cancel the term containing $T_1^{-1}T_2$, we subtract

$$y(t_2B)T_1^{-1}y^{-1}H_1 = \frac{y(t_2B)}{t_1^{-1}y}T_2 + \frac{y(t_2B)}{t_1^{-1}y}(t_1^{-1}x) + y(t_2B)T_1^{-1}T_2 + \frac{y(t_2B)}{t_1^{-1}y}(t_1^{-1}z)T_1^{-1},$$

and using (4.1), (4.15), (4.8) and the relation (4.12), we show that the coefficient in front of T_1^{-1} vanishes:

$$zB - \frac{y(t_2B)}{t_1^{-1}y}(t_1^{-1}z) = 0.$$

Hence, we see that

$$H_1A_{1,1}^1 \equiv [t_1t_2C + y(t_2A) - (t_1t_2A)(t_1y)]\,T_1T_2 + [x(t_1C) + zA - (t_1t_2A)(t_1z)]\,T_1 + \\ + \left[t_1t_2B + y(t_2C) - \frac{y(t_2B)}{t_1^{-1}y}\right]T_2 + x(t_1B) + zC - \frac{y(t_2B)}{t_1^{-1}y}(t_1^{-1}x) \mod H_1.$$

Finally, to obtain the evolution equation, we subtract $[t_1t_2C + y(t_2A) - (t_1t_2A)(t_1y)]\,H_1$ from the right hand side of the equation, and obtain the following equations:

$$-\dot{x} = x(t_1C) + zA - (t_1t_2A)(t_1z) - x\,[t_1t_2C + y(t_2A) - (t_1t_2A)(t_1y)]\,, \tag{4.17}$$

$$-\dot{y} = t_1t_2B + y(t_2C) - \frac{y(t_2B)}{t_1^{-1}y} - y\,[t_1t_2C + y(t_2A) - (t_1t_2A)(t_1y)]\,, \tag{4.18}$$

$$-\dot{z} = x(t_1B) + zC - \frac{y(t_2B)}{t_1^{-1}y}(t_1^{-1}x) - [t_1t_2C + y(t_2A) - (t_1t_2A)(t_1y)]\,. \tag{4.19}$$

Since the coefficients x, y, z of H are expressed in terms of α and β, which are in turn related by the equation $\alpha^2 - \beta^2 = 1$, it is sufficient to find one of the derivatives, for example $\dot{x}$. Expanding the expression for $\dot{x}$ and using the expressions for the coefficients x, y, z and A, B, C obtained above (using the first expression for C in t_1t_2C and using the second one in t_1C), we obtain the following equation

$$\frac{t_1\dot{b}_2^+}{t_1b_2^+} = \frac{a_1^+}{t_1a_1^+}\frac{\beta(t_1\beta)}{t_1\alpha}, \tag{4.20}$$

which is the first equation of the dmNV hierarchy.

It seems natural to replace the variables a_1^+ and b_2^+ with their logarithms, i.e. to introduce new variables $a_1^+ = e^\varphi$ and $b_2^+ = e^\psi$. Since $\alpha = t_2a_1^+/a_1^+ = t_1b_2^+/b_2^+$, these variables are related by the equation

$$t_2\varphi - \varphi = t_1\psi - \psi. \tag{4.21}$$

Writing the evolution equation (4.20) in terms of these new variables, we get

$$\frac{\partial\psi}{\partial\tau_1^1} = \sqrt{\left(e^{2t_1^{-1}t_2\varphi} - e^{2t_1^{-1}\varphi}\right)\left(e^{-2\varphi} - e^{-2t_2\varphi}\right)} \tag{4.22}$$

To derive the evolution equation for time τ_1^2, we use its equivalent form (4.4). The calculations in this case are identical to those performed above. In fact, since our problem is symmetric with respect to exchanging the marked points $P_1^\pm$ and $P_2^\pm$, we can obtained the desired equation simply by exchanging the functions a_1^+ and b_2^+ and simultaneously exchanging the shift operators t_1 and t_2 in the evolution equation in time τ_1^1 (4.22). This gives us the following equation:

$$\frac{t_2\dot{a}_1^+}{t_2a_1^+} = \frac{\beta(t_2\beta)}{t_2\alpha}\frac{b_2^+}{t_2b_2^+}. \tag{4.23}$$

In terms of the logarithmic variables, this equation reads

$$\frac{\partial\varphi}{\partial\tau_1^2} = \sqrt{\left(e^{2t_1t_2^{-1}\psi} - e^{2t_2^{-1}\psi}\right)\left(e^{-2\psi} - e^{-2t_1\psi}\right)} \tag{4.24}$$

The derivation of the equations for times τ_2^1 and τ_2^2 involves similar calculations. For time τ_2^1, we use the equivalent form (4.3):

$$-\frac{\partial H_1}{\partial\tau_2^1} = H_1A_{2,1}^1 \bmod H_1. \tag{4.25}$$

Here $A_{2,1}$ is a Laurent polynomial in T_1 with terms of degree -2 to 3. As above, we successively subtract appropriate left multiples of H_1 to cancel the terms containing $T_1^iT_2$ for $i = 3, -2, 2, -1$. At every step, the corresponding T_1^i term vanishes. Finally, canceling the T_1T_2 term gives us the following equation:

$$\frac{t_1\dot{b}_2^+}{t_1b_2^+} = \frac{\beta(t_1\beta)}{t_1\alpha}\frac{1}{t_1a_1^+}c_1^+ - \frac{\beta(t_1\beta)}{t_1\alpha}\frac{a_1^+}{(t_1a_1^+)(t_1^2a_1^+)}t_1^2c_1^+ + \frac{\beta(t_1^2\beta)}{(t_1\alpha)(t_1^2\alpha)}\frac{a_1^+}{t_1^2a_1^+} + \frac{\alpha(t_1^{-1}\beta)(t_1\beta)}{t_1\alpha}\frac{t_1^{-1}a_1^+}{t_1a_1^+}, \tag{4.26}$$

where the functions a_1^+, b_2^+, c_1^+, α and β in the equation satisfy the following relations:

$$\alpha = \frac{t_2a_1^+}{a_1^+} = \frac{t_1b_2^+}{b_2^+}, \quad \alpha^2 - \beta^2 = 1, \quad t_2c_1^+ = \alpha c_1^+ + \beta(t_1^{-1}\beta)(t_1^{-1}a_1^+). \tag{4.27}$$

5 Explicit formulas

In this section we give explicit formulas for the functions $\psi_i(n, m, \tau; P)$ in terms of the theta-functions of the surface X. Choose a basis a_j, b_j, $j = 1, \ldots, g$ of $H_1(X, \mathbb{Z})$ with canonical intersection form, i.e. such that $a_j \circ a_k = 0$, $b_j \circ b_k = 0$, $a_j \circ b_k = \delta_{jk}$. Let B be the period matrix of the curve X with respect to this basis. Let Ω_1^1 and Ω_2^1 denote Abelian differentials of the third kind with poles at $P_1^\pm$ and $P_2^\pm$:

$$\Omega_i^1 = d(k_i^\pm)^{-1}\left(\mp k_i^\pm + O(1)\right) \text{ as } P \to P_i^\pm$$

which are normalized to have zero periods over the a-cycles. Let Ω_i^s denote Abelian differentials of the second kind with poles at $P_i^\pm$ and principal parts

$$\Omega_i^s = d(k_i^\pm)^{-1}\left(\mp s(k_i^\pm)^{s+1} + O(1)\right) \text{ as } P \to P_i^\pm,$$

and with zero a-periods, and which are odd with respect to the involution σ. It is a standard fact that these differentials exist and are unique. Let U_i^1 and U_i^k denote the vectors of the b-periods of these differentials:

$$(U_i^1)_j = \frac{1}{2\pi i}\oint_{b_j}\Omega_i^1, \quad (U_i^s)_j = \frac{1}{2\pi i}\oint_{b_j}\Omega_i^s.$$

Choose a base point $P_0 \in X$ away from the marked points $P_i^{\pm}$ and the divisor D, and let $A : X \to J(X)$ denote the Abel map with base point P_0, where $J(X)$ is the Jacobian variety of X. Let $\theta(z|B)$ denote the theta function of $J(X)$ for $z \in \mathbb{C}^g$. Introduce the functions

$$r_1(P) = \frac{\theta(A(P) - A(P_2^+) - \sum_{i=2}^g A(P_i) - K|B)\theta(A(P) - \sum_{i=1}^{g+1} A(P_i) + A(P_2^+) - K|B)}{\theta(A(P) - \sum_{i=1}^g A(P_i) - K|B)\theta(A(P) - \sum_{i=2}^{g+1} A(P_i) - K|B)},$$

$$r_2(P) = \frac{\theta(A(P) - A(P_1^+) - \sum_{i=2}^g A(P_i) - K|B)\theta(A(P) - \sum_{i=1}^{g+1} A(P_i) + A(P_1^+) - K|B)}{\theta(A(P) - \sum_{i=1}^g A(P_i) - K|B)\theta(A(P) - \sum_{i=2}^{g+1} A(P_i) - K|B)}.$$

By construction, these are meromorphic functions on X whose pole divisor is $D = \sum_{i=1}^{g+1} P_i$ and whose zero divisors are $P_2^+ + D_1$ and $P_1^+ + D_2$, respectively, where D_1 and D_2 are some divisors of degree g.

We define the functions ψ_1 and ψ_2 by the following formulas:

$$\psi_i(n, m, \tau; P) = r_i(P)C_i(n, m, \tau)F_i(n, m, \tau; P)\exp\left[n\int_{P_0}^P \Omega_1^1 + m\int_{P_0}^P \Omega_2^1 + \sum_{s=1}^{\infty}\sum_{i=1}^{2}\tau_s^i\int_{P_0}^P \Omega_i^s\right], \quad (5.1)$$

where the function $F(n, m, \tau; P)$ is defined as

$$F_i(n, m, \tau; P) = \frac{\theta\left(A(P) - A(D_i) + nU_1^1 + mU_2^1 + \sum_{s=1}^{\infty}\sum_{i=1}^{2}\tau_s^i U_i^s\right)}{\theta\left(A(P) - A(D_i) - K\right)}$$

and the path of integration in the exponent is the same as in the Abel map in F_i. By construction, these are single-valued functions on the curve X, having the required meromorphic and exponential singularities at the marked points, and having pole divisor D away from the marked points.

The constants $C_i(n, m, \tau)$ are determined by the normalization conditions (2.18)-(2.20). Choose paths of integration $\gamma_i : [0, 1] \to X$ from P_0 to P_i^+ and a path γ from P_0 to $\sigma(P_0)$. We assume that the integration path in $\psi_i(P)$ is γ_i and that the path in $\psi_i(\sigma(P))$ is γ followed by the image of γ_i under σ. Writing out the expression for $\psi_i(P)\psi_i(\sigma(P))$ using (5.1), we see that we need to choose the constants $C_i(n, m, \tau)$ as follows:

$$\frac{1}{C_i(n, m, \tau)^2} = r_i(P_i^+)r_i(P_i^-)F_i(n, m, \tau; P_i^+)F_i(n, m, \tau; P_i^-)\exp\left[nI_i^1 + mI_i^2 + \sum_{s=1}^{\infty}\sum_{i=1}^{2}\tau_s^i\int_\gamma \Omega_i^s\right] \quad (5.2)$$

where the path of integration in the $F_i(n, m, \tau; P_i^-)$ factor is γ followed by $\sigma(\gamma_i)$, and the constants I_i^1 and I_i^2 are the principal values of the integrals of Ω_1^1 and Ω_2^1 along the path $-\gamma_i + \gamma + \sigma(\gamma_i)$:

$$I_i^k = \lim_{t\to 1}\left(\int_{\gamma(0)}^{\gamma(t)}\Omega_k^1 + \int_\gamma \Omega_k^1 + \int_{\sigma(\gamma(0))}^{\sigma(\gamma(t))}\Omega_k^1\right), \quad k = 1, 2. \quad (5.3)$$

Finally, we choose the signs of $C_i(n, m, \tau)$ in such a way that the functions ψ_i satisfy the equation (2.20).

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