Abelian tropical covers

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Abstract

The goal of this article is to classify unramified covers of a fixed tropical base curve Γ with an action of a finite abelian group G that preserves and acts transitively on the fibers of the cover. We introduce the notion of dilated cohomology groups for a tropical curve Γ, which generalize simplicial cohomology groups of Γ with coefficients in G by allowing nontrivial stabilizers at vertices and edges. We show that G-covers of Γ with a given collection of stabilizers are in natural bijection with the elements of the corresponding first dilated cohomology group of Γ.

Contents

1 Introduction

Class field theory is a pillar of algebraic number theory; it is mostly concerned with classifying finite abelian extensions of a fixed local or global field K. Similarly, abelian covers of a fixed Riemann surface X can be classified in terms of its first homology group $H_1(X,\mathbb{Z})$, or in terms of its Jacobian $J(X) \simeq H_1(X,\mathbb{R}/\mathbb{Z})$. André Weil, in a letter to his sister from 1940 (see [Wei79]), pointed out the analogy between these two situations, as well as a potential bridge: the theory of abelian extensions of function fields over finite fields, an area that is now known as geometric class field theory (see [Ser88]).

More recently, another analogy has entered the mathematical stage: between a Riemann surface X and a metric graph Γ, or more generally a tropical curve $\sqsubset$. Many classical geometric constructions for Riemann surfaces, such as the theory of divisors, linear equivalence, Jacobians, theta functions, and moduli spaces, have natural analogues for tropical curves, as beautifully illustrated in [MZ08].

The success of this analogy is, of course, not a coincidence. A tropical curve naturally arises as the dual graph Γ_X of a semistable degeneration $\mathcal{X}$ of an algebraic curve X (with the metric encoding the deformation parameters at the nodes of $\mathcal{X}$). Geometric constructions on Γ_X then naturally arise as combinatorial specializations of their classical counterparts on X. We refer the reader for example to [BJ16] for a survey of this story in the case of linear series.

In this article, we develop a theory of G-covers of a tropical curve $\sqsubset$, where G is a finite abelian group. A G-cover of $\sqsubset$ is an unramified harmonic morphism $\sqsubset' \to \sqsubset$ (such morphisms were studied in [CMR16] under the name of tropical admissible covers), together with an action of G on $\sqsubset'$ that preserves and acts transitively on the fibers. We show that such covers are classified by two objects. The first is a *dilation stratification* $\mathcal{S}$ of $\sqsubset$, indexed by the subgroups of G, that encodes the local stabilizer subgroups (see Def. 4.23). The second is an element of a *dilated cohomology group* $H^1(\sqsubset, S)$ associated to $\sqsubset$ and a dilation stratification $\mathcal{S}$ of $\sqsubset$ (see Def. 4.24). In the spirit of the above analogies, one may think of this work as the starting point for a tropical version of class field theory.

Our principal result is the following (see Thm. 4.25):

Theorem A. *Let $\sqsubset$ be a tropical curve, let G be a finite abelian group, and let $\mathcal{S}$ be an admissible dilation stratification of $\sqsubset$. Then there is a natural bijection between the set of unramified G-covers of $\sqsubset$ having dilation stratification $\mathcal{S}$ and the dilated cohomology group $H^1(\sqsubset,\mathcal{S})$.*

The main technical ingredient in the classification of G-covers of a tropical curve $\sqsubset$ is a theory of dilated cohomology groups of a graph marked by subgroups of G. This theory generalizes simplicial cohomology with coefficients in G and satisfies a number of natural properties such as functoriality and pullback, and admits a long exact sequence. It seems natural to generalize dilated cohomology to arbitrary simplicial complexes, but this is beyond the scope of our paper. Since our methods are cohomological, they do not readily generalize to non-abelian groups. In a future paper, we plan to treat the non-abelian case by relating dilated cohomology to Bass–Serre theory [Ser80, Bas93] and developing a Galois theory for non-abelian unramified covers of tropical curves.

Earlier and related works

A number of authors study graphs and tropical curves with a group action. The simplest example is the case of tropical hyperelliptic curves, which are $\mathbb{Z}/2\mathbb{Z}$-covers of a tree ([BN09], [Cha13], [Cap14], [ABBR15b], [Pan16], [BBC17], [Len17]). Brandt and Helminck [BH17] consider arbitrary cyclic covers of a tree, while Helminck [Hel17] looks at the tropicalization of arbitrary abelian covers of algebraic curves from a non-Archimedean perspective. Jensen and Len [JL18] classify unramified $\mathbb{Z}/2\mathbb{Z}$-covers of arbitrary tropical curves in terms of dilation cycles, which is a special case of our dilation stratification; with this article we aim to generalize this aspect of their work.

While we do not pursue this direction here, G-covers of curves may be used to produce interesting loci of special divisors and linear series. For instance, Jensen and Len [JL18] and Len and Ulirsch [LU19] develop a theory of tropical Prym varieties associated to $\mathbb{Z}/2\mathbb{Z}$-covers of tropical curves, with applications to algebraic Prym–Brill–Noether theory. In a similar vein, Song [Son19] considers G-invariant linear systems with the goal of studying their descent properties to the quotient.

From a moduli-theoretic perspective, studying degenerations of G-covers of algebraic curves is equivalent to studying the compactification of the moduli space of G-covers in terms of the moduli space of G-admissible covers, as constructed in [ACV03] and [BR11]. In [BR11, Section 7] the authors have already introduced a graph-theoretic gadget to understand the boundary strata of this moduli space: so-called *modular graphs* with an action of a finite (not necessarily abelian) group G.

This idea seems to have appeared independently in other works as well: Chiodo and Farkas [CF17] study the boundary of the moduli space of level curves, which is equivalent to a component of the moduli space of G-admissible covers for a cyclic group G, and look at cyclic covers of an arbitrary graph. Their work has been extended to an arbitrary finite group G by Galeotti in [Gal19a, Gal19b]. Finally, in [SvZ18], Schmitt and van Zelm apply a graph-theoretic approach to the boundary of the moduli space of G-admissible covers (for an arbitrary finite group G) to study their pushforward classes in the tautological ring of $\overline{\mathcal{M}}_{g,n}$.

In [CMR16] Cavalieri, Markwig, and Ranganathan develop a moduli-theoretic approach to the tropicalization of the moduli space of admissible covers (without a fixed group operation). We extend this aspect of their article to the moduli space of G-admissible covers in Section 5 below. In [CMP19], Caporaso, Melo, and Pacini study the tropicalization of the moduli space of spin curves, which, in view of the results in [JL18], is closely related to to our story in the case $G = \mathbb{Z}/2\mathbb{Z}$.

The problem of classifying covers of a graph with an action of a given group (not necessarily abelian) was studied by Corry in [Cor11, Cor12, Cor15]. However, Corry considered a different category of graph morphisms, allowing edge contraction but not dilation. To the best of our knowledge, no author has considered the problem of classifying all unramified covers of a given graph with an action of a fixed group.

Analogies in topology and algebraic geometry

It is instructive to recall the theory of abelian covers in two categories, both directly related to tropical geometry: topological covering spaces and algebraic étale covers.

Topological spaces

Let X be a path-connected, locally path-connected and semi-locally simply connected topological space, let $x_0 \in X$ be a base point, and let G be a group. A *regular* G-*cover* of (X, x_0) is a based covering space $(Y, y_0) \to (X, x_0)$ together with an G-action on Y such that G acts freely and transitively on fibers. Based regular G-covers of (X, x_0) are classified by monodromy homomorphisms $\pi_1(X, x_0) \to G$ (the cover is connected if and only if the homomorphism is surjective). If G is a finite abelian group, then we can identify the set of such homomorphisms, canonically and independently of x_0, with the cohomology group $H^1(X, G)$. We note that a G-cover is rigidified by the G-action: for example, if p is a prime number, there is a single connected degree p covering space $S^1 \to S^1$, but there are $p-1$ connected $\mathbb{Z}/p\mathbb{Z}$-covers of S^1 corresponding to the non-trivial elements of $H^1(S^1, \mathbb{Z}/p\mathbb{Z}) \simeq \mathbb{Z}/p\mathbb{Z}$.

If X is the underlying topological space of a tropical curve $\sqsubset$, then any regular G-cover $X' \to X$ can be given the structure of an unramified G-cover $\sqsubset' \to \sqsubset$ of tropical curves by pulling back the genus function from $\sqsubset$ to $\sqsubset'$. These G-covers, which we call *topological*, have the property that G-action on the fibers is free (see Ex. 3.3 and Ex. 4.10). The corresponding dilation stratification $\mathcal{S}$ on $\sqsubset$ is trivial, and the dilated cohomology group $H^1(\sqsubset, \mathcal{S})$ reduces to $H^1(X, G)$.

Algebraic varieties

Let X be an algebraic variety over a field k and x_0 a geometric base point of X. Like its topological counterpart, the étale fundamental group $\pi_1^{ét}(X, x_0)$ of X classifies finite étale covers of X. For a finite abelian group G the set of continuous homomorphisms $\mathrm{Hom}(\pi_1^{ét}(X, x_0), G)$ is equal to the set of Galois coverings of X with Galois group G. If X is a smooth projective curve, the abelian coverings of X naturally arise as pullbacks of (always abelian) coverings of its Jacobian J along the Abel-Jacobi map $X \to J$. In particular, we have an induced isomorphism $\pi_1^{ét}(X, x_0)^{ab} \simeq \pi_1^{ét}(J, x_0)$. In Section 5 we will see how our *a priori* purely combinatorial construction can be thought of a tropical limit of this well-known story.

Organization of the paper

Our paper is organized as follows. In Sec. 2, we review the necessary definitions from graph theory and tropical geometry. In Sec. 3, we introduce G-covers, G-dilation data, and dilated cohomology groups. We are primarily interested in classifying abelian covers of tropical curves, however, our constructions are purely graph-theoretic in nature and may be of interest to specialists in graph theory and topology. For this reason, we first develop the theory of G-covers for unweighted graphs. In Sec. 4 we prove our main classification results, and then extend them to weighted graphs, weighted metric graphs, and tropical curves. Finally, in Sec. 5 we relate our constructions to the tropicalization of the moduli space of admissible G-covers.

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2 Definitions and notation

We develop the theory of G-covers of graphs on several levels successively: graphs, weighted graphs, metric graphs, and tropical curves. In this section, we recall the necessary definitions from graph theory.

2.1 Graphs

We first consider unweighted graphs without a metric.

Definition 2.1. A *graph with legs* Γ, or simply a *graph*, consists of the following:

1. A finite set $X(\Gamma)$.
2. An idempotent *root map* $r : X(\Gamma) \to X(\Gamma)$.
3. An involution $\iota : X(\Gamma) \to X(\Gamma)$ whose fixed set contains the image of r.

The image $V(\Gamma)$ of r is the set of *vertices* of Γ, and its complement $H(\Gamma) = X(\Gamma) \backslash V(\Gamma)$ is the set of *half-edges* of Γ. The involution ι preserves $H(\Gamma)$ and partitions it into orbits of size 1 and 2; we call these respectively the *legs* and *edges* of Γ and denote the corresponding sets by $L(\Gamma)$ and $E(\Gamma)$. The root map assigns one root vertex to each leg and two root vertices to each edge. A *loop* is an edge whose root vertices coincide.

We note that, from a graph-theoretic point of view, there is essentially no difference between a leg and an extremal edge. This distinction is important, however, from a tropical viewpoint: legs are the tropicalizations of marked points, while an extremal edge represents a rational tail. Note that, unlike an extremal edge, a leg does not have a vertex at its free end.

The *tangent space* $T_v\Gamma$ and *valency* $\mathrm{val}(v)$ of a vertex $v \in V(\Gamma)$ are defined by

$$T_v\Gamma = \left\{ h \in H(\Gamma) \middle| r(h) = v \right\} \text{ and } \mathrm{val}(v) = \#(T_v\Gamma).$$

Definition 2.2. Let Γ be a graph. A *subgraph* Δ of Γ is a subset of $X(\Gamma)$ closed under the root and involution maps. Given a subgraph $\Delta \subset \Gamma$ and a vertex $v \in V(\Delta)$, we denote $\mathrm{val}_\Delta(v)$ the valency of v viewed as a vertex of Δ. A subgraph $\Delta \subset \Gamma$ is called a *cycle* if $\mathrm{val}_\Delta(v)$ is even for every $v \in V(\Delta)$. A subgraph $\Delta \subset \Gamma$ is called *edge-maximal* if every edge $e \in E(\Gamma)$ having both root vertices in Γ lies in Δ.

It is clear that a subgraph of Γ is edge-maximal if and only if it is the largest subgraph of Γ with a given set of vertices.

Definition 2.3. Let Γ be a graph. An *orientation* on Γ is a choice of order (h, h') on each edge $e = \{h, h'\} \in E(\Gamma)$. We call $s(e) = r(h)$ and $t(e) = r(h')$ the *source* and *target* vertices of e.

Definition 2.4. A *finite morphism* of graphs $\varphi : \Gamma' \to \Gamma$, or simply a *morphism*, is a map of sets $\varphi : X(\Gamma') \to X(\Gamma)$ which commutes with the root and involution maps, such that edges map to edges and legs map to legs.

An *automorphism* $\varphi : \Gamma \to \Gamma$ of a graph Γ is a morphism with an inverse. We denote the group of automorphisms of Γ by $\mathrm{Aut}(\Gamma)$. We remark that a nontrivial graph automorphism may act trivially on the vertex and edge sets. For example, the graph Γ consisting of one vertex v and one loop $e = \{h, h'\}$ has a nontrivial automorphism fixing v and exchanging h and h'. To form quotients of graphs by group actions, we need to exclude such automorphisms from consideration.

Definition 2.5. Let Γ be a graph, and let G be a group. A G-*action* on Γ is a homomorphism of G to the automorphism group $\mathrm{Aut}(\Gamma)$ such that for every $g \in G$, the corresponding automorphism does not flip edges. In other words, for every edge $e = \{h, h'\} \in E(\Gamma)$ either $\varphi(e) \neq e$, or $\varphi(h) = h$ and $\varphi(h') = h'$. Given a G-action on Γ, we define the *quotient graph* Γ/G by setting $X(\Gamma/G) = X(\Gamma)/G$. The root and involution maps on Γ are G-invariant and descend to Γ/G. It is clear that $V(\Gamma/G) = V(\Gamma)/G$ and $H(\Gamma/G) = H(\Gamma)/G$, and the no-flipping assumption implies that the G-action does not identify the two half-edges of any edge of Γ. Therefore $E(\Gamma/G) = E(\Gamma)/G$ and $L(\Gamma/G) = L(\Gamma)/G$, and the quotient map $\pi : \Gamma \to \Gamma/G$ is a finite morphism.

2.2 Weighted graphs, harmonic morphisms, and ramification

We now consider graphs with vertex weights. Heuristically, one may think of a vertex of weight g as an infinitesimally small graph with g loops (cf. [AC13, Section 5]).

Definition 2.6. A *weighted graph* (Γ, g) is a pair consisting of a graph Γ and a vertex weight function $g : V(\Gamma) \to \mathbb{Z}_{\geq 0}$.

We will usually suppress g and denote weighted graphs by Γ. We define the *Euler characteristic* $\chi(v)$ of a vertex $v \in V(\Gamma)$ on a weighted graph Γ as

$$\chi(v) = 2 - 2g(v) - \mathrm{val}(v).$$

The *genus* of a connected graph Γ is defined to be

$$g(\Gamma) = \#(E(\Gamma)) - \#(V(\Gamma)) + 1 + \sum_{v \in V(\Gamma)} g(v).$$

We define the *Euler characteristic* $\chi(\Gamma)$ of a graph Γ by

$$\chi(\Gamma) = \sum_{v \in V(\Gamma)} \chi(v);$$

this is not to be confused with the topological Euler characteristic of Γ. An easy calculation shows that, if Γ is connected, then

$$\chi(\Gamma) = 2 - 2g(\Gamma) - \#(L(V)).$$

A subgraph $\Delta \subset \Gamma$ of a weighted graph Γ is naturally given the structure of a weighted graph by restricting the weight function g. In this case, denote $\chi_\Delta(v) = 2 - 2g(v) - \mathrm{val}_\Delta(v)$ the Euler characteristic of a vertex v of Δ.

We say that a vertex $v \in \Gamma$ of a weighted graph Γ is *unstable* if $\chi(v) \geq 1$, *semistable* if $\chi(v) \leq 0$, and *stable* if $\chi(v) \leq -1$. An unstable vertex has genus zero and is either isolated or extremal. A semistable vertex that is not unstable is either an isolated vertex of genus one or a valency two vertex of genus zero, in which case we call it *simple*. We say that a graph Γ is *semistable* if all of its vertices are semistable, and *stable* if all of its vertices are stable.

Let Γ be a connected weighted graph with $\chi(\Gamma) < 0$. Following [ACP15, Section 8.2], we construct a stable graph Γ_{st}, called the *stabilization* of Γ, as follows. First, we construct the *semistabilization* Γ_{sst} of Γ by inductively removing all extremal edges ending at an extremal vertex of genus zero (but not the legs). The graph Γ_{sst} is a semistable subgraph of Γ, and it is clear that $\chi(\Gamma_{sst}) = \chi(\Gamma)$, and that any vertices of Γ_{sst} that are not stable are simple. We then construct Γ_{st} by gluing together the two half-edges at each simple vertex v of Γ_{sst}. Specifically, if v is an endpoint of two edges e_1 and e_2, we replace v, e_1, and e_2 with a new edge connecting the other endpoints of e_1 and e_2. If v is an endpoint of an edge e and a leg l, we replace v, e, and l with a new leg rooted at the other endpoint of e. The result is a stable graph Γ_{st} with $\chi(\Gamma_{st}) = \chi(\Gamma_{sst}) = \chi(\Gamma)$.

Definition 2.7. Let Γ and Γ' be graphs. A *finite harmonic morphism* $\varphi : \Gamma' \to \Gamma$, or simply a *harmonic morphism*, consists of a finite morphism $\Gamma' \to \Gamma$ and a map $d_\varphi : X(\Gamma') \to \mathbb{Z}_{>0}$, called the *degree* of φ, such that the following properties are satisfied:

1. If $e' = \{h'_1, h'_2\} \in E(\Gamma')$ is an edge then $d_\varphi(h'_1) = d_\varphi(h'_2)$. We call this number the *degree* of φ along e' and denote it $d_\varphi(e')$.

2. For every vertex $v' \in V(\Gamma')$ and every tangent direction $h \in T_{\varphi(v)}\Gamma$, we have

$$d_\varphi(v') = \sum_{\substack{h' \in T_{v'}\Gamma', \\ \varphi(h') = h}} d_\varphi(h').$$

   In particular, this sum does not depend on the choice of h.

Let $\varphi : \Gamma' \to \Gamma$ be a harmonic morphism of graphs, where Γ is connected. The sum

$$\deg(\varphi) = \sum_{\substack{v' \in V(\Gamma'), \\ \varphi(v') = v}} d_\varphi(v') = \sum_{\substack{e' \in E(\Gamma'), \\ \varphi(e') = e}} d_\varphi(e') = \sum_{\substack{l' \in L(\Gamma'), \\ \varphi(l') = l}} d_\varphi(l')$$

does not depend on the choice of $v \in V(\Gamma)$, $e \in E(\Gamma)$ or $l \in L(\Gamma)$ and is called the *degree* of φ (see Section 2 of [ABBR15a]).

Definition 2.8. Let $\varphi : \Gamma' \to \Gamma$ be a harmonic morphism of weighted graphs. The *ramification degree* $\mathrm{Ram}_\varphi(v')$ of φ at a vertex $v' \in V(\Gamma')$ is equal to

$$\mathrm{Ram}_\varphi(v') = d_\varphi(v')\chi(\varphi(v')) - \chi(v').$$

We say that φ is *effective* if

$$\mathrm{Ram}_\varphi(v') \geq 0$$

for all $v' \in V(\Gamma')$, and *unramified* if

$$\mathrm{Ram}_\varphi(v') = 0 \tag{1}$$

for all $v' \in V(\Gamma')$.

Remark 2.9. Unramified morphisms were studied extensively in [CMR16], where they were called *tropical admissible covers*. We partly preserve this terminology: for example, we call a dilation stratification admissible if it corresponds to an unramified cover. A simple calculation shows that our definition of ramification degree agrees with the standard one in the literature (see, for example, Sec. 2.2 in [ABBR15b] or Def. 16 in [CMR16]):

$$\mathrm{Ram}_\varphi(v') = d_\varphi(v')\left(2 - 2g(\varphi(v'))\right) - \left(2 - 2g(v')\right) - \sum_{h' \in T_{v'}\Gamma'} \left(d_\varphi(h') - 1\right).$$

For an unramified harmonic morphism $\varphi : \Gamma' \to \Gamma$, we call equation (1) the *local Riemann–Hurwitz condition* at $v' \in V(\Gamma')$. Adding together these conditions at all $v' \in V(\Gamma')$, we obtain the *global Riemann–Hurwitz condition*

$$\chi(\Gamma') = \deg(\varphi)\chi(\Gamma). \tag{2}$$

Example 2.10. Let $\varphi : \Gamma' \to \Gamma$ be an unramified harmonic morphism, and suppose that $d_\varphi(v') = 1$ for some v'. By the harmonicity condition, each $h \in T_{\varphi(v')}\Gamma$ has a unique preimage in $T_{v'}\Gamma'$, hence $\mathrm{val}(v') = \mathrm{val}(\varphi(v'))$. Furthermore, we have $\chi(\varphi(v')) = \chi(v')$, which implies that $g(v') = g(\varphi(v'))$. It follows that φ is a local isomorphism of weighted graphs in a neighborhood of v'. In particular, an unramified harmonic morphism of degree one is a graph isomorphism, and vice versa.

We observe that if $\varphi : \Gamma' \to \Gamma$ is an effective harmonic morphism and $\Delta \subset \Gamma$ is a subgraph with preimage $\Delta' = \varphi^{-1}(\Delta)$, then the induced map $\varphi|_{\Delta'} : \Delta' \to \Delta$ is also an effective harmonic morphism, since the ramification degree does not decrease when a half-edge and its preimages are removed. However, if φ is unramified, then $\varphi|_{\Delta'}$ is not necessarily unramified. We now show that unramified morphisms naturally restrict to stabilizations.

Let $\varphi : \Gamma' \to \Gamma$ be an unramified harmonic morphism of connected graphs, and assume that $\chi(\Gamma) < 0$ (or, equivalently by (2), that $\chi(\Gamma') < 0$). For any two vertices $v' \in V(\Gamma')$ and $v = \varphi(v') \in V(\Gamma)$, Eq. (1) implies that v' is unstable if and only if v is unstable, in which case $\chi(v') = \chi(v) = 1$ and $d_\varphi(v') = 1$. Let $v \in V(\Gamma)$ be an extremal vertex of genus 0, let $e \in E(\Gamma)$ be the unique edge rooted at v, and let $u \in V(\Gamma)$ be the other root vertex of v. By the above, we see that $v \in V(\Gamma)$ has $\deg(\varphi)$ preimages v'_i in Γ', each of which is a root vertex of a unique extremal edge e'_i mapping to e with local degree 1. For any $u' \in \varphi^{-1}(u)$, $d_\varphi(u')$ of the edges e'_i are rooted at u'. Therefore, removing v'_i, e'_i, v, and e increases $\chi(u)$ by 1 and increases each $\chi(u')$ by $d_\varphi(u')$, hence does not change the local Riemann–Hurwitz condition at u'. Proceeding in this way, we remove all unstable vertices of Γ' and Γ and obtain an unramified harmonic morphism $\varphi_{sst} : \Gamma'_{sst} \to \Gamma_{sst}$.

Similarly, we see that for two vertices $v' \in V(\Gamma'_{sst})$ and $v = \varphi(v') \in V(\Gamma_{sst})$, Eq. (1) implies that one is simple if and only if the other is. Furthermore, by the harmonicity condition, the degrees of φ at the two half-edges at v' are equal, hence we can remove v' and v, glue together the free half-edges, and extend φ; this does not change the local Riemann–Hurwitz condition at any remaining vertex of Γ'_{sst}. Proceeding in this way, we obtain an unramified morphism $\varphi_{st} : \Gamma'_{st} \to \Gamma_{st}$.

Definition 2.11. Let $\varphi : \Gamma' \to \Gamma$ be an unramified harmonic morphism of connected weighted graphs, such that $\chi(\Gamma) < 0$ (or, equivalently, $\chi(\Gamma') < 0$). The unramified morphism $\varphi_{st} : \Gamma'_{st} \to \Gamma_{st}$ constructed above is called the *stabilization* of φ.

Finally, we define the contraction of a graph along a subset of its edges; this can be viewed as a non-finite harmonic morphism of degree one.

Definition 2.12. Let Γ be a weighted graph, and let $S \subset E(\Gamma)$ be a set of edges of Γ. We define the *weighted edge contraction* Γ/S of Γ along S as follows. Let Δ be the minimal subgraph of Γ whose edge set contains S, and let $\Delta_1, \ldots, \Delta_k$ be the connected components of Δ. We obtain Γ/S from Γ by contracting each Δ_i to a vertex v_i of genus $g(\Delta_i)$.

Given a harmonic morphism $\varphi : \Gamma' \to \Gamma$ of weighted graphs, we can contract a subset of edges $S \subset E(\Gamma)$ of Γ, and their preimages in Γ'. Connected components of graphs map to connected components, and degree is constant when restricted to a connected component, so there is a natural harmonic morphism $\varphi_S : \Gamma'/\varphi^{-1}(S) \to \Gamma/S$. A simple calculation shows that if φ is unramified, then so is φ_S:

Proposition 2.13 (Proposition 19 in [CMR16])**.** *Let $\varphi : \Gamma' \to \Gamma$ be an unramified harmonic morphism of unweighted graphs, let $S \subset E(\Gamma)$ be a subset of the edges of Γ, and let $\Gamma'/\varphi^{-1}(S)$ and Γ/S be the weighted edge contractions. Then $\varphi_S : \Gamma'/\varphi^{-1}(S) \to \Gamma/S$ is unramified.*

2.3 Metric graphs and tropical curves

Finally, we consider weighted graphs with a metric, as well as tropical curves.

Definition 2.14. A *weighted metric graph* consists of a weighted graph (Γ, g) and a function $\ell : E(\Gamma) \to \mathbb{R}_{>0}$. A *finite harmonic morphism* of weighted metric graphs $\varphi : (\Gamma', \ell') \to (\Gamma, \ell)$, or simply a *harmonic morphism*, is a finite harmonic morphism $\varphi : \Gamma' \to \Gamma$ of the underlying weighted graphs such that for every edge $e' \in E(\Gamma')$ we have

$$\ell(\varphi(e')) = d_\varphi(e')\ell'(e'). \tag{3}$$

In other words, φ dilates each edge $e' \in E(\Gamma')$ by a factor of $d_\varphi(e')$. A harmonic morphism $\varphi : \Gamma' \to \Gamma$ of weighted metric graphs is called *effective* or *unramified* if it is so as a map of weighted graphs.

Remark 2.15. Given a finite harmonic morphism $\varphi : \Gamma' \to \Gamma$ of weighted graphs and a length function ℓ on Γ, there is a unique length function ℓ' on Γ' satisfying the dilation condition (3). Similarly, a length function on Γ' uniquely induces a length function on Γ. It follows that the classification of unramified covers of weighted metric graphs, in particular abelian covers, is independent of the choice of metric. For this reason, in this paper we mostly work with graphs and weighted graphs without metrics.

Given a connected weighted metric graph Γ with $\chi(\Gamma) < 0$, we give Γ_{st} the structure of a weighted metric graph in the obvious way, by setting $\ell(e) = \ell(e_1) + \ell(e_2)$ whenever we replace two edges e_1 and e_2 with a new edge e. It is clear that an unramified morphism of weighted metric graphs $\varphi : \Gamma' \to \Gamma$ induces an unramified morphism $\varphi_{st} : \Gamma'_{st} \to \Gamma_{st}$.

Definition 2.16. Let (Γ, ℓ) be a weighted metric graph. We define a metric space $|\Gamma|$, called the *metric realization* of (Γ, l), as follows. Consider a closed interval $I_e \subset \mathbb{R}$ of length $\ell(e)$ for each edge $e \in E(\Gamma)$, and a half-open interval $I_l = [0, \infty)$ for each leg $l \in L(\Gamma)$. We obtain $|\Gamma|$ from the I_e and the I_l by treating their endpoints as the root vertices and gluing accordingly. We then give $|\Gamma|$ the path metric.

A harmonic morphism $\varphi : (\Gamma', \ell') \to (\Gamma, \ell)$ of weighted metric graphs naturally induces a continuous map $|\varphi| : |\Gamma'| \to |\Gamma|$ where, for a pair of edges $e = \varphi(e')$, the map is given by dilation by a factor of $d_\varphi(e')$, and similarly for a pair of legs $l = \varphi(l')$. The map is piecewise-linear with integer slope with respect to the metric structure.

A basic inconvenience of tropical geometry is that different weighted metric graphs may have the same metric realizations. This motivates the following definition.

Definition 2.17. A *tropical curve* $(\sqsubset, g)$ is a pair consisting of a metric space $\sqsubset$ and a weight function $g : \sqsubset \to \mathbb{Z}_{\geq 0}$ such that there exists a weighted metric graph (Γ, g, ℓ) and an isometry $m : |\Gamma| \to \sqsubset$ of its metric realization with Γ such that the weight functions agree:

$$g(x) = \begin{cases} g(v) & \text{if } x = m(v) \text{ for a } v \in V(\Gamma), \\ 0 & \text{otherwise.} \end{cases}$$

We call a quadruple (Γ, g, ℓ, m) satisfying these properties a *model* for $\sqsubset$.

The *genus* of a connected tropical curve $\sqsubset$ is given by

$$g(\sqsubset) = b_1(\sqsubset) + \sum_{x \in \sqsubset} g(x)$$

and is equal to the genus of any model of $\sqsubset$.

For a point $x \in \sqsubset$ on a tropical curve $\sqsubset$ with model (Γ, g, l, m), we define its *valency* $\mathrm{val}(x)$ to be $\mathrm{val}(v)$ if $x = m(v)$ for some $v \in V(\Gamma)$ and 2 otherwise. We similarly define the Euler characteristic as $\chi(x) = 2 - 2g(x) - \mathrm{val}(x)$; these numbers do not depend on the choice of model. We define the Euler characteristic of a tropical curve $\sqsubset$ to be $\chi(\sqsubset) = \chi(\Gamma)$ for any model Γ. It is clear that

$$\chi(\sqsubset) = \sum_{x \in \sqsubset} \chi(x),$$

where $\chi(x) = 0$ for all but finitely many $x \in \sqsubset$.

Remark 2.18. Our definition differs from Def. 2.14 in [ABBR15b], where a tropical curve was defined as an equivalence class of weighted metric graphs up to tropical modifications.

Given a tropical curve $\sqsubset$ with model Γ, we can form another model Γ' by splitting any edge or leg of Γ at a new vertex. Conversely, any tropical curve $\sqsubset$ (other than $\mathbb{R}$ and S^1) has a unique *minimal model* Γ_{min} having no simple vertices. We say that a connected tropical curve $\sqsubset$ is *stable* if $\chi(x) \leq 0$ for all $x \in \sqsubset$, or, equivalently, if its minimal model is a stable graph. We define the *stabilization* of a connected tropical curve $\sqsubset$ with $\chi(\sqsubset) < 0$ by removing all trees of edges having no vertices of positive genus, or, equivalently, as the geometric realization of the stabilization of any model of $\sqsubset$.

Any tropical curve other than the real line has a well-defined set of maximal legs. A morphism of tropical curves is a continuous, piecewise-linear map that sends legs to legs and is eventually linear on each leg.

Definition 2.19. A *morphism* $\tau : \sqsubset' \to \sqsubset$ of tropical curves is a continuous, piecewise-linear map with integer slopes such that for any leg $l' \subset \sqsubset'$, there exists a leg $l \subset \sqsubset$ and numbers $a \in \mathbb{Z}_{>0}$ and $b \in \mathbb{R}$ such that, identifying l' and l with $[0, +\infty)$, we have $\tau(x) = ax + b \in l$ for $x \in l'$ sufficiently large. We note that τ may map a finite section of l' to $\sqsubset \backslash l$.

Let $\tau : \sqsubset' \to \sqsubset$ be a morphism of tropical curves. A *model* for τ is a pair of models (Γ', g', ℓ', m') and (Γ, g, ℓ, m) for $\sqsubset'$ and $\sqsubset$, respectively, and a morphism $\varphi : \Gamma' \to \Gamma$ of weighted metric graphs such that $m \circ |\varphi| = \tau \circ m'$. We say that τ is *harmonic*, *effective* or *unramified* if φ has the corresponding property.

Given a morphism $\tau : \sqsubset' \to \sqsubset$ of tropical curves, we construct a model $\varphi : \Gamma' \to \Gamma$ by choosing the vertex set $V(\Gamma')$ to contain the finite set of points where τ changes slope, and then enlarging $V(\Gamma')$ and $V(\Gamma)$ to ensure that the image and the preimage of a vertex is a vertex. We let the degree of φ on each edge and leg be the slope of τ. Given a model $\varphi : \Gamma' \to \Gamma$ of τ, we can produce another model by adding more vertices to Γ' and Γ. Conversely, any morphism $\tau : \sqsubset' \to \sqsubset$ to a tropical curve $\sqsubset$ with $\chi(\sqsubset) < 0$ has a unique *minimal model* $\varphi_{min} : \Gamma'_{min} \to \Gamma_{min}$ with the property that every simple vertex $v \in V(\Gamma_{min})$ has at least one preimage that is not simple.

Example 2.20. Let $\tau : \sqsubset' \to \sqsubset$ be an unramified morphism of tropical curves of local degree one. Then φ is a topological covering space of degree $\deg \tau$. Conversely, if $\sqsubset$ is a tropical curve and $f : \sqsubset' \to \sqsubset$ is a covering space of finite degree, then there is a unique way to give $\sqsubset'$ the structure of a tropical curve such that f is unramified: we define the genus function on $\sqsubset'$ as the pullback of the genus function on $\sqsubset$.

3 Dilated cohomology

In the following two sections, we fix a finite abelian group G and classify the G-covers of a given unweighted graph Γ. These are defined as surjective finite morphisms $\varphi : \Gamma' \to \Gamma$ together with an G-action on Γ' that preserves and acts transitively on the fibers. We will see that a G-cover of Γ is uniquely determined by two objects. The first is a G-*dilation datum* D on Γ (equivalently, a G-*stratification* $\mathcal{S}$ of Γ), recording the fibers of φ in terms of local stabilizer subgroups of G. The second is an element of a *dilated cohomology group* $H^1(\Gamma, D)$ (or $H^1(\Gamma, \mathcal{S})$), which generalizes the first simplicial cohomology group $H^1(\Gamma, G)$ by taking the local stabilizers into account.

We introduce G-covers, G-dilation data and G-stratifications in Sec. 3.1. In Sec. 3.2, we introduce the dilated cohomology groups $H^i(\Gamma, D)$ of a pair (Γ, D), where Γ is a graph and D is a G-dilation datum on Γ. In Sec. 3.3 we introduce the long exact sequence in dilated cohomology and study the cohomology groups of a subgraph $\Delta \subset \Gamma$. Once all the relevant definitions have been established, we reach Sec. 4, which is mostly dedicated to proving our classification results.

3.1 G-covers, dilation data, and stratifications

Throughout this section, we only consider unweighted graphs with legs. We now give the main definition of our paper.

Definition 3.1. Let Γ be a graph. A G-*cover* of Γ is a finite surjective morphism $\varphi : \Gamma' \to \Gamma$ together with an action of G on Γ', such that the following properties are satisfied:

1. The action is invariant with respect to φ.
2. For each $x \in X(\Gamma)$, the group G acts transitively on the fiber $\varphi^{-1}(x)$.

Example 3.2. Let Γ be a graph with a G-action (see Def. 2.5), then the quotient map $\pi : \Gamma \to \Gamma/G$ is a G-cover.

Example 3.3. Let Γ be a graph. Viewing Γ as a topological space, an element of $H^1(\Gamma, G)$ determines a covering space $\varphi : \Gamma' \to \Gamma$ with a G-action. It is clear that we can equip Γ' with the structure of a graph such that φ is a G-cover of graphs. Such G-covers, which we call *topological* G-*covers*, are distinguished by the property that G acts freely on each fiber $\varphi^{-1}(x)$. For such covers, the G-dilation datum is trivial, while the the dilated cohomology group is $H^1(\Gamma, G)$. An example with G the Klein group is given below in Fig. 1a.

Our goal is to describe all G-covers $\varphi : \Gamma' \to \Gamma$ of a given graph Γ. We begin our description by considering the local stabilizer subgroups.

Definition 3.4. Let Γ be a graph. A G-*dilation datum* D on Γ is a choice of a subgroup $D(x) \subset G$ for every $x \in X(\Gamma)$, such that $D(h) \subset D(r(h))$ for every half-edge $h \in H(\Gamma)$, and such that $D(h) = D(h')$ for each edge $e = \{h, h'\} \in E(\Gamma)$. Given G-dilation data D and D' on Γ, we say that D is a *refinement* of D' if $D(x) \subset D'(x)$ for all $x \in X(\Gamma)$. A G-*dilated graph* is a pair (Γ, D) consisting of a graph Γ and a G-dilation datum D on Γ.

We call $D(x)$ the *dilation group* of $x \in X(\Gamma)$, and for an edge $e = \{h, h'\} \in E(\Gamma)$ we call $D(e) = D(h) = D(h')$ the *dilation group* of e. If e is an edge with root vertices u and v (which may be the same), then $D(e) \subset D(u) \cap D(v)$. We call $C(e) = D(u) + D(v)$ the *vertex dilation group* of the edge e.

Remark 3.5. A G-dilation datum on a graph Γ is an example of a *graph of groups*, as defined by Bass (see Def. 1.4 in [Bas93]). In a future paper, we plan to explore the relationship between the cohomology groups $H^i(\Gamma, D)$ and the fundamental group of the graph of groups defined by D, with the goal of extending our theory to the non-abelian case.

Definition 3.6. Let $\varphi : \Gamma' \to \Gamma$ be a G-cover. We define the G-*dilation datum* D_φ of φ by setting $D_\varphi(x)$ for $x \in X(\Gamma)$ to be the stabilizer group of any $x' \in \varphi^{-1}(x)$.

The group G is assumed to be abelian, therefore the stabilizer group of $x' \in \varphi^{-1}(x)$ does not depend on the choice of x'.

Remark 3.7. If D_φ is the G-dilation datum of a G-cover φ that is the tropicalization of a G-cover of algebraic curves, then the dilation subgroup of every half-edge is cyclic (this follows, for instance, from [SvZ18, Lemma 3.1]). As a result, many of the covers described throughout this paper are not algebraically realizable, e.g. the cover 1f below. Our approach is to develop, as far as possible, an independent theory of G-covers of graphs, so we do not impose this condition from the start. In any case, as we shall see, the dilation groups of the half-edges play a secondary role in the classification of G-covers.

For any $x \in X(\Gamma)$, the fiber $\varphi^{-1}(x)$ of a G-cover $\varphi : \Gamma' \to \Gamma$ is a $G/D_\varphi(x)$-torsor. If $h \in T_v(\Gamma)$ is a half-edge rooted at $v \in V(\Gamma)$, then the root map $r : \varphi^{-1}(h) \to \varphi^{-1}(v)$ is an equivariant map of transitive G-sets, which implies that $D_\varphi(h) \subset D_\varphi(v)$. Furthermore, it is clear that $D_\varphi(h) = D_\varphi(h')$ for any edge $e = \{h, h'\} \in E(\Gamma)$. Therefore, D_φ is a G-dilation datum.

The cardinality of each fiber $\varphi^{-1}(x)$ equals the index of $D_\varphi(x)$ in G:

$$\#(\varphi^{-1}(x)) = [G : D_\varphi(x)].$$

Furthermore, for a half-edge $h \in H(\Gamma)$ rooted at $r(h) = v \in V(\Gamma)$, the $[G : D_\varphi(h)]$ half-edges in the fiber $\varphi^{-1}(h)$ are partitioned by their root vertices into $\#(\varphi^{-1}(v)) = [G : D_\varphi(v)]$ subsets, each containing $[D_\varphi(v) : D_\varphi(h)]$ elements.

Example 3.8 (Klein covers)**.** We now give several of examples of G-covers in the simplest non-cyclic case, when $G = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ is the Klein group. The base graph Γ consists of two vertices u and v joined by two edges e and f.

We use the following notation to describe a G-cover $\varphi : \Gamma' \to \Gamma$. We denote the elements of G by 00, 10, 01, and 11, and denote the subgroups generated by 10, 01, and 11 by respectively H_1, H_2, and H_3. The vertices of Γ' lying above u and v are labeled (non-uniquely if the corresponding stabilizer is non-trivial) u_{ij} and v_{ij} for $ij \in G$, and the action of G on $\varphi^{-1}(u)$ and $\varphi^{-1}(v)$ is the natural additive action on the indices. We color the edges $\varphi^{-1}(e)$ and $\varphi^{-1}(f)$ red and blue, respectively, and label them with indices ij in such a way that e_{ij} and f_{ij} are attached to u_{ij}. The sizes of the vertices and the thickness of the edges of Γ' denote the size of the dilation subgroup. In the caption, we indicate the nontrivial dilation groups. In Ex. 4.7, we will enumerate all Klein covers of Γ.

We now give an alternative way to record a G-dilation datum on Γ, by means of a stratification of Γ indexed by the subgroups of G. This description is often easier to visualize, and generalizes more naturally to tropical curves.

Definition 3.9. Let Γ be a graph. A G-*stratification* $\mathcal{S} = \{\Gamma_H | H \in S(G)\}$ on Γ is a collection of subgraphs $\Gamma_H \subset \Gamma$ indexed by the set $S(G)$ of subgroups of G, such that

$$\begin{aligned} \Gamma_0 &= \Gamma, \\ \Gamma_K &\subset \Gamma_H \text{ if } H \subset K, \text{ and} \\ \Gamma_H \cap \Gamma_K &= \Gamma_{H+K} \text{ for all } H, K \in S(G). \end{aligned} \tag{4}$$

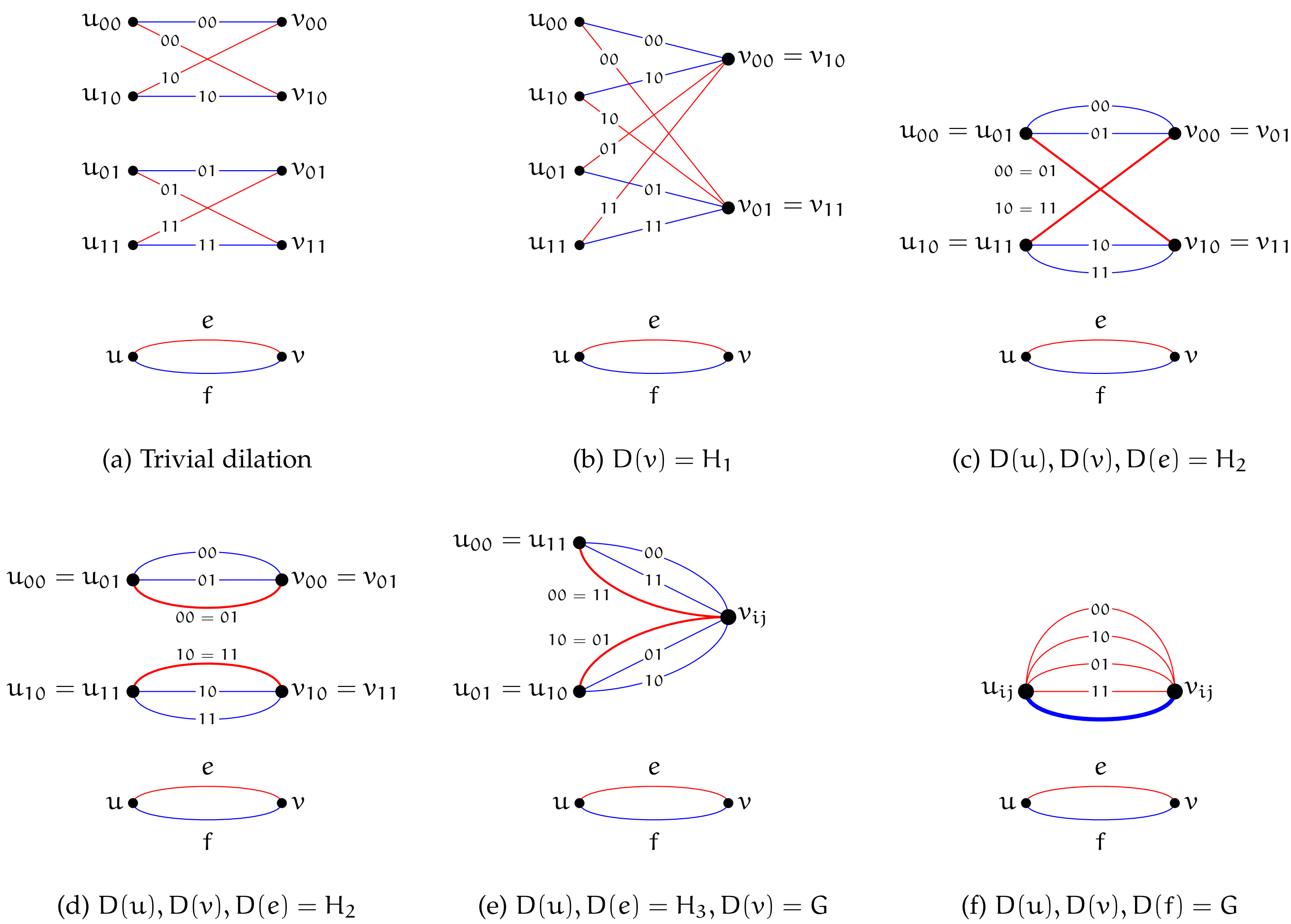
We allow the Γ_H to be empty or disconnected for $H \neq 0$. The union of the Γ_H for $H \neq 0$ is called the *dilated subgraph* of Γ and is denoted Γ_{dil}.

We can associate a G-stratification of Γ to a G-dilation datum D, and vice versa.

Definition 3.10. Let Γ be a graph, and let D be a G-dilation datum on Γ. We define the G-*stratification* $\mathcal{S}(D) = \{\Gamma_H : H \in S(G)\}$ *associated to* D as follows:

$$\Gamma_H = \{x \in X(\Gamma) | H \subset D(x)\}.$$

We observe that for any half-edge $h \in H(\Gamma)$ we have $D(h) \subset D(r(h))$, therefore each Γ_H is indeed a subgraph of Γ.



(a) Trivial dilation

(b) $D(v) = H_1$

(c) $D(u), D(v), D(e) = H_2$

(d) $D(u), D(v), D(e) = H_2$

(e) $D(u), D(e) = H_3, D(v) = G$

(f) $D(u), D(v), D(f) = G$

Figure 1: Klein covers of a genus 1 graph

Remark 3.11. Let D_φ be the G-dilation datum associated to a G-cover $\varphi : \Gamma' \to \Gamma$. Then for any $H \in S(G)$, Γ_H is the image under φ of the subgraph of Γ' fixed under the action of H.

A G-dilation datum D can be uniquely recovered from a G-stratification $\mathcal{S}$ as follows. Condition (4) implies that the set $X(\Gamma)$ is partitioned into disjoint subsets (which are not subgraphs in general)

$$X(\Gamma) = \coprod_{H \in S(G)} \Gamma_H \backslash \Gamma_H^0, \text{ where } \Gamma_H^0 = \bigcup_{H \subsetneq K} \Gamma_K.$$

For any $x \in X(\Gamma)$ we set $D(x) = H$, where H is the unique subgroup of G such that $x \in \Gamma_H \backslash \Gamma_H^0$.

We also define a dual stratification associated to a G-dilation datum.

Definition 3.12. Let D be a G-dilation datum on Γ. The *dual stratification* $\mathcal{S}^*(D) = \{\Gamma^H : H \in S(G)\}$ of $\mathcal{S}$ is defined as follows. For $H \in S(G)$, we define Γ^H to be the edge-maximal subgraph of Γ whose vertex set is

$$V(\Gamma^H) = \bigcup_{K \subset H} V(\Gamma_K \backslash \Gamma_K^0) = \left\{ v \in V(\Gamma) \middle| D(v) \subset H \right\}.$$

In other words, a leg of Γ with root vertex v lies in Γ^H if and only if $D(v) \subset H$, and an edge $e \in E(\Gamma)$ with root vertices u and v lies in Γ^H if and only if $C(e) = D(u) + D(v) \subset H$.

The dual stratification satisfies the following properties:

$$\begin{aligned} \Gamma^G &= \Gamma, \\ \Gamma^H &\subset \Gamma^K \text{ if } H \subset K, \text{ and} \\ \Gamma_H \cap \Gamma_K &= \Gamma_{H \cap K} \text{ for all } H, K \in S(G). \end{aligned}$$

Remark 3.13. Unlike $\mathcal{S}(D)$, the dual stratification $\mathcal{S}^*(D)$ of a G-dilation datum does not uniquely determine D. For a vertex $v \in V(\Gamma)$, we can recover $D(v)$ as the smallest subgroup $H \subset G$ such that $v \in V(\Gamma^H)$, but the dilation groups $D(h)$ of the edges cannot be determined. For example, let Γ be the graph consisting of a vertex v and a loop e, let $D(v) = H$ be a subgroup of G, and let $D(e)$ be any subgroup of H. The dual stratification is

$$\Gamma^K = \begin{cases} \Gamma & \text{if } H \subseteq K, \\ \emptyset & \text{if } H \subsetneq K, \end{cases}$$

so we can recover H but not $D(e)$.

Finally, we define morphisms of G-covers of Γ.

Definition 3.14. Let $\varphi_1 : \Gamma_1' \to \Gamma$ and $\varphi_2 : \Gamma_2' \to \Gamma$ be G-covers. A *morphism of* G*-covers* from φ_1 to φ_2 is a G-equivariant morphism $\psi : \Gamma_1' \to \Gamma_2'$ such that $\varphi_1 = \varphi_2 \circ \psi$.

We observe that if $\psi : \Gamma_1' \to \Gamma_2'$ is a morphism of G-covers from $\varphi_1 : \Gamma_1' \to \Gamma$ to $\varphi_2 : \Gamma_2' \to \Gamma$, then for any $x \in X(\Gamma)$ the restriction of τ to the fiber $\varphi_1^{-1}(x)$ is a G-equivariant surjective map onto $\varphi_2^{-1}(x)$, which implies that $D_{\varphi_1}(x) \subset D_{\varphi_2}(x)$, in other words D_{φ_1} is a refinement of D_{φ_2}.

Remark 3.15. In this paper, we only consider G-covers of a fixed base graph Γ (except that we do consider restrictions of covers to a subgraph). It is also possible to define morphisms of G-covers of graphs that are related by a morphism. For example, given a G-cover $\varphi : \Gamma' \to \Gamma$

and a morphism $\psi : \Delta \to \Gamma$, we define the pullback G-cover $\varphi' : \Delta' \to \Delta$ by taking Δ' to be the fiber product $\Gamma' \times_\Gamma \Delta'$ (defined by $X(\Delta') = X(\Gamma') \times_{X(\Gamma)} X(\Delta')$ with coordinatewise involution and root maps), and letting G act on the first factor. All of the constructions of this chapter are functorial with respect to such operators, so for example the G-dilation datum $D_{\varphi'}$ on Δ is equal to the pullback G-dilation datum $\psi^* D_\varphi = D_\varphi \circ \psi$.

3.2 Cohomology of G-data

In this subsection, we define the cohomology groups $H^0(\Gamma, D)$ and $H^1(\Gamma, D)$ of a G-dilated graph (Γ, D). These groups generalize the simplicial cohomology groups $H^i(\Gamma, G)$ of Γ with coefficients in G. The groups $H^i(\Gamma, D)$ do not depend on the legs of Γ, so we assume for simplicity that Γ has no legs. The legs of Γ will again play a role in Sec. 4.2, when we classify unramified G-covers of weighted graphs.

Rather than only considering G-dilation data on a graph Γ, we work in a larger category of G-*data* on Γ, a G-datum being simply a choice of a G-group at every vertex and every edge of Γ that is consistent with the root maps (see Definition 3.5). A G-datum D_φ arising from a G-cover φ is always a G-dilation datum. However, cohomology groups of the more general G-data appear in the long exact sequence (12) that relates the cohomology groups $H^i(\Gamma, D)$ of a G-dilation datum D on Γ to the cohomology groups $H^i(\Delta, D|_\Delta)$ of the restriction of D to a subgraph $\Delta \subset \Gamma$.

We begin by recalling the simplicial cohomology groups of a graph Γ with coefficients in G. Choose an orientation on the edges, and let $s, t : E(\Gamma) \to V(\Gamma)$ be the source and target maps. The simplicial chain complex of Γ is

$$0 \longrightarrow \mathbb{Z}^{E(\Gamma)} \xrightarrow{\delta} \mathbb{Z}^{V(\Gamma)} \longrightarrow 0,$$

with the boundary map defined on the generators of $\mathbb{Z}^{E(\Gamma)}$ by $\delta(e) = t(e) - s(e)$. Applying the functor $\mathrm{Hom}(-, G)$ and identifying

$$\mathrm{Hom}(\mathbb{Z}^{V(\Gamma)}, G) = G^{V(\Gamma)} \quad \text{and} \quad \mathrm{Hom}(\mathbb{Z}^{E(\Gamma)}, G) = G^{E(\Gamma)},$$

we obtain the simplicial cochain complex of Γ with coefficients in G:

$$0 \longrightarrow G^{V(\Gamma)} \xrightarrow{\delta^*} G^{E(\Gamma)} \longrightarrow 0. \tag{5}$$

We identify elements of $G^{V(\Gamma)}$ and $G^{E(\Gamma)}$ with functions $\xi : V(\Gamma) \to G$ and $\eta : E(\Gamma) \to G$, respectively. Under this identification, the duals $s^*, t^* : G^{V(\Gamma)} \to G^{E(\Gamma)}$ of the maps s and t are

$$s^*(\xi)(e) = \xi(s(e)) \quad \text{and} \quad t^*(\xi)(e) = \xi(t(e)),$$

and the coboundary map is equal to

$$\delta^* = t^* - s^*. \tag{6}$$

The *simplicial cohomology groups* of Γ with coefficients in G are

$$H^0(\Gamma, G) = \mathrm{Ker}\, \delta^* \quad \text{and} \quad H^1(\Gamma, G) = \mathrm{Coker}\, \delta^*.$$

We now generalize this construction by replacing every copy of G in the cochain complex (5) with an arbitrary G-group. We recall that a G-*group* is a map of abelian groups $f : G \to H$, and a *morphism of* G-*groups* from $f_1 : G \to H_1$ to $f_2 : G \to H_2$ is a group homomorphism $g : H_1 \to H_2$ such that $f_2 = g \circ f_1$.

Definition 3.16. A G-*datum* A on an oriented graph Γ consists of the following:

1. For every vertex $v \in V(\Gamma)$, a G-group $f_v : G \to A(v)$.
2. For every edge $e \in E(\Gamma)$, a G-group $f_e : G \to A(e)$ and morphisms of G-groups $s_e : A\big(s(e)\big) \to A(e)$ and $t_e : A\big(t(e)\big) \to A(e)$ such that $s_e \circ f_{s(e)} = t_e \circ f_{t(e)} = f_e$, i.e. for which the diagram

$$\begin{tikzcd} & G \arrow[dl, "f_{s(e)}"'] \arrow[d, "f_e"] \arrow[dr, "f_{t(e)}"] & \\ A(s(e)) \arrow[r, "s_e"'] & A(e) & A(t(e)) \arrow[l, "t_e"] \end{tikzcd}$$

commutes.

In other words, a G-datum on Γ is a functor to the category of G-groups from the category whose objects are $V(\Gamma) \cup E(\Gamma)$, and whose non-trivial morphisms are the source and target maps. In contrast with Remark 3.5, a G-datum is not necessarily a graph of groups in the sense of [Bas93], since the maps $f_{s(e)}$ and $f_{t(e)}$ are not required to be injective.

To verify that G-data, in fact, generalize the notion of G-dilation data, we associate a G-datum A^D to each G-dilation datum D. First, let H_1 and H_2 be subgroups of G, let $f_i : G \to G/H_i$ be the projections, and let $\iota_i : G/H_i \to G/H_1 \oplus G/H_2$ be the embeddings. The coproduct of f_1 and f_2 is the G-group

$$G/H_1 \sqcup_G G/H_2 = (G/H_1 \oplus G/H_2)/(\operatorname{Im} f_1 \oplus -f_2).$$

The natural map $f_1 \sqcup f_2 : G \to G/H_1 \sqcup_G G/H_2$ is equal to $\pi \circ \iota_1 \circ f_1 = \pi \circ \iota_2 \circ f_2$, where $\pi : G/H_1 \oplus G/H_2 \to G/H_1 \sqcup_G G/H_2$ is the projection. It is clear that $f_1 \sqcup f_2$ is surjective and that $\operatorname{Ker} f_1 \sqcup f_2 = H_1 + H_2$, hence the G-group $G \to G/H_1 \sqcup_G G/H_2$ can be identified with the quotient $G \to G/(H_1 + H_2)$.

Definition 3.17. Let Γ be an oriented graph, and let D be a G-dilation datum on Γ. We define the *associated* G-*datum* A^D as follows. For each $v \in V(\Gamma)$, we set $A^D(v) = G/D(v)$, and let f_v be the natural projection map:

$$f_v : G \to A^D(v) = G/D(v).$$

For an edge $e \in E(\Gamma)$, we let $f_e = f_{s(e)} \sqcup f_{t(e)}$ be the coproduct. In other words, we let

$$A^D(e) = [G/D(s(e)) \oplus G/D(t(e))]/(\operatorname{Im} f_{s(e)} \oplus -f_{t(e)}) \simeq G/C(e),$$

where $C(e) = D(s(e)) + D(t(e))$ is the edge dilation group. We let

$$f_e : G \to A^D(e) \simeq G/C(e)$$

be the quotient map, and we let

$$s_e : A^D(s(e)) \to A^D(e) \quad \text{and} \quad t_e : A^D(t(e)) \to A^D(e)$$

be the natural quotient maps $G/D(s(e)) \to G/C(e)$ and $G/D(t(e)) \to G/C(e)$.

We now define the cochain complex and cohomology groups of a G-datum A on an oriented graph Γ.

Definition 3.18. Let G be an oriented graph, and let A be a G-datum on Γ. We define the *cochain groups* of the pair (Γ, A) as follows:

$$C^0(\Gamma, A) = \prod_{v \in V(\Gamma)} A(v) = \Big\{ \xi : V(\Gamma) \to \coprod_{v \in V(\Gamma)} A(v) : \xi(v) \in A(v) \Big\},$$

$$C^1(\Gamma, A) = \prod_{e \in E(\Gamma)} A(e) = \Big\{ \eta : E(\Gamma) \to \coprod_{e \in E(\Gamma)} A(e) : \eta(e) \in A(e) \Big\}.$$

We define the morphisms $s^*, t^* : C^0(\Gamma, A) \to C^1(\Gamma, A)$ by

$$s^*(\xi)(e) = s_e(\xi(s(e))) \quad \text{and} \quad t^*(\xi)(e) = t_e(\xi(t(e))).$$

We define the *cochain complex* of the pair (Γ, A) as

$$0 \longrightarrow C^0(\Gamma, A) \xrightarrow{\delta^*_{\Gamma,A}} C^1(\Gamma, A) \longrightarrow 0,$$

where the *coboundary map* $\delta^*_{\Gamma,A}$ is

$$\delta^*_{\Gamma,A} = t^* - s^*. \tag{7}$$

We define the *cohomology groups* of the pair (Γ, A) as

$$H^0(\Gamma, A) = \operatorname{Ker} \delta^*_{\Gamma,A} \quad \text{and} \quad H^1(\Gamma, A) = \operatorname{Coker} \delta^*_{\Gamma,A}.$$

Specializing to G-dilation data, we obtain the main definition of this section.

Definition 3.19. Let (Γ, D) be a G-dilated graph, and let A^D be the G-datum associated to D. The *cochain complex* of (Γ, D) is the cochain complex of the pair (Γ, A^D):

$$0 \longrightarrow C^0(\Gamma, D) \xrightarrow{\delta^*_{\Gamma,D}} C^1(\Gamma, D) \longrightarrow 0,$$

where

$$C^i(\Gamma, D) = C^i(\Gamma, A^D) \quad \text{and} \quad \delta^*_{\Gamma,D} = \delta^*_{\Gamma,A^D}.$$

The *dilated cohomology groups* $H^i(\Gamma, D)$ are the cohomology groups of (Γ, A^D):

$$H^0(\Gamma, D) = \operatorname{Ker} \delta^*_{\Gamma,D} = H^0(\Gamma, A^D) \quad \text{and} \quad H^1(\Gamma, D) = \operatorname{Coker} \delta^*_{\Gamma,D} = H^1(\Gamma, A^D). \tag{8}$$

For the sake of clarity, and for future use, we give an explicit description of $H^1(\Gamma, D)$ as a quotient. The cochain group $C^1(\Gamma, D)$ is the direct product of $A^D(e)$ over all $e \in E(\Gamma)$, where each $A^D(e)$ is the coproduct $G/C(e)$ of $G \to G/D(s(e))$ and $G \to G/D(t(e))$. In other words, each $\eta \in C^1(\Gamma, D)$ is given by choosing a pair of elements $(\eta_s(e), \eta_t(e)) \in G/D(s(e)) \oplus G/D(t(e))$ for each $e \in E(\Gamma)$. A tuple $(\eta_s(e), \eta_t(e))_{e \in E(\Gamma)}$ is equivalent to $(\widetilde{\eta}_s(e), \widetilde{\eta}_t(e))_{e \in E(\Gamma)}$ if and only if there exist elements $\omega(e) \in G$ for all $e \in E(\Gamma)$ such that

$$\eta_s(e) = \widetilde{\eta}_s(e) + \omega(e) \bmod D(s(e))$$
$$\eta_t(e) = \widetilde{\eta}_t(e) - \omega(e) \bmod D(t(e)).$$

Note that, instead of assuming that $\omega(e) \in G$, we may assume that $\omega(e)$ lies in any quotient group between G and $G/(D(s(e)) \cap D(t(e)))$, and it is natural to assume that in fact $\omega(e) \in G/D(e)$.

An element of $C^0(\Gamma, D)$ is given by choosing $\xi(v) \in G/D(v)$ for each $v \in V(\Gamma)$. Putting everything together, we see that an element $[\eta] \in H^1(\Gamma, D)$ is given by choosing a pair of elements $(\eta_s(e), \eta_t(e)) \in G/D(s(e)) \oplus G/D(t(e))$ for each $e \in E(\Gamma)$, and that two choices $(\eta_s(e), \eta_t(e))_{e\in E(\Gamma)}$ and $(\tilde\eta_s(e), \tilde\eta_t(e))_{e\in E(\Gamma)}$ represent the same element of $H^1(\Gamma, D)$ if and only if there exist elements $\omega(e) \in G/D(e)$ for all $e \in E(\Gamma)$ and elements $\xi(v)$ for all $v \in V(\Gamma)$ such that

$$\begin{aligned} \eta_s(e) &= \tilde\eta_s(e) - \xi(s(e)) + \omega(e) \bmod D(s(e)) \\ \eta_t(e) &= \tilde\eta_t(e) + \xi(t(e)) - \omega(e) \bmod D(t(e)) \end{aligned} \tag{9}$$

for all $e \in E(\Gamma)$.

Remark 3.20. The dilated cochain complex of (Γ, D), and hence the cohomology groups $H^i(\Gamma, D)$, depend only on the dilation groups $D(v)$ of the vertices $v \in V(\Gamma)$, and do not depend on the edge groups $D(e)$. Specifically, given a graph Γ, we can choose the dilation groups $D(v)$ of the vertices $v \in V(\Gamma)$ arbitrarily, and for each edge $e \in E(\Gamma)$ choose $D(e)$ to be any subgroup of $D(s(e)) \cap D(t(e))$. The resulting groups $H^0(\Gamma, D)$ and $H^1(\Gamma, D)$ are independent of the choice of the $D(e)$. In other words, the dilated cohomology groups of (Γ, D) only depend on the dual stratification $\mathcal{S}^*(D)$.

For the remainder of the paper, with the exception of Sec. 3.3 below, we restrict our attention to G-dilation data and their cohomology groups. Before we proceed, we calculate our first example, showing that we have in fact generalized simplicial cohomology.

Example 3.21. Let Γ be a graph, and let A_G be the *trivial* G-*datum*, namely $A_G(v) = G$ and $A_G(e) = G$ for all $v \in V(\Gamma)$ and all $e \in E(\Gamma)$, with all structure maps being the identity. Alternatively, A_G is the G-datum associated to the *trivial* G-*dilation datum* D_0 given by $D_0(x) = 0$ for all $x \in X(\Gamma)$. It is clear that $C^i(\Gamma, A_G) = C^i(\Gamma, G)$, and that the coboundary map δ^*_{Γ, A_G} given by (7) is equal to δ^* given by (6). Hence $H^i(\Gamma, A_G) = H^i(\Gamma, D_0) = H^i(\Gamma, G)$.

We now work out several explicit examples of the cohomology groups $H^i(\Gamma, D)$ of G-dilated graphs (Γ, D). In the previous example, we saw that the cohomology of the trivial G-dilation datum on Γ is the simplicial cohomology of Γ with coefficients in G. In particular, $H^1(\Delta, G)$ is trivial for any tree Δ. We now show that $H^1(\Delta, D) = 0$ for any G-dilation datum D on a tree Δ.

Proposition 3.22. *Let Δ be a tree. Then $H^1(\Delta, D) = 0$ for any G-dilation datum D on Δ.*

Proof. Let Γ be an arbitrary graph, and suppose that D and D' are two G-dilation data on Γ, such that D is a refinement of D'. In this case, we can define natural surjective maps $\pi^i : C^i(\Gamma, D) \to C^i(\Gamma, D')$ by taking coordinatewise quotients. These maps commute with the coboundary maps and induce maps $\pi^i : H^i(\Gamma, D) \to H^i(\Gamma, D')$, and furthermore the map π^1 is surjective.

Now suppose that Δ is a tree, and D is a G-dilation datum on Δ. Let D_0 be the trivial G-dilation datum on Δ. Then D_0 is a refinement of D, so there is a surjective map $H^1(\Delta, D_0) \to H^1(\Delta, D)$. But by Ex. 3.21 we know that $H^1(\Delta, D_0) = H^1(\Delta, G) = 0$, hence $H^1(\Delta, D) = 0$. $\square$

We now work out an example of $H^i(\Gamma, D)$ for a topologically non-trivial graph Γ.

Example 3.23. Let Γ be the graph consisting of two vertices v_1 and v_2 joined by n edges $e_1, \ldots, e_n$, oriented such that $s(e_i) = v_1$ and $t(e_i) = v_2$. Let H_1 and H_2 be two subgroups of G, and consider the following G-dilation datum on Γ:

$$D(v_1) = H_1, \quad D(v_2) = H_2 \quad \text{and} \quad D(e_i) \text{ are arbitrary subgroups of } H_1 \cap H_2.$$

We see that $C(e_i) = H_1 + H_2$ for all i, therefore

$$C^0(\Gamma, D) = G/H_1 \oplus G/H_2 \quad \text{and} \quad C^1(\Gamma, D) = [G/(H_1 + H_2)]^n.$$

The coboundary map $\delta^*_{\Gamma,D}$ is the composition of the projection

$$\pi : G/H_1 \oplus G/H_2 \to G/H_1 \sqcup G/H_2 \simeq G/(H_1 + H_2)$$

and the diagonal map. Therefore

$$H^0(\Gamma, D) = \operatorname{Ker} \pi \simeq G/(H_1 \cap H_2) \quad \text{and} \quad H^1(\Gamma, D) \simeq [G/(H_1 + H_2)]^{n-1}. \tag{10}$$

We also show that cohomology of G-dilation data can be used to compute simplicial cohomology of edge-maximal subgraphs of Γ, with coefficients in any quotient group of G.

Example 3.24. Let Γ be a graph, let $\Delta \subset \Gamma$ be an edge-maximal subgraph, and let $H \subset G$ be a subgroup. Consider the following G-dilation datum on Γ:

$$D_{\Delta,H}(x) = \begin{cases} H, & x \in X(\Delta), \\ G, & x \notin X(\Delta). \end{cases}$$

By definition, an edge $e \in E(\Gamma)$ lies in Δ if and only if both of its root vertices do. It follows that the dilated cochain complex $C^*(\Gamma, D_{\Delta,H})$ is equal to the simplicial cochain complex $C^*(\Delta, G/H)$, and hence

$$H^i(\Gamma, D_{\Delta,H}) = H^i(\Delta, G/H) \text{ for } i = 0, 1.$$

Remark 3.25. We will show in Sec. 4.1 that the group $H^1(\Gamma, D)$ classifies G-covers of Γ with dilation datum D. We do not know of a similar geometric interpretation of the group $H^0(\Gamma, D)$. For the trivial dilation datum $D = 0$, the group $H^0(\Gamma, D) = H^0(\Gamma, G)$ is equal to G for any connected graph Γ. In general, the group $H^0(\Gamma, D)$ can be quite large, even on a connected graph. For example, let Γ be a chain of $2n$ vertices, let p and q be distinct prime numbers, let $G = \mathbb{Z}/pq\mathbb{Z}$, and let $H_1 = \mathbb{Z}/p\mathbb{Z}$ and $H_2 = \mathbb{Z}/q\mathbb{Z}$ be the nontrivial subgroups of G. Label the vertices of Γ by H_1 and H_2 in an alternating fashion. Then $C(e) = G/(H_1 + H_2) = 0$ for any edge of e, hence $C^1(\Gamma, D) = 0$ and therefore $H^0(\Gamma, D) = C^0(\Gamma, D) = H_1^n \oplus H_2^n = G^n$.

We have already noted (see Rem. 3.15) that we restrict our attention to a fixed base graph Γ. It is possible to define morphisms between pairs consisting of a graph and a G-datum on it (it is necessary to require that the graph morphism be finite). Such morphisms define natural pullback maps on the cochain and cohomology groups. In the next section, we work out these pullback maps for a single example, namely the relationship (Prop. 3.30) between the cohomology groups $H^i(\Gamma, A)$ of a G-datum A on Γ, and the cohomology groups $H^i(\Delta, A|_\Delta)$ of the restriction of A to a subgraph $\Delta \subset \Gamma$.

3.3 Relative cohomology and reduced cohomology

This section is somewhat technical in nature, and deals with a single question: how to relate the cohomology groups $H^i(\Gamma, D)$ of a G-dilated graph (Γ, D) to the cohomology groups $H^i(\Delta, D|_\Delta)$ of the restriction of D to a subgraph $\Delta \subset \Gamma$. This question is natural from the point of view of tropical geometry: we often study tropical curves by contracting edges and forming simpler graphs, and hence we may need to understand the classification of G-covers of a graph in terms of G-covers of its contractions.

We fix a graph Γ, a subgraph $\Delta \subset \Gamma$, and a G-datum A on Γ. We will see that the cohomology groups $H^i(\Gamma, A)$ and $H^i(\Delta, A|_\Delta)$ fit into an exact sequence, which is the analogue of the long exact sequence of the cohomology groups of a pair of topological spaces. The relative cohomology groups occurring in this sequence can be computed as reduced cohomology groups of an induced G-datum $A_{\Gamma/\Delta}$ on the quotient graph Γ/Δ. Unfortunately, the G-datum $A_{\Gamma/\Delta}$ is not in general the G-datum associated to a G-dilation datum on Γ/Δ, even when A is associated to a G-dilation datum on Γ.

Definition 3.26. Let Γ be an oriented graph, let $\Delta \subset \Gamma$ be a subgraph, and let A be a G-datum on Γ. Then $A|_\Delta$ is a G-datum on Δ. Viewing $C^0(\Gamma, A)$ and $C^0(\Delta, A|_\Delta)$ as sets of $A(v)$-valued maps from $V(\Gamma)$ and $V(\Delta)$, respectively, we define a surjective map $\iota^0 : C^0(\Gamma, A) \to C^0(\Delta, A|_\Delta)$ by restricting from $V(\Gamma)$ to $V(\Delta)$. We similarly define a surjective restriction map $\iota^1 : C^1(\Gamma, A) \to C^1(\Delta, A|_\Delta)$, and define the *relative cochain complex* of the triple (Γ, Δ, A):

$$0 \longrightarrow C^0(\Gamma, \Delta, A) \xrightarrow{\delta^*_{\Gamma,\Delta,A}} C^1(\Gamma, \Delta, A) \longrightarrow 0,$$

by setting $C^i(\Gamma, \Delta, A) = \operatorname{Ker} \iota^i$ for $i = 0, 1$ and $\delta^*_{\Gamma,\Delta,A}$ to be the restriction of $\delta^*_{\Gamma,A}$ to $C^0(\Gamma, \Delta, A)$. The *relative cohomology groups* of the triple (Γ, Δ, A) are

$$H^0(\Gamma, \Delta, A) = \operatorname{Ker} \delta^*_{\Gamma,\Delta,A} \quad \text{and} \quad H^1(\Gamma, \Delta, A) = \operatorname{Coker} \delta^*_{\Gamma,\Delta,A}.$$

We note that $\delta^*_{\Gamma,A} \circ \iota^0 = \iota^1 \circ \delta^*_{\Delta,A|_\Delta}$, in other words the ι^i form a cochain map. Hence we have a short exact sequence of cochain complexes:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & C^0(\Gamma,\Delta,A) \arrow[r, "\delta^*_{\Gamma,\Delta,A}"] \arrow[d] & C^1(\Gamma,\Delta,A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C^0(\Gamma,A) \arrow[r, "\delta^*_{\Gamma,A}"] \arrow[d, "\iota^0"] & C^1(\Gamma,A) \arrow[r] \arrow[d, "\iota^1"] & 0 \\
0 \arrow[r] & C^0(\Delta,A|_\Delta) \arrow[r, "\delta^*_{\Delta,A|_\Delta}"] \arrow[d] & C^1(\Delta,A|_\Delta) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 &
\end{tikzcd} \tag{11}$$

By the snake lemma, the cohomology groups of (Γ, A), $(\Delta, A|_\Delta)$ and the triple (Γ, Δ, A) fit into an exact sequence

$$
\begin{aligned}
0 \longrightarrow H^0(\Gamma,\Delta,A) \longrightarrow H^0(\Gamma,A) \longrightarrow H^0(\Delta,A|_\Delta) \longrightarrow \\
\longrightarrow H^1(\Gamma,\Delta,A) \longrightarrow H^1(\Gamma,A) \longrightarrow H^1(\Delta,A|_\Delta) \longrightarrow 0.
\end{aligned} \tag{12}
$$

We now show that the relative cohomology groups of the triple (Γ,Δ,A) are equal to the reduced cohomology groups of the contracted graph Γ/Δ with a certain induced G-datum. First, we define the reduced cohomology groups of a pair (Γ,A).

Definition 3.27. Let Γ be an oriented graph and let A be a G-datum on Γ. Let $d : G \to C^0(\Gamma,A)$ be the diagonal morphism given by $d(g) = \xi_g \in C^0(\Gamma,A)$, where

$$\xi_g(v) = f_v(g) \in A(v) \text{ for all } v \in V(\Gamma) \text{ and all } g \in G.$$

For any $g \in G$ and any $e \in E(\Gamma)$ we have

$$\delta^*_{\Gamma,A}(d(g))(e) = t^*(\xi_g)(e) - s^*(\xi_g)(e) = t_e(f_{t(e)}(g)) - s_e(f_{s(e)}(g)) = f_e(g) - f_e(g) = 0,$$

hence $\operatorname{Im} d \subset \operatorname{Ker} \delta^*_{\Gamma,A}$. Therefore we can define the *reduced cochain complex* of the pair (Γ,A)

$$0 \longrightarrow \widetilde{C}^0(\Gamma,A) \xrightarrow{\widetilde{\delta}^*_{\Gamma,A}} \widetilde{C}^1(\Gamma,A) \longrightarrow 0$$

by

$$\widetilde{C}^0(\Gamma,A) = C^0(\Gamma,A)/\operatorname{Im} d \quad \text{and} \quad \widetilde{C}^1(\Gamma,A) = C^1(\Gamma,A),$$

and the *reduced cohomology groups* of (Γ,A):

$$\widetilde{H}^0(\Gamma,A) = \operatorname{Ker} \widetilde{\delta}^*_{\Gamma,A} = H^0(\Gamma,A) \bmod G \quad \text{and} \quad \widetilde{H}^1(\Gamma,A) = \operatorname{Coker} \widetilde{\delta}^*_{\Gamma,A} = H^1(\Gamma,A).$$

We define the quotient of a graph Γ by a subgraph $\Delta \subset \Gamma$ by contracting Δ to a single vertex. Note that this definition comes from topology, and differs from weighted edge contraction (see Def. 2.12), wherein each connected component of Δ is contracted to a separate vertex.

Definition 3.28. Let Γ be an oriented graph and let Δ be a subgraph. We define the graph Γ/Δ as follows:

$$V(\Gamma/\Delta) = V(\Gamma)\backslash V(\Delta) \cup \{w\} \quad \text{and} \quad E(\Gamma/\Delta) = E(\Gamma)\backslash E(\Delta),$$

as well as

$$s(e) = \begin{cases} w & \text{if } s(e) \in V(\Delta), \\ s(e) & \text{if } s(e) \notin V(\Delta), \end{cases} \qquad t(e) = \begin{cases} w & \text{if } t(e) \in V(\Delta), \\ t(e) & \text{if } t(e) \notin V(\Delta). \end{cases}$$

Now let Γ be an oriented graph, let Δ be a subgraph, and let A be a G-datum on Γ. We define the G-datum $A_{\Gamma/\Delta}$ on Γ/Δ by restricting A to all vertices except w and all edges, and by placing the trivial G-datum at w. Specifically, the G-groups $f'_v : G \to A_{\Gamma/\Delta}(v)$ corresponding to the vertices $v \in V(\Gamma/\Delta)$ are

$$f'_v : G \to A_{\Gamma/\Delta}(v) = \begin{cases} f_v : G \to A(v) & \text{if } v \in V(\Gamma)\backslash V(\Delta), \\ \mathrm{Id} : G \to G & \text{if } v = w. \end{cases}$$

The G-groups $f'_e : G \to A_{\Gamma/\Delta}(e)$ corresponding to $e \in E(\Gamma/\Delta)$ are the same as $f_e : G \to A(e)$. Finally, the source and target maps $s'_e : A_{\Gamma/\Delta}(s(e)) \to A_{\Gamma/\Delta}(e)$ and $t'_e : A_{\Gamma/\Delta}(t(e)) \to A_{\Gamma/\Delta}(e)$ are

$$s'_e = \begin{cases} s_e : A\big(s(e)\big) \to A(e) & \text{if } s(e) \neq w, \\ f_e : G \to A(e) & \text{if } s(e) = w, \end{cases}$$

and

$$t'_e = \begin{cases} t_e : A\big(t(e)\big) \to A(e) & \text{if } t(e) \neq w, \\ f_e : G \to A(e) & \text{if } t(e) = w. \end{cases}$$

Remark 3.29. If $A = A^D$ is the G-datum associated to a G-dilation datum D, then so is $A|_\Delta$, but not, in general, $A_{\Gamma/\Delta}$. Specifically, the edge groups of A^D are the coproducts of the vertex groups, which is no longer the case for $A_{\Gamma/\Delta}$. In other words, the relationship between the dilated cohomology groups $H^i(\Gamma, D)$ and $H^i(\Delta, D|_\Delta)$ cannot be expressed without using the more general framework of G-data and their cohomology. The edge groups of $A_{\Gamma/\Delta}$ retain a record of the dilation groups $D(v)$ of the edges $v \in V(\Delta)$ that are contracted in Γ/Δ.

Proposition 3.30. *Let Γ be an oriented graph, let Δ be a subgraph, and let A be a G-datum on Γ. The relative cohomology groups of the triple (Γ, Δ, A) are equal to the reduced cohomology groups of $(\Gamma/\Delta, A_{\Gamma/\Delta})$:*

$$H^i(\Gamma, \Delta, A) = \widetilde{H}^i(\Gamma/\Delta, A_{\Gamma/\Delta}). \tag{13}$$

Proof. The group G acts diagonally on $C^0(\Gamma/\Delta, A_{\Gamma/\Delta})$, and the action is free and transitive on the w-coordinate, since by definition $A_{\Gamma/\Delta}(w) = G$. Therefore any element $[\xi]$ in the quotient group $\widetilde{C}^0(\Gamma/\Delta, A_{\Gamma/\Delta})$ has a unique representative $\xi \in C^0(\Gamma/\Delta, A_{\Gamma/\Delta})$ satisfying $\xi(w) = 0$. Since $A_{\Gamma/\Delta}(v) = A(v)$ for $v \in V(\Gamma\backslash\Delta)$, we can define an extension by zero map $j^0 : \widetilde{C}^0(\Gamma/\Delta, A_{\Gamma/\Delta}) \to C^0(\Gamma, A)$ by

$$j^0([\xi])(v) = \begin{cases} \xi(v) & \text{if } v \in V(\Gamma)\backslash V(\Delta), \\ 0 & \text{if } v \in V(\Delta). \end{cases}$$

Similarly, since $A_{\Gamma/\Delta}(e) = A(e)$ for all $e \in E(\Gamma/\Delta)$, we can define an extension by zero map $j^1 : \widetilde{C}^1(\Gamma/\Delta, A_{\Gamma/\Delta}) = C^1(\Gamma/\Delta, A_{\Gamma/\Delta}) \to C^1(\Gamma, A)$ by

$$j^1(\eta)(e) = \begin{cases} \eta(e) & \text{if } e \in E(\Gamma)\backslash E(\Delta), \\ 0 & \text{if } e \in E(\Delta). \end{cases}$$

We claim that the j^i form a chain map. Denote for simplicity $\delta^* = t^* - s^* = \delta^*_{\Gamma,\Delta}$ and $\widetilde{\delta}^* = \widetilde{\delta}^*_{\Gamma/\Delta, \alpha_{\Gamma/\Delta}}$. For $[\xi] \in \widetilde{C}^0(\Gamma/\Delta, A_{\Gamma/\Delta})$ let $\xi \in C^0(\Gamma/\Delta, A_{\Gamma/\Delta})$ be the representative satisfying $\xi(w) = 0$. Let $e \in E(\Gamma)$ be an edge. If $e \in E(\Delta)$ then $(j^1 \circ \widetilde{\delta}^*)\big([\xi]\big)(e) = 0$. If $e \in E(\Gamma)\backslash E(\Delta)$ has root vertices $u = s(e)$ and $v = t(e)$, then, using

$$(j^1 \circ \widetilde{\delta}^*)\big([\xi]\big)(e) = t'_e\big(\xi(v)\big) - s'_e\big(\xi(u)\big)$$

we find

$$(j^1 \circ \widetilde{\delta}^*)\big([\xi]\big)(e) = \begin{cases} t_e(\xi(v)) - s_e(\xi(u)) & \text{if } v \in V(\Gamma)\backslash V(\Delta) \text{ and } u \in V(\Gamma)\backslash V(\Delta), \\ t_e\big(\xi(v)\big) & \text{if } v \in V(\Gamma)\backslash V(\Delta) \text{ and } u \in V(\Delta), \\ -s_e\big(\xi(u)\big) & \text{if } v \in V(\Delta) \text{ and } u \in V(\Gamma)\backslash V(\Delta), \\ 0 & \text{if } v \in V(\Delta) \text{and} u \in V(\Delta), \end{cases} \tag{14}$$

because $\xi(w) = 0$.

On the other hand, $j^0([\xi])$ is the element of $C^0(\Gamma, \Delta)$ obtained by setting $j^0([\xi])(v) = \xi(v)$ for all vertices $v \in V(\Gamma)\backslash V(\Delta)$ and $\xi(v) = 0$ for all $v \in V(\Delta)$. It is clear that $(\delta^* \circ j^0)([\xi])(e)$ is given by (14) for any $e \in E(\Gamma)\backslash E(\Delta)$, and $(\delta^* \circ j^0)([\xi])(e) = 0$ for any $e \in E(\Delta)$, because any such edge has root vertices in Δ and ξ vanishes at those vertices. It follows that $\delta^* \circ j^0 = j^1 \circ \widetilde{\delta}^*$, hence the j^i form a chain map.

We now consider the diagram (11). By definition,

$$C^0(\Gamma, \Delta, A) = \left\{\xi \in C^0(\Gamma, A) : \xi(v) = 0 \text{ for all } v \in V(\Delta)\right\}$$

and

$$C^1(\Gamma, \Delta, A) = \left\{\eta \in C^1(\Gamma, A) : \chi(e) = 0 \text{ for all } e \in E(\Delta)\right\}.$$

It is clear that j^i maps $\widetilde{C}^i(\Gamma/\Delta, A_{\Gamma/\Delta})$ bijectively onto $C^i(\Gamma, \Delta, A)$ for $i = 0, 1$. It follows that j^i is a chain isomorphism from $\widetilde{C}^i(\Gamma/\Delta, A_{\Gamma/\Delta})$ to $C^i(\Gamma, \Delta, A)$, which completes the proof.

□

4 Classification of G-covers of graphs and tropical curves

In this section, we use dilated cohomology groups, defined in the previous section, to classify G-covers of graphs and tropical curves. In Sec. 4.1 we give the main classification result for unweighted graphs, Thm. 4.1, which identifies the set of G-covers of Γ with a given dilation datum D with the group $H^1(\Gamma, D)$. Among these covers, we characterize the connected ones in Prop. 4.6, and give examples. The case of unramified G-covers of a weighted graph is treated in Sec. 4.2, the only novelty being a numerical restriction on the dilation datum D imposed by the local Riemann–Hurwitz condition (1). Finally, the case of weighted metric graphs and tropical curves is summarized in Sec. 4.3.

4.1 G-covers of graphs

In this section, we determine all G-covers of an unweighed graph Γ with a given G-dilation datum D. Our theorem generalizes the standard result that the set of topological G-covers of Γ (i.e. with trivial stabilizers) is identified with $H^1(\Gamma, G)$ (see Ex. 3.3 and Ex. 3.21).

Theorem 4.1. *Let Γ be a graph, let G be a finite abelian group, and let D be a G-dilation datum on Γ. Then there is a natural bijection between $H^1(\Gamma, D)$ and the set of G-covers having dilation datum D.*

Proof. We first explain how to associate an element $[\eta_\varphi] \in H^1(\Gamma, D)$ to a G-cover $\varphi : \Gamma' \to \Gamma$. Pick an orientation on $E(\Gamma)$ and a consistent orientation on $E(\Gamma')$, and denote $s, t : E(\Gamma') \to V(\Gamma')$ and $s, t : E(\Gamma) \to V(\Gamma)$ the source and target maps. For each $x \in X(\Gamma)$, the preimage $\varphi^{-1}(x)$ is a $G/D(x)$-torsor, so pick a G-equivariant bijection $f_x : \varphi^{-1}(x) \to G/D(x)$. Namely, for every $x' \in \varphi^{-1}(x)$ and every $g \in G$ we have

$$f_x(gx') = f_x(x') + g \mod D(x).$$

We require that if $e = (h_1, h_2) \in E(\Gamma)$, then, under the identification $\varphi^{-1}(h_1) = \varphi^{-1}(h_2) = \varphi^{-1}(e)$, the two maps f_{h_1} and f_{h_2} are equal, in which case we denote them by f_e.

If $l \in L(\Gamma)$ is a leg, then $D(l) \subset D(r(l))$ and we have the following diagram of G-sets:

$$\begin{tikzcd}
\varphi^{-1}(l) \arrow[r, "r"] \arrow[d, "f_l"] & \varphi^{-1}(r(l)) \arrow[d, "f_{r(l)}"] \\
G/D(l) \arrow[r] & G/D(r(l))
\end{tikzcd}$$

The vertical maps are bijections, while the horizontal maps are surjections. Adding a constant to f_l if necessary, we can assume that the lower horizontal map is reduction modulo $D(r(l))/D(l)$.

Now let $e \in E(\Gamma)$ be an edge, then $D(s(e))$ and $D(t(e))$ are subgroups of G containing $D(e)$. The source and target maps restrict to G-equivariant surjections $s : \varphi^{-1}(e) \to \varphi^{-1}(s(e))$ and $t : \varphi^{-1}(e) \to \varphi^{-1}(t(e))$, and we have a commutative diagram of G-sets

$$\begin{tikzcd}
\varphi^{-1}(s(e)) \arrow[d, "f_{s(e)}"] & \varphi^{-1}(e) \arrow[l, "s"'] \arrow[r, "t"] \arrow[d, "f_e"] & \varphi^{-1}(t(e)) \arrow[d, "f_{t(e)}"] \\
G/D(s(e)) & G/D(e) \arrow[l, "+\eta_s(e)"'] \arrow[r, "+\eta_t(e)"] & G/D(t(e))
\end{tikzcd}$$

where the vertical arrows are bijections. The lower horizontal arrows are surjections, and are therefore given by adding certain elements $\eta_t(e) \in G/D(t(e))$ and $\eta_s(e) \in G/D(s(e))$, and then reducing modulo $D(t(e))$ and $D(s(e))$, respectively. Hence the cover φ determines an element

$$\eta_\varphi = \big(\eta_s(e), -\eta_t(e)\big)_{e\in E(\Gamma)} \in \prod_{e\in E(\Gamma)} G/D\big(s(e)\big) \oplus G/D\big(t(e)\big).$$

Denote $[\eta_\varphi]$ its class in $H^1(\Gamma, D)$.

We need to verify that the association $\varphi \mapsto [\eta_\varphi]$ is independent of all choices. Suppose that we chose different bijections $\widetilde{f}_x : \varphi^{-1}(x) \to G/D(x)$. For any leg $l \in L(\Gamma)$ we can assume, as above, that the induced map $G/D(l) \to G/D(r(l))$ is reduction modulo $D(r(l))/D(l)$. Now let $e \in E(\Gamma)$ be an edge. We have a diagram of G-sets

$$\begin{tikzcd}
G/D(s(e)) & G/D(e) \arrow[l, "+\widetilde{\eta}_s(e)"'] \arrow[r, "+\widetilde{\eta}_t(e)"] & G/D(t(e)) \\
\varphi^{-1}(s(e)) \arrow[u, "\widetilde{f}_{s(e)}"] \arrow[d, "f_{s(e)}"] & \varphi^{-1}(e) \arrow[l, "s"'] \arrow[r, "t"] \arrow[u, "\widetilde{f}_e"] \arrow[d, "f_e"] & \varphi^{-1}(t(e)) \arrow[u, "\widetilde{f}_{t(e)}"] \arrow[d, "f_{t(e)}"] \\
G/D(s(e)) & G/D(e) \arrow[l, "+\eta_s(e)"'] \arrow[r, "+\eta_t(e)"] & G/D(t(e))
\end{tikzcd}$$

The top horizontal maps define an element $\widetilde{\eta}_\varphi = (\widetilde{\eta}_s(e), -\widetilde{\eta}_t(e))_{e\in E(\Gamma)}$ and a corresponding class $[\widetilde{\eta}_\varphi]$ in $H^1(\Gamma, D)$.

The middle column consists of isomorphisms of G-sets, hence the map $\widetilde{f}_e \circ f_e^{-1} : G/D(e) \to G/D(e)$ is the addition of an element $\omega(e) \in G/D(e)$. Similarly, the isomorphisms $\widetilde{f}_{s(e)} \circ f_{s(e)}^{-1} : G/D\big(s(e)\big) \to G/D\big(s(e)\big)$ and $\widetilde{f}_{t(e)} \circ f_{t(e)}^{-1} : G/D\big(t(e)\big) \to G/D\big(t(e)\big)$ are given by adding certain elements $\xi\big(s(e)\big) \in G/D\big(s(e)\big)$ and $\xi\big(t(e)\big) \in G/D\big(t(e)\big)$. The two vertical rectangles give the following relations on all of these elements:

$$\eta_s(e) + \xi(s(e)) = \omega(e) + \widetilde{\eta}_s(e) \bmod D(s(e))$$
$$\eta_t(e) + \xi(t(e)) = \omega(e) + \widetilde{\eta}_t(e) \bmod D(t(e)).$$

Comparing this with Eq. (9), we see that $[\eta_\varphi] = [\widetilde{\eta}_\varphi]$, so the G-cover φ determines a well-defined element of $H^1(\Gamma, D)$.

Conversely, let D be a G-dilation datum on Γ, let $[\eta] \in H^1(\Gamma, D)$ be an element, and let $(\eta_t(e), \eta_s(e))_{e\in E(\Gamma)}$ be a lift of $[\eta]$. Running the above construction in reverse, we obtain a G-cover $\varphi : \Gamma' \to \Gamma$ with associated G-dilation datum D. Specifically, let

- $V(\Gamma') = \coprod_{v\in V(\Gamma)} G/D(v)$,
- $E(\Gamma') = \coprod_{e\in E(\Gamma)} G/D(e)$, and
- $L(\Gamma') = \coprod_{l\in L(\Gamma)} G/D(l)$.

We define $\varphi : \Gamma' \to \Gamma$ by sending each $G/D(x)$ to the corresponding $x \in X(\Gamma)$. For a leg $l \in L(\Gamma)$, we define the lifting $r : G/D(l) \to G/D\big(r(l)\big)$ of the root map to Γ' as reduction modulo $D(r(l))/D(l)$. Finally, for an edge $e \in L(\Gamma)$, we define the liftings $s : G/D(e) \to G/D(s(e))$ and $t : G/D(e) \to G/D(t(e))$ of the source and target maps as

$$s(g) = g + \eta_s(e) \bmod D(s(e))/D(e) \quad \text{and} \quad t(g) = g - \eta_t(e) \bmod D(t(e))/D(e).$$

□

We observe that there is at least one G-cover associated to any G-dilation datum D on Γ, namely the *trivial G-cover with dilation datum* D, corresponding to the identity element of $H^1(\Gamma, D)$. Explicitly, the source graph Γ' is the union of the sets $G/D(x)$ for all $x \in X(\Gamma)$, and the root maps $G/D(x) \to G/D\big(r(x)\big)$ are the quotient maps corresponding to the injections $D(x) \subset D\big(r(x)\big)$. Note also that the set of G-covers with dilation datum D depends only on the vertex groups $D(v)$ for $v \in V(\Gamma)$, or, alternatively, on the dual stratification $\mathcal{S}^*(D)$.

Remark 4.2 (Functoriality)**.** The correspondence $\varphi \mapsto \eta_\varphi$ between G-covers of Γ with dilation datum D and elements of $H^1(\Gamma, D)$ given in Thm. 4.1 is functorial, in the following sense. Let $\varphi_1 : \Gamma_1' \to \Gamma$ and $\varphi_2 : \Gamma_2' \to \Gamma$ be G-covers, and let $\tau : \Gamma_1' \to \Gamma_2'$ be a morphism of G-covers (in the sense of Def. 3.14). Then the dilation datum D_{φ_1} is a refinement of D_{φ_2}, and by the proof of Prop. 3.22 there is a surjective map $\pi : H^1(\Gamma, D_{\varphi_1}) \to H^1(\Gamma, D_{\varphi_2})$. It is easy to check that $\pi(\eta_{\varphi_1}) = \eta_{\varphi_2}$. More generally, the correspondence $\varphi \mapsto \eta_\varphi$ is functorial with respect to pullback maps induced by finite harmonic morphisms $\Delta \to \Gamma$, which, as we have already remarked, are beyond the scope of our paper.

Remark 4.3 (Trivialization along a tree)**.** We saw in Prop. 3.22 that $H^1(\Delta, D) = 0$ for any G-dilation datum on a tree Δ. In other words, any G-cover of a tree is isomorphic to the trivial G-cover associated to some dilation datum D. This statement allows us to give a somewhat explicit description of G-covers of an arbitrary graph Γ. Let $\varphi : \Gamma' \to \Gamma$ be a G-cover with dilation datum D. Pick a spanning tree $\Delta \subset \Gamma$, and let $\{e_1, \ldots, e_n\} = E(\Gamma)\backslash E(\Delta)$ be the remaining edges. The restricted G-cover $\varphi|_\Delta$ is isomorphic to the trivial G-cover of Δ with dilation datum $D|_\Delta$, in other words there is a G-equivariant bijection

$$\tau : \varphi^{-1}(\Delta) \to \coprod_{x\in X(\Delta)} G/D(x), \quad \tau\big(\varphi^{-1}(x)\big) = G/D(x).$$

The cover φ is then completely determined by the way that the fibers $G/D(e_i)$ are attached to the fibers $G/D(s(e_i))$ and $G/D(t(e_i))$. As we saw in the proof above, this attachment datum can be recorded (in general, non-uniquely) by an n-tuple of elements of $\eta_i \in A^D(e_i) = G/C(e_i)$. In terms of the dilated cohomology group, we have shown that any element $[\eta] \in H^1(\Gamma, D)$ can be represented by a cochain $\eta \in C^1(\Gamma, D)$ such that $\eta(e) = 0$ unless $e = e_i$ for some $i = 1, \ldots, n$ (cf. Lemma 2.3.4 in [LU19]).

Connected covers

Given a connected graph Γ, it is natural to ask which of its G-covers constructed above are connected. To answer this question, we first consider the following construction. Let H be a proper subgroup of G, and let D be an H-dilation datum on a connected graph Γ. We can then view D as a G-dilation datum, which we denote by D^G to prevent confusion. There are natural injective chain maps $\iota^i : C^i(\Gamma, D) \to C^i(\Gamma, D^G)$ that induce maps $\iota^i : H^i(\Gamma, D) \to H^i(\Gamma, D^G)$.

Lemma 4.4. *The maps $\iota^i : H^i(\Gamma, D) \to H^i(\Gamma, D^G)$ are injective.*

Proof. The cochain groups $C^0(\Gamma, D)$ and $C^0(\Gamma, D^G)$ are the products of $H/D(v)$ and $G/D(v)$, respectively, over all $v \in V(\Gamma)$. It follows that $\operatorname{Coker} \iota^0$ can be identified with the cochain group $C^0(\Gamma, G/H)$, and similarly $\operatorname{Coker} \iota^1 = C^1(\Gamma, G/H)$. By the snake lemma, we have a long exact sequence of cohomology groups:

$$0 \longrightarrow H^0(\Gamma, D) \xrightarrow{\iota^0} H^0(\Gamma, D^G) \longrightarrow H^0(\Gamma, G/H) \longrightarrow H^1(\Gamma, D) \xrightarrow{\iota^1} H^1(\Gamma, D^G).$$

Therefore ι^0 is injective. To prove that ι^1 is injective, we show that the map $\pi : H^0(\Gamma, D^G) \to H^0(\Gamma, G/H)$ is surjective. All of our chain complexes split into direct sums over the connected components of Γ, so we assume that Γ is connected. In this case $H^0(\Gamma, G/H) = G/H$, and moreover any $[\xi] \in H^0(\Gamma, G/H)$ is represented by a constant cochain $\xi(v) = \overline{g}$ for some $\overline{g} \in G/H$. Pick $g \in G$ representing $\overline{g}$, then the constant cochain $\xi'(v) = g \bmod G/D(v)$ in $C^0(\Gamma, D^G)$ lies in $\operatorname{Ker} \delta^0_{\Gamma, D^G}$, hence represents a class $[\xi'] \in H^0(\Gamma, D^G)$, and $\pi\big([\xi']\big) = [\xi]$. Therefore π is surjective, so ι^0 is injective. $\square$

There is a natural way to associate a G-cover of Γ to an H-cover of Γ that corresponds, under the bijection of Thm. 4.1, to the injective map $\iota^1 : H^1(\Gamma, D) \to H^1(\Gamma, D^G)$.

Definition 4.5. Let Γ be a graph, let $H \subset G$ be abelian groups, and let $\varphi \colon \Gamma' \to \Gamma$ be an H-cover with H-dilation datum D. We define the G-cover $\varphi^G : \Gamma'^G \to \Gamma$ with G-dilation datum D^G, called the *extension of* φ *by* G, as follows. For each $x \in X(\Gamma)$, pick an identification of H-sets, as in the proof of Thm. 4.1, of $\varphi^{-1}(x)$ with $H/D(x)$, and for every edge $e \in E(\Gamma)$ let $\eta_t(e) \in H/D\big(t(e)\big)$ and $\eta_s(e) \in H/D\big(s(e)\big)$ be the elements that determine the root maps $t : \varphi^{-1}(e) \to \varphi^{-1}\big(t(e)\big)$ and $s : \varphi^{-1}(e) \to \varphi^{-1}\big(s(e)\big)$. We define φ^G by identifying each fiber $(\varphi^G)^{-1}(x)$ with the G-set $G/D(x)$, and rooting $(\varphi^G)^{-1}(e)$ to $(\varphi^G)^{-1}\big(t(e)\big)$ and $(\varphi^G)^{-1}\big(s(e)\big)$ using $\eta_t(e)$ and $\eta_s(e)$, viewed, respectively, as elements of $G/D\big(t(e)\big)$ and $G/D\big(s(e)\big)$.

Looking at the proof of 4.1, it is clear that $\iota^1(\eta_\varphi) = \eta_{\varphi^G}$. Furthermore, the cover φ^G is disconnected (unless $H = G$), since the root maps $t : G/D(e) \to G/D(t(e))$ and $s : G/D(e) \to G/D(s(e))$ preserve the decomposition into H-cosets.

We now show that all disconnected G-covers of a connected graph Γ arise in this way. Indeed, let $\varphi : \Gamma' \to \Gamma$ be a G-cover of a connected graph with G-dilation datum D, and let $\Gamma' = \Gamma'_1 \sqcup \cdots \sqcup \Gamma'_n$ be the connected components of Γ'. The group G acts on the connected components by permutation. Let $H = \{g \in G \mid g(\Gamma'_1) = \Gamma'_1\}$, then $D(v) \in H$ for all $v \in V(\Gamma)$. We view D as an H-dilation datum, which we denote D_H. It is clear that the restriction $\varphi_{\Gamma'_1} \colon \Gamma'_1 \to \Gamma$ is a connected H-cover with H-dilation datum D_H, and that φ is isomorphic to the G-extension of $\varphi|_{\Gamma'_1}$ by G. In other words, every disconnected G-cover of Γ is the extension of an H-cover, where $H \subset G$ is some proper subgroup. We have proved the following result, which classifies connected G-covers of a connected graph Γ.

Proposition 4.6. *Let Γ be a connected graph, and let D be a G-dilation datum on Γ. If the groups $D(v)$ span G, then every G-cover with dilation datum D is connected. If not, then the set of disconnected G-covers with dilation datum D is the union of the images of the maps $H^1(\Gamma, D_H) \to H^1(\Gamma, D)$ over all proper subgroups $H \subset G$ such that $D(v) \subset H$ for all $v \in V(\Gamma)$, where for each such H, D_H denotes D viewed as an H-dilation datum.*

Example 4.7 (Klein covers continued)**.** We now apply the results of this section to enumerate all G-covers of the graph Γ consisting of two vertices u and v joined by two edges e and f, when $G = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ is the Klein group. In particular, we describe the covers of Ex. 3.8 in terms of dilated cohomology. We recall that we denote 00, 10, 01, and 11 the elements of G, and H_1, H_2, and H_3 the subgroups of G generated respectively by 10, 01, and 11. We enumerate the covers in the following way: first, we enumerate the choices for $D(u)$ and $D(v)$, then, for each choice, we consider the possible $D(e), D(f) \subset D(u) \cap D(v)$, and finally $\#H^1(\Gamma, D)$ counts the G-covers with such G-dilation data (note that the last two steps are independent, since $H^1(\Gamma, D)$ does not depend on the edge dilation groups).

We saw in Ex. 3.23 that $H^1(\Gamma, D) = G/(D(u) + D(v))$ for any G-dilation datum on Γ. We now make this identification more explicit. Orient Γ so that $s(e) = s(f) = u$ and $t(e) = t(f) = v$. An element $[\eta] \in H^1(\Gamma, D)$ is represented by two pairs of elements

$$\big(\eta_s(e), \eta_t(e)\big), \big(\eta_s(f), \eta_t(f)\big) \in G/D(u) \oplus G/D(v),$$

modulo the relations (9). It is clear that for any $[\eta]$ we can pick a representative with $\eta_s(e) = 0$, $\eta_s(f) = 0$, and $\eta_t(f) = 0$ (in other words, we trivialize $[\eta]$ along the spanning tree $\{u, v, f\}$), so we can represent $[\eta]$ with a single element $\eta_t(e) \in G/D(v)$. Furthermore, the class of this $\eta_t(e)$ in $G/(D(u)+D(v))$ is equal to $[\eta]$ under the isomorphism $G/\big(D(u)+D(v)\big) = H^1(\Gamma, D)$. Explicitly, the cover corresponding to $[\eta]$ is constructed as follows: define the sets $\{u_{ij}\} = G/D(u)$, $\{v_{ij}\} = G/D(v)$, $\{e_{ij}\} = G/D(e)$, and $\{f_{ij}\} = G/D(f)$ (where the labeling is non-unique for a nontrivial dilation group), attach f_{ij} to u_{ij} and v_{ij}, and attach e_{ij} to u_{ij} and $v_{ij+\eta_t(e)}$.

1. $D(u) = D(v) = 0$. This is the topological case, with trivial dilation. Here $D(e) = D(f) = 0$, $H^1(\Gamma, D) = H^1(\Gamma, G) = G$, and there are four covers, three of them non-trivial. All of these covers are disconnected, since there are no surjective maps $\pi_1(\Gamma) = \mathbb{Z} \to G$. The cover corresponding to $\eta_t(e) = 10$ is given in Fig. 1a.

2. $D(u) = 0$, $D(v) = H_i$ for $i = 1, 2, 3$. In this case $D(e) = D(f) = 0$, $H^1(\Gamma, D) = G/H_i$, so for each i there is one trivial and one nontrivial cover. For example, Fig. 1b shows the non-trivial cover with $D(v) = H_1$ and $\eta_t(e) = 01$. There are a total of six covers of this type: three trivial disconnected covers and three non-trivial connected covers.

3. $D(u) = H_i$ for $i = 1, 2, 3$, $D(v) = 0$. This case is symmetric to the one above, with three connected and three disconnected covers.

4. $D(u) = D(v) = H_i$ for $i = 1, 2, 3$. Each of the groups $D(e)$ and $D(f)$ can be chosen to be 0 or H_i. Since $H^1(\Gamma, D) = G/H_i$, there is one trivial and one non-trivial cover for each choice. For example, when $D(u) = D(v) = D(e) = H_2$ and $D(f) = 0$, we obtain the non-trivial cover of Fig. 1c by choosing $\eta_t(e) = 10$, and the trivial cover of Fig. 1d by choosing $\eta_t(e) = 00$. There are a total of 24 such covers, 12 connected and 12 disconnected.

5. $D(u) = H_i$, $D(v) = H_j$, $i \neq j$. The only possibility is $D(e) = D(f) = 0$, and $H^1(\Gamma, D) = 0$, so for each $i \neq j$ there is a unique trivial cover, for a total of six covers, all connected.

6. If one or both of the groups $D(u)$ and $D(v)$ are equal to G, then $H^1(\Gamma, D) = 0$. Picking $D(e)$ and $D(f)$ to be arbitrary subgroups of $D(u) \cap D(v)$, we obtain 51 connected trivial covers. Two such covers are given in Figs. 1e and 1f. We note that 9 of these covers, including the one on 1f, have non-cyclic edge dilation groups, and are therefore not algebraically realizable.

In total, there are 97 Klein covers of Γ, including 75 connected covers.

4.2 Weighted graphs and unramified G-covers

We now consider the category of weighted graphs and finite harmonic morphisms between them. Given a weighted graph Γ and a G-cover $\varphi : \Gamma' \to \Gamma$ (where we view Γ as an unweighted graph and φ as a morphism), there is a natural way to promote φ to a harmonic morphism of degree equal to $\#(G)$. Since the action of G is transitive on the fibers, the genera of all vertices of Γ' lying in a single fiber are equal. Therefore a G-cover of Γ with a given dilation datum D is uniquely specified by an element of $H^1(\Gamma, D)$ and a weight function $g' : V(\Gamma) \to \mathbb{Z}_{\geq 0}$ (which we lift to Γ'). There is a natural way to specify this weight: require φ to be unramified. This condition imposes a numerical restriction on the G-dilation datum D.

Definition 4.8. Let Γ be a weighted graph, and let G be a finite abelian group. A G-*cover* of Γ is a finite harmonic morphism $\varphi : \Gamma' \to \Gamma$ together with an action of G on Γ', such that the following properties are satisfied:

1. The action is invariant with respect to φ.

2. For each $x \in X(\Gamma)$, the group G acts transitively on the fiber $\varphi^{-1}(x)$.

3. $\#(G) = \deg \varphi$.

We say that a G-cover φ is *effective* or *unramified* if it is so as a harmonic morphism.

Remark 4.9. This definition is similar to Definition 7.1.2 in [BR11].

Example 4.10. Let Γ be a weighted graph. In Example 3.3, we saw that an element $\eta \in H^1(\Gamma, G)$ determines a topological G-cover $\varphi : \Gamma' \to \Gamma$. We now weight Γ' by setting $g(v') = g(v)$ for all $v \in V(\Gamma)$ and all $v' \in \varphi^{-1}(v)$. Setting $\deg_\varphi(x) = 1$ for all $x \in X(\Gamma')$, we see that φ is an unramified G-cover. Conversely, it is clear that a G-cover $\varphi : \Gamma' \to \Gamma$ is a topological G-cover if and only if $\deg_\varphi(x) = 1$ for all $x \in X(\Gamma')$.

We now classify all G-covers and unramified G-covers of a given weighted graph Γ. We first note that there is no difference between studying G-covers of a weighted graph and G-covers of the underlying unweighted graph. Indeed, let $\varphi : \Gamma' \to \Gamma$ be a G-cover of a weighted graph Γ, and let D_φ be the associated G-dilation datum. An element $g \in G$ determines an automorphism of Γ', which in particular is an unramified cover of degree one. Therefore, for any $x \in X(\Gamma)$, the harmonic morphism φ has the same degree at x and at $g(x)$. Since G acts transitively on $\varphi^{-1}(x)$, we see that $d_\varphi(x')$ is the same for all $x' \in \varphi^{-1}(x)$. Since

$$\deg \varphi = \sum_{x' \in \varphi^{-1}(x)} d_\varphi(x') = d_\varphi(x')\#\big(\varphi^{-1}(x)\big) = d_\varphi(x')\big[G : D_\varphi(x)\big],$$

we see that

$$d_\varphi(x') = \#\big(D_\varphi(x)\big) \tag{15}$$

for all $x' \in \varphi^{-1}(x)$. Therefore, the local degrees of φ are uniquely defined by the associated dilation datum. Conversely, if $\varphi : \Gamma' \to \Gamma$ is a G-cover of Γ viewed as an unweighted graph, then Eq. (15) gives the unique way to promote φ to a harmonic morphism of degree $\#(G)$. As a result, the classification of G-covers of weighted graphs reduces trivially to the unweighted case, except that we need to manually specify the weights on the cover.

Theorem 4.11. *Let Γ be a weighted graph, let G be a finite abelian group, let D be a G-dilation datum on Γ, and let $g' : V(\Gamma) \to \mathbb{Z}_{\geq 0}$ be a function. Then the there is a natural bijection between $H^1(\Gamma, D)$ and the set of G-covers $\varphi : \Gamma' \to \Gamma$ having dilation datum D, such that $g(v') = g'(\varphi(v'))$ for all $v' \in V(\Gamma')$.*

Proof. This follows immediately from Thm. 4.1, since G acts transitively on each fiber $\varphi^{-1}(v)$ and therefore the numbers $g(v')$ for $v' \in \varphi^{-1}(v)$ are all equal to some $g'(v)$. ☐

For the remainder of this section, we restrict our attention to unramified G-covers, which are the graph-theoretic analogues of étale maps. Given such a cover $\varphi : \Gamma' \to \Gamma$, we consider the Riemann–Hurwitz condition (1) at all vertices $v' \in V(\Gamma')$. This condition uniquely specifies the genera of the vertices of Γ'. However, these genera may fail to be non-negative integers, which imposes a numerical constraint on the G-dilation data on Γ that are associated to unramified G-covers.

Definition 4.12. Let (Γ, D) be a G-dilated graph. We define the *index function* $a_{\Gamma,D} : V(\Gamma) \times S(G) \to \mathbb{Z}_{\geq 0}$ of (Γ, D) by

$$a_{\Gamma,D}(v; H) = \#\big\{h \in T_v\Gamma \big| D(h) = H\big\}. \tag{16}$$

Proposition 4.13. *Let $\varphi : \Gamma' \to \Gamma$ be an unramified G-cover, let D_φ be the associated G-dilation datum, and let $a_{\Gamma,D}$ be the index function of (Γ, D_φ). Let $v \in V(\Gamma)$ be a vertex with dilation group $D(v)$, and let $S(D(v))$ be the set of subgroups of $D(v)$. Then*

$$2 - 2g'(v) - \sum_{K \in S(D(v))} a_{\Gamma,D}(v;K)\big[D(v) : K\big] = \#\big(D(v)\big)\Big[2 - 2g(v) - \sum_{K \in S(D(v))} a_{\Gamma,D}(v;K)\Big] \tag{17}$$

where $g'(v)$ is the genus of any vertex $v' \in \varphi^{-1}(v)$.

Proof. For any half-edge $h \in T_v\Gamma$, the dilation group $D(h)$ is a subgroup of $D(v)$. Hence

$$\mathrm{val}(v) = \sum_{K \in S(D(v))} a_{\Gamma,D}(v;K).$$

As noted above, each $h \in T_v\Gamma$ has $[D(v):D(h)]$ preimages in Γ' attached to v'. Therefore

$$\mathrm{val}(v') = \sum_{K \in S(D(v))} a_{\Gamma,D}(v;K)\big[D(v):K\big].$$

Plugging this into (1), we obtain (17).

$\square$

Definition 4.14. Let Γ be a weighted graph. A G-dilation datum D on Γ is called *admissible* if for every $v \in V(\Gamma)$ the number

$$g'(v) = \#\big(D(v)\big)\big[g(v)-1\big] + 1 + \frac{1}{2}\sum_{K \in S(D(v))} a_{\Gamma,D}(v;K)\big(\#(D(v)) - [D(v):K]\big) \tag{18}$$

determined by (17) is a non-negative integer. A G-stratification $\mathcal{S}$ is called *admissible* if the associated G-stratification D is admissible.

It is clear that the G-dilation datum associated to an unramified G-cover is admissible. Conversely, if D is an admissible G-dilation datum, then (18) uniquely specifies the weight function on any G-cover of Γ with dilation datum D. Hence we obtain the following result.

Theorem 4.15. *Let (Γ, D) be G-dilated weighted graph. If D is admissible, then there is a natural bijection between the set of unramified G-covers of Γ having dilation datum D and $H^1(\Gamma, D)$. Otherwise, there are no unramified covers of Γ having dilation datum D.*

Proof. The result follows immediately from Thm. 4.11 and Prop. 4.13. $\square$

Condition (18) imposes two restrictions on a G-stratification $\mathcal{S}$ (equivalently, on a G-dilation datum D): a stability condition ($g'(v)$ is non-negative) and a parity condition ($g'(v)$ is an integer). We make a number of general observations. First, we note that the admissibility condition is trivially satisfied at each undilated vertex $v \in V(\Gamma)\backslash V(\Gamma_{dil})$. Indeed, if $D(v) = 0$, then equation (18) reduces to $g'(v) = g(v)$. We also observe that $g'(v)$ is positive, and hence the stability condition is satisfied, if $g(v) \geq 1$. We also note that $g'(v)$ is an integer if $\#(D(v))$ is odd, so (18) does not impose a parity condition if the order of G is odd.

Remark 4.16. Equation (18) is the only role that the weight function on Γ plays in the classification of unramified G-covers of Γ. Furthermore, when checking the admissibility condition at a vertex $v \in V(\Gamma)$, we only need to know whether $g(v)$ is positive or not, the actual value is not important. Therefore, for example, two weighted graphs having the same underlying unweighted graph, and having the same set of genus zero vertices with respect to the two weight functions, will have the same set of unramified G-covers.

We now show that an admissible G-stratification has the following semistability properties.

Proposition 4.17. *Let $\mathcal{S}$ be an admissible G-stratification of a graph Γ. For every simple vertex $v \in V(\Gamma)$ and for each $H \in S(G)$, either $v \in V(\Gamma_H)$ and $\mathrm{val}_{\Gamma_H}(v) = 2$, or $v \notin V(\Gamma_H)$.*

Proof. Suppose that $\mathcal{S}$ is admissible. Let $v \in V(\Gamma)$ be a vertex with $g(v) = 0$ and two tangent directions h_1 and h_2. We write condition (18) at v:

$$g'(v) = 1 - \frac{1}{2}\left([D(v) : D(h_1)] + [D(v) : D(h_2)]\right).$$

The only way that this number can be a non-negative integer is $D(v) = D(h_1) = D(h_2)$, hence $v \in V(\Gamma_H)$ and $\operatorname{val}_{\Gamma_H}(v) = 2$ if $H \subset D(v)$ and $v \notin V(\Gamma_H)$ otherwise.

□

Proposition 4.18. *Let $\mathcal{S}$ be an admissible G-stratification of a graph Γ. Then the dilated subgraph $\Gamma_{dil} \subset \Gamma$ is semistable.*

Proof. We recall that Γ_{dil} is the union of the Γ_H for all subgroups $H \subset G$ except $H = 0$. Let D be the G-dilation datum associated to $\mathcal{S}$, and let $v \in V(\Gamma_{dil})$ be a vertex, so that $D(v) \neq 0$, and assume that $g(v) = 0$. If v is an isolated vertex of Γ_{dil}, then $a_{\Gamma,D}(v; K) = 0$ for all subgroups $K \subset D(v)$ such that $K \neq 0$. It follows that the sum in the right hand side of (18) vanishes, hence $g'(v) = -\#\big(D(v)\big) + 1 < 0$. Similarly, suppose that v is an extremal vertex of Γ_{dil}, so that there exists a unique edge $h \in T_v\Gamma_{dil}$ with $H = D(h) \neq 0$. It follows that $a_{\Gamma,D}(v; H) = 1$ and $a_{\Gamma,D}(v; K) = 0$ for all $K \neq 0, H$, hence

$$g'(v) = -\#\big(D(v)\big) + 1 + \frac{1}{2}\big(\#(D(v)) - [D(v) : H]\big) = 1 - \frac{\#\big(D(v)\big)}{2}\left(1 + \frac{1}{\#(H)}\right) < 0,$$

since $\#\big(D(v)\big) \geq \#(H) \geq 2$. Therefore $\operatorname{val}_{\Gamma_{dil}}(v) \geq 2$ and Γ_{dil} is semistable.

□

Unramified G-covers and stability

Let Γ be a weighted graph, and let Γ_{st} be its stabilization. We have seen in Def. 2.11 that any unramified cover $\varphi : \Gamma' \to \Gamma$ descends to an unramified cover $\varphi_{st} : \Gamma'_{st} \to \Gamma_{st}$. It follows that we can restrict unramified G-covers of Γ to its stabilization, and vice versa.

Proposition 4.19. *Let Γ be a weighted graph. Then there is a natural bijection between the unramified G-covers of Γ and the unramified G-covers of Γ_{st}.*

Proof. Let $\varphi : \Gamma' \to \Gamma$ be an unramified G-cover. The G-action descends to the subgraph $\Gamma'_{sst} \subset \Gamma'$, hence $\varphi_{sst} : \Gamma'_{sst} \to \Gamma_{sst}$ is a G-cover. We note that the supporting arguments for Def. 2.11 show that φ is undilated on $\Gamma' \backslash \Gamma_{sst}$; alternatively, this follows from Prop. 4.18, since any semistable subgraph of Γ is contained in Γ_{sst}. Therefore, for any vertex $v \in V(\Gamma_{sst})$, any adjacent half-edge $h \in H(\Gamma) \backslash H(\Gamma_{sst})$ has $\deg \varphi$ preimages in $H(\Gamma')$, evenly split among the preimages of v. It follows that φ_{sst} is an unramified G-cover. It is then clear how to descend the G-action to $\varphi_{st} : \Gamma'_{st} \to \Gamma_{st}$: for any $g \in G$ and any simple vertex $v' \in V(\Gamma'_{sst})$ that is replaced by an edge or a leg, g maps that edge or leg to the edge or leg that replaces $g(v)$.

Conversely, let Γ be a weighted graph, and let $\varphi_{st} : \Gamma'_{st} \to \Gamma_{st}$ be an unramified G-cover, where Γ'_{st} is a stable weighted graph. The semistabilization Γ_{sst} is obtained from Γ_{st} by splitting edges and legs at new vertices of genus 0. Performing the same operation on the preimages of these vertices in Γ'_{st}, we obtain an unramified G-cover $\varphi_{sst} : \Gamma'_{sst} \to \Gamma_{sst}$. The graph Γ is obtained from Γ_{sst} by attaching trees having no vertices of positive genus. For each such tree T attached at $v \in V(\Gamma_{sst})$, we attach $\#(G)$ copies of T to Γ'_{sst} at the fiber $\varphi^{-1}(v)$, and extend the G-action in the obvious way. We obtain an unramified G-cover $\varphi : \Gamma' \to \Gamma$ whose stabilization is φ_{st}.

□

4.3 G-covers of weighted metric graphs and tropical curves

In this final section, we reformulate our classification results for weighted metric graphs and tropical curves. There is essentially no new mathematical content obtained by adding metrics to graphs, so this section is essentially a restatement and a summary of the results of the previous sections, and is included for the reader's convenience.

First, we introduce G-covers of weighted metric graphs:

Definition 4.20. Let (Γ, ℓ) be a weighted metric graph. A G-*cover* of (Γ, ℓ) is a finite harmonic morphism $\varphi : (\Gamma', \ell') \to (\Gamma, \ell)$ together with an action of G on (Γ', ℓ'), such that the following properties are satisfied:

1. The action is invariant with respect to φ.
2. For each $x \in X(\Gamma)$, G acts transitively on the fiber $\varphi^{-1}(x)$.
3. $\#(G) = \deg \varphi$.

We say that a G-cover φ is *unramified* if it is an unramified harmonic morphism.

In other words, a G-cover $\varphi : (\Gamma', \ell') \to (\Gamma, \ell)$ is a G-cover $\varphi : \Gamma' \to \Gamma$ of the underlying weighted graph Γ that satisfies the dilation condition (3). Given φ and ℓ, there is a unique way to choose ℓ' such that the dilation condition is satisfied (see Rem. 2.15). It follows that the classification of G-covers of (Γ, ℓ) is identical to the classification of G-covers of Γ. Specifically, such a cover is uniquely determined by choosing the dilation subgroups, an element of the corresponding dilated cohomology group, and a genus assignment on Γ which is then lifted to Γ'. To obtain unramified G-covers, we require the dilation data to be admissible, and pick the genus using Eq. (18):

Theorem 4.21. *Let (Γ, ℓ) be a weighted metric graph. There is a natural bijection between the set of G-covers of (Γ', ℓ') and the set of triples (D, η, g'), where*

1. *D is a G-dilation datum on the underlying weighted graph Γ,*
2. *η is an element of $H^1(\Gamma, D)$,*
3. *g' is a map from $V(\Gamma)$ to $\mathbb{Z}_{\geq 0}$.*

The set of unramified G-covers is obtained by choosing D to be an admissible G-dilation datum, and defining g' by Eq. (18).

Proof. This follows immediately from Thms. 4.11 and 4.15, and Rem. 2.15.

□

Finally, we describe G-covers of tropical curves.

Definition 4.22. Let $\sqsubset$ be a tropical curve. A G-*cover* of $\sqsubset$ is a finite harmonic morphism $\tau : \sqsubset' \to \sqsubset$ together with an action of G on $\sqsubset'$ such that the following properties are satisfied:

1. The action is invariant with respect to τ.

2. For each $x \in \sqsubset$, G acts transitively on the fiber $\tau^{-1}(x)$.

3. $\#(G) = \deg \tau$.

We say that a G-cover τ is *unramified* if it is an unramified harmonic morphism.

To describe G-covers of a tropical curve $\sqsubset$, we need to define G-dilation data on $\sqsubset$. We can define this to be a G-dilation datum on some model of $\sqsubset$. It is more convenient to define dilation in terms of the associated stratification, which does not involve choosing a model. The following definition generalizes Defs. 3.9, 4.12, and 4.14 to the case of tropical curves.

Definition 4.23. Let $\sqsubset$ be a tropical curve. A G-*stratification* $\mathcal{S} = \{\sqsubset_H : H \in S(G)\}$ on $\sqsubset$ is a collection of subcurves $\sqsubset_H \subset \sqsubset$, indexed by the set $S(G)$ of subgroups of G, such that

- $\sqsubset_0 = \sqsubset$,
- $\sqsubset_K \subset \sqsubset_H$ if $H \subset K$,
- $\sqsubset_H \cap \sqsubset_K = \sqsubset_{H+K}$ for all $H, K \in S(G)$.

We allow $\sqsubset_H$ to be empty or disconnected for $H \neq 0$. A G-stratification $\mathcal{S}$ partitions $\sqsubset$ into disjoint subsets

$$\sqsubset = \coprod_{H \in S(G)} \sqsubset_H \backslash \sqsubset_H^0 \quad \text{and} \quad \sqsubset_H^0 = \bigcup_{H \subsetneq K} \sqsubset_K.$$

For $x \in \sqsubset$ we define the *dilation subgroup* $D(x)$ to be the unique subgroup $H \subset G$ such that $x \in \sqsubset_H \backslash \sqsubset_H^0$. We define the *index function* $a_{\mathcal{S}} : \sqsubset \times S(G) \to \mathbb{Z}_{\geq 0}$ of $\mathcal{S}$ by setting $a_{\mathcal{S}}(x, H)$ to be the number of connected components of the intersection of $\sqsubset_H \backslash \sqsubset_H^0$ with a sufficiently small punctured neighborhood of x. We say that $\mathcal{S}$ is *admissible* if for every $x \in \sqsubset$ the number

$$g'(x) = \#(D(x))\big[g(x) - 1\big] + 1 + \frac{1}{2} \sum_{K \in S(D(x))} a_{\mathcal{S}}(x; K)\big(\#(D(x)) - [D(x) : K]\big) \tag{19}$$

is a non-negative integer.

Finally, we define the dual stratification $\mathcal{S}^*$ of a stratification $\mathcal{S}$ of $\sqsubset$ as follows. Choose a model Γ of $\sqsubset$ minimal with respect to the property that each element $\sqsubset_H$ of $\mathcal{S}$ corresponds to a subgraph Γ_H of Γ. Then the Γ_H form a G-stratification of the weighted metric graph Γ, and we let $\mathcal{S}^*$ be the dual of this stratification. We note that choosing a larger model Γ' will result in a larger dual stratification, which will, however, retract to $\mathcal{S}^*$.

Similarly, we can define the dilated cohomology groups of a tropical curve with a G-stratification:

Definition 4.24. Let $\mathcal{S}$ be a stratification of a tropical curve $\sqsubset$. Pick a model (Γ, ℓ) for $\sqsubset$ such that each $\sqsubset_H$ corresponds to a subgraph Γ_H of Γ, then $\mathcal{S}$ is a G-stratification of Γ and induces a G-dilation datum D on Γ. We define the *dilated cohomology group* $H^1(\sqsubset, \mathcal{S})$ as the cohomology group $H^1(\Gamma, D)$; it is clear that this group does not depend on the choice of model.

We can now state our main classification result for G-covers of tropical curves, which is simply a restatement of Thm. 4.21 using the equivalent description of dilation by means of a stratification:

Theorem 4.25. *Let Γ be a tropical curve. There is a natural bijection between the set of G-covers of Γ and the set of triples $(\mathcal{S}, \eta, g')$, where*

1. *$\mathcal{S}$ is a G-stratification of Γ,*

2. *η is an element of $H^1(\Gamma, \mathcal{S})$,*

3. *g' is a function from Γ to $\mathbb{Z}_{\geq 0}$.*

The set of unramified G-covers of Γ is obtained by requiring $\mathcal{S}$ to be an admissible G-stratification, and defining g' by (19).

We also restate Prop. 4.19 for tropical curves.

Proposition 4.26. *Let Γ be a tropical curve. Then there is a natural bijection between the unramified G-covers of Γ and the unramified G-covers of Γ^{st}.*

Remark 4.27. Any tropical curve Γ has infinitely many G-covers for any nontrivial group G, since we can choose a dilation stratification with arbitrarily many connected components. However, the number of *unramified* G-covers of Γ is finite. Indeed, Prop. 4.17 shows that if $\mathcal{S}$ is an admissible stratification of Γ, then no Γ_H can contain any simple point $x \in \Gamma$ as an unstable extremal point. Since any tropical curve has only finitely many non-simple points, it follows that the number of admissible stratifications of a tropical curve is finite, and hence so is the number of unramified G-covers.

Cyclic covers of prime order

We now classify the unramified G-covers of a tropical curve Γ in the case when $G = \mathbb{Z}/p\mathbb{Z}$, where p is prime. These covers were studied in [JL18] and [BBC17] for $p = 2$, and in [BH17] for arbitrary p in the case when Γ is a tree.

Let Γ be a tropical curve, let p be a prime number, and let $G = \mathbb{Z}/p\mathbb{Z}$. A G-stratification $\mathcal{S} = \{\Gamma_0, \Gamma_G\}$ of Γ has a single nontrivial element $\Gamma_G = \Gamma_{dil}$, the dilated subcurve. Condition (19) is trivially satisfied at any non-dilated point. If $x \in \Gamma_G$, then $D(x) = \mathbb{Z}/p\mathbb{Z}$ and $a_{\mathcal{S}}(x; \mathbb{Z}/p\mathbb{Z}) = \operatorname{val}_{\Gamma_G}(x)$, and condition (19) is

$$g'(x) = \big[g(x) - 1\big]p + 1 + \frac{p-1}{2}\operatorname{val}_{\Gamma_G}(x).$$

We see that $g'(x)$ is non-negative if $g(x) > 0$, or if $g(x) = 0$ and $\operatorname{val}_{\Gamma_G}(x) \geq 2$. Similarly, $g'(x)$ is an integer if $p \geq 3$, or if $p = 2$ and $\operatorname{val}_{\Gamma_G}(x)$ is even. We therefore have the following result:

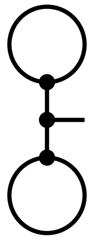
1. For $p \geq 3$, a $\mathbb{Z}/p\mathbb{Z}$-stratification Γ is admissible if and only if the dilated subcurve Γ_G is semistable.

2. For $p = 2$, a $\mathbb{Z}/p\mathbb{Z}$-stratification Γ is admissible if and only if the dilated subcurve Γ_G is a semistable cycle. This was observed in [JL18] (see Corollary 5.5).

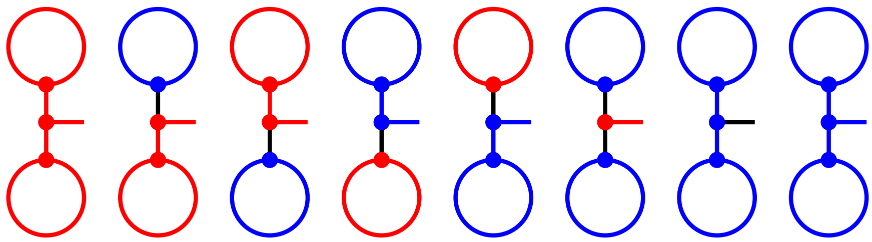
If $\mathcal{S}$ is an admissible $\mathbb{Z}/p\mathbb{Z}$-stratification of Γ, Ex. 3.24 and Thm. 4.15 shows that the set of unramified G-covers of Γ having dilation stratification $\mathcal{S}$ is equal to $H^1(\Gamma^0, \mathbb{Z}/p\mathbb{Z})$, where Γ^0 is the nontrivial element of the dual stratification $\mathcal{S}^*(D)$. Specifically, Γ^0 is obtained from Γ

by removing Γ_G, and then removing any edges or legs that are missing an endpoint. In other words, an unramified $\mathbb{Z}/p\mathbb{Z}$-cover of Γ is uniquely specified by choosing a (possibly empty) semistable subcurve $\Gamma_G \subset \Gamma$, which is required to be a cycle when $p = 2$, and an element of $H^1(\Gamma^0, \mathbb{Z}/p\mathbb{Z})$.

As an example, we count the number of unramified $\mathbb{Z}/p\mathbb{Z}$-covers of the following genus two tropical curve Γ with one leg (the edge lengths are arbitrary and irrelevant):



This curve has eight admissible $\mathbb{Z}/p\mathbb{Z}$-stratifications, listed below, with the last one being admissible only when p is odd. We draw the semistable dilated subgraph Γ_G in blue, and the corresponding element Γ^0 of the dual stratification in red. Below each stratification we list the number $p^{b_1(\Gamma^0)}$ of $\mathbb{Z}/p\mathbb{Z}$-covers with the given stratification.



p^2 p p p p 1 1 1

Hence there are a total of $p^2 + 4p + 3$ covers when p is odd, and 14 covers when $p = 2$.

5 Tropicalizing the moduli space of admissible G-covers

In this section we explain how unramified tropical G-covers naturally arise as tropicalizations of algebraic G-covers from a moduli-theoretic perspective, expanding on [ACP15] and [CMR16] (recall that tropical unramified covers are called *tropical admissible covers* in [CMR16]).

Throughout this section we assume that the genus $g \geq 2$ and we work over an algebraically closed field k of characteristic zero endowed with the trivial absolute value. In this section, we do not need to assume that G is abelian.

5.1 Compactifying the moduli space of G-covers

Let G be a finite group and let $X \to S$ be a family of smooth projective curves of genus g. A G-cover of X is a finite unramified Galois morphism $f\colon X' \to X$ together with an isomorphism $\mathrm{Aut}(X'/X) \simeq G$. Denote by $\mathcal{H}_{g,G}$ the moduli space of connected G-covers of smooth curves of genus g (see e.g. [RW06] for a construction). There is a good notion of a limit object as X degenerates to a stable curve, as introduced in [ACV03]. The definition below generalizes this construction, by allowing a fixed ramification profile along marked points.

Definition 5.1. Let G be a finite group and let $X \to S$ be a family of stable curves of genus g with n marked disjoint sections $s_1, \dots, s_n$. Let $\mu = (r_1, \dots, r_n)$ be a n-tuple of natural numbers that divide $\#(G)$, and denote $k_i = \#(G)/r_i$ for $i = 1, \dots, n$. An *admissible* G*-cover* of X consists of a finite morphism $f\colon X' \to X$ from a family of stable curves $X' \to S$ that is Galois and unramified away from the sections of X, an action of G on X', and disjoint sections s'_{ij} of X' over S for $i = 1, \dots, n$ and $j = 1, \dots, k_i$, subject to the following conditions:

(i) The map $f : X' \to X$ is a principal G-bundle away from the nodes and sections of X.

(ii) The preimage of the set of nodes in X is precisely the set of nodes of X'.

(iii) The preimage of a section s_i is precisely given by the sections $s'_{i1}, \dots, s'_{ik_i}$.

(iv) Let p be a node in X and p' a node of X' above p. Then étale-locally p' is given by $x'y' = t$ for $t \in \mathcal{O}_S$ and p is étale-locally given by $xy = t^r$ for some integer $r \geq 1$ with $x' = x^r$ and $y' = y^r$, and the stabilizer of G at p' is cyclic of order r and operates via

$$(x', y') \longmapsto (\zeta x', \zeta^{-1} y')$$

for an r-th root of unity $\zeta \in \mu_r$.

(v) Étale-locally near the sections s_i and s'_{ij} respectively, the morphism f is given by $\mathcal{O}_S[t_i] \to \mathcal{O}_S[t'_{ij}]$ with $(t'_{ij})^{r_i} = t_i$, and the stabilizer of G along s_{ij} is cyclic or order r_i and operates via $t'_{ij} \mapsto \zeta t$, for an r_i-th root of unity $\zeta \in \mu_{r_i}$.

We emphasize that the G-action is part of the data, in particular, an isomorphism between two admissible G-covers has to be a G-equivariant isomorphism. As explained in [ACV03], the moduli space $\overline{\mathcal{H}}_{g,G}(\mu)$ of G-admissible covers of stable n-marked curves of genus g is a smooth and proper Deligne-Mumford stack that contains the locus $\mathcal{H}_{g,G}(\mu)$ of G-covers of smooth curves of ramification type μ as an open substack. The complement of $\mathcal{H}_{g,G}(\mu)$ is a normal crossing divisor.

Remark 5.2. Although closely related, the moduli space $\overline{\mathcal{H}}_{g,G}(\mu)$ is not quite the same as the one constructed in [ACV03]. The quotient

$$\left[\overline{\mathcal{H}}_{g,G}(\mu)/S_{k_1} \times \dots \times S_{k_n}\right]$$

which forgets about the order of the marked sections on s'_{ij} of X' over S for $i = 1, \dots, n$ and $j = 1, \dots, k_i$, is equivalent to a connected component of the moduli space of twisted stable maps to $\mathbf{B}G$ in the sense of [AV02, ACV03], indexed by ramification profile and decomposition into connected components. Our variant of this moduli space $\overline{\mathcal{H}}_{g,G}(\mu)$, with ordered sections on X', has also appeared in [SvZ18] and in [JKK05] (the latter permitting admissible covers with possibly disconnected domains).

An object in this stack is technically not a admissible G-cover $X' \to X$ but rather a G-cover $X' \to \mathcal{X}$ of a twisted stable curve $\mathcal{X}$. A *twisted stable curve* $\mathcal{X} \to S$ is a Deligne-Mumford stack $\mathcal{X}$ with sections $s_1, \dots, s_n\colon S \to \mathcal{X}$ whose coarse moduli space $X \to S$ is a family of stable curves over S with n marked sections (also denoted by $s_1, \dots, s_n$) such that

- The smooth locus of $\mathcal{X}$ is representable by a scheme,

- The singularities are étale-locally given by $[\{x'y' = t\}/\mu_r]$ for $t \in \mathcal{O}_S$, where $\zeta \in \mu_r$ acts by $\zeta \cdot (x', y') = (\zeta x', \zeta^{-1} y')$. In this case the singularity in X' is locally given by $xy = t^r$.
- The stack $\mathcal{X}$ is a root stack $[\sqrt[r_i]{s_i/X}]$ along the section s_i for all $i = 1, \ldots n$.

Both notions are naturally equivalent: given a G-admissible cover $X' \to X$ the associated twisted G-cover is given by $X' \to [X'/G]$. Conversely, given a twisted G-cover $X' \to \mathcal{X}$ in the corresponding connected component, the composition $X' \to \mathcal{X} \to X$ with the morphism to the coarse moduli space X is a G-admissible cover. We refer the interested reader to [BR11] for an alternative approach to this construction.

5.2 The moduli space of unramified tropical G-covers

We now construct a moduli space $H_{g,G}^{trop}(\mu)$ of unramified G-covers of stable tropical curves of genus g with n marked points and ramification profile $\mu = (r_1, \ldots, r_n)$, where each r_i divides $\#(G)$. Denote $k_i = \#(G)/r_i$, as well as $k = k_1 + \cdots + k_n$, and assume that $n \cdot \#(G) - k$ is even. A point $[\varphi, l, l']$ of $H_{g,G}^{trop}(\mu)$ consists of the following data:

1. A G-equivariant isomorphism class of an unramified G-cover $\varphi \colon \sqsubset' \to \sqsubset$ of a stable tropical curve $\sqsubset$ of genus g with n legs, by a stable tropical curve $\sqsubset'$ of genus
$$g' = (g-1) \cdot \#(G) + 1 + (n \cdot \#(G) - k)/2$$
with k legs.
2. A marking $l : \{1, \ldots, n\} \simeq L(\sqsubset)$ of the legs of $\sqsubset$.
3. A marking $l' \colon \{(1, \ldots, k_1), \ldots, (1, \ldots, k_n)\} \simeq L(\sqsubset')$ of the legs of $\sqsubset'$ such that $\varphi(l'_{ij}) = l_i$, where we denote $l_i = l(i)$ and $l'_{ij} = l(i, j)$.

Proposition 5.3. *The moduli space $H_{g,G}^{trop}(\mu)$ naturally carries the structure of a generalized cone complex.*

Proof. We need to show that $H_{g,G}^{trop}(\mu)$ is naturally the colimit of a diagram of rational polyhedral cones connected by (not necessarily proper) face morphisms. We first construct an index category $J_{g,G}(\mu)$ as follows:

- The objects are tuples $(\varphi \colon \Gamma' \to \Gamma, l, l')$, where Γ' and Γ are stable weighted graphs of genera g' and g having respectively k and n legs, φ is an unramified G-cover, and l' and l are markings of the legs of Γ' and Γ, respectively, such that $\varphi(l'_{ij}) = l_i$.
- The morphisms are generated by the automorphisms of $\Gamma' \to \Gamma$ that preserve the markings on both Γ and Γ', and weighted edge contractions (see Def. 2.12) of the target graph Γ. We recall that a weighted edge contraction of the target graph Γ induces a weighted edge contraction of the source graph Γ' along the preimages of the contracted edges. Moreover, the G-action on Γ' induces a G-action on its weighted edge contraction, which is an unramified G-cover by Prop. 2.13.

We then consider a functor $\Sigma_{g,G}(\mu) \colon J_{g,G}(\mu) \to \mathbf{RPC}_{face}$ to the category $\mathbf{RPC}_{face}$ of rational polyhedral cones with (not necessarily proper) face morphisms defined as follows:

- An object $(\varphi : \Gamma' \to \Gamma, l, l')$ is sent to the rational polyhedral cone $\sigma_\varphi = \mathbb{R}_{\geq 0}^{E(\Gamma)}$.
- An automorphism of $(\Gamma' \to \Gamma, l, l')$ induces an automorphism of σ_φ that permutes the entries according to the induced permutation of the edges of Γ; for a set of edges $S \subset E(\Gamma)$, a weighted edge contraction $\varphi_S : \Gamma'/\varphi^{-1}(S) \to \Gamma/S$ of $\varphi : \Gamma' \to \Gamma$ induces a morphism $\sigma_{\varphi_S} \hookrightarrow \sigma_\varphi$ that sends σ_{φ_S} to the face of σ_φ given by setting all entries of the contracted edges equal to zero.

The natural maps $\sigma_\varphi \to H_{g,G}^{trop}(\mu)$ are given by associating to a point $(a_e)_{e \in E(\Gamma)} \in \mathbb{R}_{\geq 0}^{E(\Gamma)}$ an unramified G-cover $[\Gamma' \to \Gamma]$ defined as follows:

- In the special case that $a_e \neq 0$ for all $e \in E(\Gamma)$, the tropical curve Γ is given by the graph Γ with the metric $\ell(e) = a_e$. In general, the tropical curve Γ is given by contracting those edges $e \in E(\Gamma)$ for which $a_e = 0$ and then by endowing the contracted weighted graph with the induced edge length given by the $a_e \neq 0$.
- The tropical curve Γ' is defined accordingly: we first contract all edges that map to an edge e with $a_e = 0$, and then endow Γ' with the edge length $\ell'(e') = \ell(\varphi(e'))/d_\varphi(e)$ so that the induced map $\Gamma' \to \Gamma$ is an unramified G-cover of tropical curves.

The maps $\sigma_\varphi \to H_{g,G}^{trop}(\mu)$ naturally commute with the morphisms induced by $J_{g,G}(\mu)$, and therefore descend to a map

$$\varinjlim_{(\varphi,l,l') \in J_{g,G}(\mu)} \sigma_\varphi \simeq H_{g,G}^{trop}(\mu)$$

that is easily checked to be a bijection. This realizes $H_{g,G}^{trop}(\mu)$ as a colimit of a diagram of (not necessarily proper) face morphisms and therefore endows it with the structure of a generalized cone complex. □

There are natural *source* and *target morphisms*

$$\mathrm{src}_{g,G}^{trop}(\mu) \colon H_{g,G}^{trop}(\mu) \longrightarrow M_{g',k}^{trop} \qquad \text{and} \qquad \mathrm{tar}_{g,G}^{trop}(\mu) \colon H_{g,G}^{trop}(\mu) \longrightarrow M_{g,n}^{trop}$$

that are given by the associations

$$[\Gamma' \to \Gamma, l, l'] \longmapsto [\Gamma', l'] \qquad \text{and} \qquad [\Gamma' \to \Gamma, l, l'] \longmapsto [\Gamma, l]$$

respectively. By Rem. 4.27, the map $\mathrm{tar}_{g,G}^{trop}(\mu)$ has finite fibers.

Remark 5.4. The functor $J_{g,G}(\mu) \to \mathbf{RPC}_{face}$ in the proof of Proposition 5.3 defines a category fibered in groupoids over $\mathbf{RPC}_{face}$, i.e. a *combinatorial cone stack* in the sense of [CCUW17]. So we may think of $H_{g,G}^{trop}(\mu)$ as a "coarse moduli space" of a *cone stack* $\mathcal{H}_{g,G}^{trop}(\mu)$, a geometric stack over the category of rational polyhedral cones (see [CCUW17] for details), that parametrizes families of unramified tropical G-covers over rational polyhedral cones.

5.3 A modular perspective on tropicalization

Denote by $\mathcal{H}_{g,G}^{an}(\mu)$ the Berkovich analytic space[1] associated to $\mathcal{H}_{g,G}(\mu)$. We define a natural *tropicalization map*

$$\begin{aligned} \mathrm{trop}_{g,G}(\mu)\colon \mathcal{H}_{g,G}^{an}(\mu) &\longrightarrow H_{g,G}^{trop}(\mu) \\ [X' \to X, s_i, s'_{ij}] &\longmapsto \big[\Gamma_{X'} \to \Gamma_X, l, l'\big] \end{aligned}$$

that associates to an admissible G-cover $X' \to X$ of smooth curves over a non-Archimedean extension K of k an unramified tropical G-cover $\Gamma_{X'} \to \Gamma_X$ of the dual tropical curve Γ_X of X that is defined in the following way.

Let X be a smooth projective curve of genus g over a non-Archimedean extension K of k with n marked sections $s_1, \ldots, s_n$ over K. Let $(X' \to X, s'_{ij})$ be a G-cover of X, where $i = 1, \ldots, n$ and $j = 1, \ldots, k_i$. By the valuative criterion for properness, applied to the stack $\overline{\mathcal{H}}_{g,G}(\mu)$, there is a finite extension L of K such that $X'_L \to X_L$ extends to a family of admissible G-covers $f: \mathcal{X}' \to \mathcal{X}$ defined over the valuation ring R of L (with marked sections also denoted by s_i and s'_{ij}). The *dual tropical curve* (Γ_X, l) of $\mathcal{X}$ (and similarly $(\Gamma_{X'}, l')$ of $\mathcal{X}'$) is given by the following data:

- The dual graph $\Gamma_{\mathcal{X}_0}$ of the special fiber $\mathcal{X}_0$ of $\mathcal{X}$: the components of $\mathcal{X}_0$ correspond to vertices, nodes correspond to edges, and the sections correspond to legs.

- A vertex weight $V(\Gamma_{\mathcal{X}_0}) \to \mathbb{Z}_{\geq 0}$ that associates to a vertex v the genus of the normalization of the corresponding component of $\mathcal{X}_0$.

- A marking $l\colon \{1, \ldots n\} \simeq L(\Gamma_{\mathcal{X}_0})$ of the legs of $\Gamma_{\mathcal{X}_0}$ according to the full order of $s_1, \ldots, s_n$.

- An edge length $\ell\colon E(\Gamma_{\mathcal{X}_0}) \to \mathbb{R}_{>0}$ that associates to an edge e the positive real number $r \cdot \mathrm{val}(t)$, where the corresponding node is étale-locally given by an equation $xy = t^r$ for $t \in R$.

The map $f: \mathcal{X}' \to \mathcal{X}$ induces a map $\varphi: \Gamma_{\mathcal{X}'_0} \to \Gamma_{\mathcal{X}_0}$:

- Every component $X'_{v'}$ of $\mathcal{X}'_0$ is mapped to exactly one component X_v of $\mathcal{X}_0$.

- Every node $p_{e'}$ of $\mathcal{X}'_0$, over a node p_e of $\mathcal{X}_0$ given by $xy = t^r$ for $t \in R$ on the base, has a local equation $x'y' = t$ which determines the dilation factor $r = d_\varphi(e')$.

- Étale-locally around the sections s_i and s'_{ij} respectively, the morphism f is given by $\mathcal{O}_S[t_i] \to \mathcal{O}[t'_{ij}]$ with $(t'_{ij})^{r_i} = t_i$, so the dilation factor $d_\varphi(l'_{ij})$ is given by r_i.

The map $f: \Gamma_{\mathcal{X}'_0} \to \Gamma_{\mathcal{X}_0}$ is harmonic by [ABBR15a, Theorem A] (identifying both $\Gamma_{X'}$ and Γ_X with the non-Archimedean skeletons of $(X')^{an}$ and X^{an} respectively). Applying the Riemann–Hurwitz formula to $X_{v'} \to X_v$ shows that it is unramified. The operation of G on $\mathcal{X}'_0$ induces an operation of G on $\Gamma_{\mathcal{X}'_0}$ for which the map $\Gamma_{X'} \to \Gamma_X$ is G-invariant. The stabilizer at every edge e'_i and of every leg l'_{ij} is a cyclic group of order r_i and r_{ij} respectively by Definition 5.1 (iii) and (iv). Since $\mathcal{X}'_0 \to \mathcal{X}_0$ is a principal G-bundle away from the nodes, the operation of G on the fiber over each point in Γ_X is transitive and so $\Gamma_{X'} \to \Gamma_X$ is a G-cover.

[1] We implicitly work with the underlying topological space of the Berkovich analytic stack $\mathcal{H}_{g,G}^{an}(\mu)$, as introduced in [Uli17, Section 3].

Remark 5.5. In the language of twisted stable curves, the stabilizers of the G-operation on the nodes and legs of $\mathcal{X}_0'$ give rise to the dilation datum on the dual graph. So one may think of a dilation datum as a stack-theoretic enhancement of a tropical curve.

Since the boundary of $\overline{\mathcal{H}}_{g,G}(\mu)$ has normal crossings, the open immersion $\mathcal{H}_{g,G}(\mu) \hookrightarrow \overline{\mathcal{H}}_{g,G}(\mu)$ is a toroidal embedding in the sense of [KKMSD73]. Therefore, as explained in [Thu07, ACP15], there is a natural strong deformation retraction $\rho_{g,G}\colon \mathcal{H}_{g,G}^{an}(\mu) \to \mathcal{H}_{g,G}^{an}(\mu)$ onto a closed subset of $\mathcal{H}_{g,G}^{an}(\mu)$ that carries the structure of a generalized cone complex, the *non-Archimedean skeleton* $\Sigma_{g,G}(\mu)$ of $\mathcal{H}_{g,G}^{an}(\mu)$. Expanding on [CMR16, Theorem 1 and 4], we have:

Theorem 5.6. *The tropicalization map* $\mathrm{trop}_{g,G}(\mu)\colon \mathcal{H}_{g,G}^{an}(\mu) \longrightarrow H_{g,G}^{trop}(\mu)$ *factors through the retraction to the non-Archimedean skeleton* $\Sigma_{g,G}(\mu)$ *of* $\mathcal{H}_{g,G}^{an}(\mu)$*, so that the restriction*

$$\mathrm{trop}_{g,G}(\mu)\colon \Sigma_{g,G}(\mu) \longrightarrow H_{g,G}^{trop}(\mu)$$

to the skeleton is a finite strict morphism of generalized cone complexes. Moreover, the diagram

$$\begin{tikzcd}
\mathcal{H}_{g,G}^{an}(\mu) \arrow[rr, "\mathrm{src}_{g,G}^{an}(\mu)"] \arrow[dd, "\mathrm{tar}_{g,G}^{an}(\mu)"'] \arrow[dr, "\mathrm{trop}_{g,G}(\mu)"] & & \mathcal{M}_{g',k}^{an} \arrow[d, "\mathrm{trop}_{g',k}"] \\
& H_{g,G}^{trop}(\mu) \arrow[r, "\mathrm{src}_{g,G}^{trop}(\mu)"] \arrow[d, "\mathrm{tar}_{g,G}^{trop}(\mu)"'] & M_{g',k}^{trop} \\
\mathcal{M}_{g,n}^{an} \arrow[r, "\mathrm{trop}_{g,n}"'] & M_{g,n}^{trop} &
\end{tikzcd} \tag{20}$$

commutes.

In other words, the restriction of $\mathrm{trop}_{g,G}(\mu)$ onto a cone in $\Sigma_{g,G}(\mu)$ is an isomorphism onto a cone in $H_{g,G}^{trop}(\mu)$ and every cone in $H_{g,G}^{trop}(\mu)$ has at most finitely many preimages in $\Sigma_{g,G}(\mu)$.

Proof of Theorem 5.6. Let x be a closed point in $\overline{\mathcal{H}}_{g,G}(\mu)$, which corresponds to an admissible G-cover $X' \to X$ over k. Denote by $\varphi : \Gamma_{X'} \to \Gamma_X$ the corresponding unramified G-cover of the dual graphs. Denote by $\mathfrak{o}_k$ either k when $\mathrm{char}\, k = 0$ or the unique complete local ring with residue field k when $\mathrm{char}\, k = p > 0$ (using Cohen's structure theorem). The complete local ring at x is given by

$$\widehat{\mathcal{O}}_{\overline{\mathcal{H}}_{g,G}(\mu),x} \simeq \mathfrak{o}_k[\![t_1, \ldots, t_{3g-3+n}]\!]$$

where $t_i = 0$ for $i = 1, \ldots, r$ cuts out the locus where the corresponding node q_i of X remains a node. The retraction to the skeleton is locally given by

$$\begin{aligned} \big(\mathrm{Spec}\, \widehat{\mathcal{O}}_{\overline{\mathcal{H}}_{g,G}(\mu),x}\big)^{\beth} &\longrightarrow \overline{\mathbb{R}}_{\geq 0}^{r} \\ x &\longmapsto \big(-\log|t_1|_x, \ldots, -\log|t_r|_x\big)\ , \end{aligned} \tag{21}$$

where $(.)^{\beth}$ denotes the generic fiber functor constructed in [Thu07, Prop./Déf. 1.3] and $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$. We find that under the isomorphism $\sigma_\varphi \simeq \mathbb{R}_{\geq 0}^r$ the restriction of (21) to the preimage of $\mathbb{R}_{\geq 0}^r$ is nothing but the tropicalization map $\mathrm{trop}_{g,G}(\mu)$ defined above.

We observe the following:

- A degeneration of $X' \to X$ in $\overline{\mathcal{H}}_{g,G}(\mu)$ to another admissible G-cover $X'_0 \to X_0$ with corresponding $\varphi_0 : \Gamma_{X'_0} \to \Gamma_{X_0}$ may be described by additional coordinates $t_{r+1}, \ldots, t_{r_0}$ that encode the new nodes $q_{r+1}, \ldots, q_{r_0}$ in the degeneration. The induced map $\mathbb{R}^r_{\geq 0} \hookrightarrow \mathbb{R}^{r_0}_{\geq 0}$ describes σ_φ as the face of σ_{φ_0} that corresponds to letting the edges $e_{r+1}, \ldots, e_{r_0}$ have length zero.

- Denote by $E \subset \overline{\mathcal{H}}_{g,G}(\mu)$ the toroidal stratum containing x and by $\widetilde{E}$ and $\widetilde{x}$ respectively their images in $\overline{\mathcal{M}}_{g,n}$. The operation of the fundamental group $\pi_1(E, x)$ of E on $\mathbb{R}^r_{\geq 0} \simeq \mathrm{Hom}(\Lambda^+_E, \mathbb{R}_{\geq 0})$, where Λ^+_E denotes the monoid of effective divisors supported on the closure of E, naturally factors through the operation of $\pi_1(\widetilde{E}, \widetilde{x})$ on $\mathbb{R}^r_{\geq 0} \simeq \mathrm{Hom}(\Lambda^+_{\widetilde{E}}, \mathbb{R}_{\geq 0})$. Analogously, the operation of the automorphisms of $\Gamma_{X'} \to \Gamma_X$ on $\mathbb{R}^r_{\geq 0}$ naturally factors through the operation of the automorphisms of Γ_X on $\mathbb{R}^r_{\geq 0} = \mathbb{R}^{E(\Gamma_X)}_{\geq 0}$. Therefore, by [ACP15, Proposition 7.2.1], the images of the automorphism groups of both $\pi_1(E, x)$ and $\mathrm{Aut}(\Gamma_{X'} \to \Gamma_X)$ in the permutation group of the entries of $\mathbb{R}^r_{\geq 0}$ are equal.

This shows that the isomorphisms $\mathbb{R}^r_{\geq 0} \simeq \sigma_\varphi$ induce a necessarily strict morphism of generalized cone complexes $\Sigma_{g,G}(\mu) \to H^{trop}_{g,G}(\mu)$ that factors the tropicalization map as $\mathcal{H}^{an}_{g,G}(\mu) \to \Sigma_{g,G}(\mu) \to H^{trop}_{g,G}(\mu)$. Its fibers are finite, since above every toroidal stratum of $\overline{\mathcal{M}}_{g,n}$ there are only finitely many toroidal strata of $\overline{\mathcal{H}}_{g,G}(\mu)$. Finally, the commutativity of (20) is an immediate consequence of the definition of $\mathrm{trop}_{g,G}(\mu)$. □

Remark 5.7. In general, not every unramified tropical cover is realizable. We may, for example, consider an unramified cover for which the local ramification profile is not of Hurwitz type (e.g. when $d = 4$ and the ramification profile at a vertex is given by $\{(3,1), (2,2), (2,2)\}$). This explains why the tropicalization map on the moduli space of admissible covers (without the G-action), as considered in [CMR16], is not surjective. We refer the reader to [Cap14, Section 2.2] for a discussion of this issue in the context of comparing algebraic and tropical gonality and to [PP06] for a survey of the underlying widely open problem, the so-called *Hurwitz existence problem*.

We do not know whether, for a general finite abelian group G, every unramified tropical G-cover (with cyclic stablizers at the nodes) is realizable. When G itself is cyclic and there are no marked legs (along which ramification is possible), we do expect every G-admissible cover to be realizable, since by [JL18, Theorem 3.1] the tropicalization map on the level of n-torsion points on Jacobians is surjective. We will return to this topic in its proper setting in the upcoming [LUZ19].

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