TOPOLOGICAL RECURSION RELATIONS FROM PIXTON'S FORMULA

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Abstract. For any genus $g \leq 26$, and for $n \leq 3$ in all genus, we prove that every degree-g polynomial in the ψ-classes on $\overline{\mathcal{M}}_{g,n}$ can be expressed as a sum of tautological classes supported on the boundary with no κ-classes. Such equations, which we refer to as topological recursion relations, can be used to deduce universal equations for the Gromov–Witten invariants of any target.

1. Introduction

The tautological rings are $\mathbb{Q}$-subalgebras of the Chow ring of the moduli spaces of curves,

$$R^*(\overline{\mathcal{M}}_{g,n}) \subset A^*(\overline{\mathcal{M}}_{g,n}),$$

defined as the minimal system of such subalgebras closed under push-forward by the gluing morphisms

$$\overline{\mathcal{M}}_{g_1,n_1+1} \times \overline{\mathcal{M}}_{g_2,n_2+1} \to \overline{\mathcal{M}}_{g_1+g_2,n_1+n_2},$$
$$\overline{\mathcal{M}}_{g,n+2} \to \overline{\mathcal{M}}_{g+1,n}$$

and the forgetful morphisms

$$\overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}.$$

This elegant definition is due to Faber and Pandharipande [FP05], who also proved that $R^*(\overline{\mathcal{M}}_{g,n})$ admits an explicit set of additive generators that we call *strata classes*. The strata classes are constructed in terms of the ψ-classes

$$\psi_i = c_1(s_i^*(\omega_\pi)), \quad i = 1, \ldots, n$$

and the κ-classes

$$\kappa_d = \pi_*(\psi_{n+1}^{d+1}), \quad d \geq 0$$

on the boundary strata, where $\pi : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ forgets the last marked point and s_i is the section of π given by the ith marked point.

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Products of strata classes are described by an explicit multiplication rule, so the study of the tautological ring amounts to a search for *tautological relations*: linear equations satisfied by the strata classes. Of particular interest in our work are the equations in which no κ-classes appear, which we refer to as *topological recursion relations*, or TRRs. The well-known WDVV equations, for example, are TRRs in genus zero, and other specific examples in genus $g \leq 4$ were discovered by Getzler [Ge97, Ge98], Belorousski–Pandharipande [BP00], Kimura–Liu [KL06, KL15], and the third author [W18]. Liu–Pandharipande [LP11], in addition, proved TRRs expressing ψ_1^k on $\overline{\mathcal{M}}_{g,1}$ in terms of κ-free boundary classes whenever $k \geq 2g$.

One reason that TRRs are worthy of special interest is that they can be translated into universal equations for the descendent Gromov–Witten invariants of any target. For targets with semisimple quantum cohomology, these universal equations are known to be sufficient to determine all Gromov–Witten invariants in genus one or two in terms of genus-zero data; this follows from work of Dubrovin–Zhang [DZ98] in genus one and of Liu [L02] in genus two. The analogue in genus three does not follow from the currently known universal equations, though, and very little is understood in any genus outside of the semisimple setting. One might hope that a more complete picture of the TRRs on $\overline{\mathcal{M}}_{g,n}$ would fill some of these gaps.

The main theorem of our work is the following:

Theorem 1. *Suppose that $g > 0$, and either $g \leq 26$ or $n \leq 3$. Then there exists a topological recursion relation for every degree-g monomial in the ψ-classes on $\overline{\mathcal{M}}_{g,n}$.*

More explicitly, a topological recursion relation for $\alpha = \psi_1^{l_1} \cdots \psi_n^{l_n}$ refers to a tautological relation expressing α in terms of strata classes supported on the boundary of $\overline{\mathcal{M}}_{g,n}$ with no κ-classes; a precise definition is given in Section 2.1 below.

This theorem can be seen as a next step in a series of vanishing results for the tautological ring. Looijenga [L95] proved that the tautological ring $R^*(\mathcal{M}_g)$, which is generated by monomials in κ-classes, vanishes in degrees greater than or equal to $g-1$. Ionel proved in [I02] that $R^*(\mathcal{M}_{g,n})$, which is additively generated by monomials in the ψ- and κ-classes, vanishes (in cohomology) for degrees greater than $g-1$. The results of Graber–Vakil [GV05] and Faber–Pandharipande [FP05] imply that any monomial in the ψ- and κ-classes of degree greater than or equal to g can be expressed, in $R^*(\overline{\mathcal{M}}_{g,n})$, as a boundary class. This raises the natural question of finding explicit boundary formulas for such monomials on $\overline{\mathcal{M}}_{g,n}$.

In [CGJZ18], the first, second and fourth author described an explicit algorithm for finding such formulas (thus reproving these vanishing results in an effective manner). We considered a family of relations in $R^*(\overline{\mathcal{M}}_{g,n})$, which are referred to as *Pixton's relations* in what follows and which arise from the study of the double ramification cycle on $\overline{\mathcal{M}}_{g,n}$. These relations were conjectured by Pixton and proven by the first and second authors in [CJ16], and are not to be confused with Pixton's r-spin relations (see below). We observed in [CGJZ18] that Pixton's relations are a natural candidate for producing TRRs, for two reasons. First, Pixton's relations start in degree $g+1$, which is only one degree greater than the TRRs that we seek. Second, Pixton's relations do not involve any κ classes to begin with. To obtain relations in degree g, we multiply Pixton's relations by appropriate ψ-classes and push them forward under forgetful maps. We make the non-boundary contributions to these relations explicit in a family of cases. From here, simple linear algebra and computer verification shows that for (g, n) as in the statement of the theorem, the resulting equations are sufficient to produce TRRs for any ψ-monomial. Thus, the results of this paper are a sharpening of the algorithm described in [CGJZ18].

Note. At the final stages of writing this article, the paper [KLLS18] was posted. The authors of [KLLS18] studied the tautological rings $R^*(\overline{\mathcal{M}}_{g,n})$ using a different set of relations, known as r-spin relations (also conjectured by Pixton and proved in [PPZ15] and [J16]). Although it is not specifically noted in [KLLS18], our main result, Theorem 1, follows from [KLLS18, Lemma 5.2] for all n and all $g > 0$. Indeed, Lemma 5.2 reproves the vanishing in degrees greater than g of the tautological ring of $\mathcal{M}_{g,n}$, which is generated by monomials in ψ-classes. Specifically, it is proven using linear combinations of the polynomial r-spin relations [PPZ15] under the substitution $r = \frac{1}{2}$. As the authors note [KLLS18, Section 2.4], "graphs without dilaton leaves do not contribute to the tautological relations", which is another way of saying that the resulting relations do not involve κ-markings and are thus topological recursion relations.

We do not know the relationship between the explicit topological recursion relations established in [KLLS18] and those established in our paper. In view of the continued interest in this problem, we believe that both sets of relations deserve further investigation.

1.1. **Plan of the paper.** Section 2 contains the relevant preliminaries on the strata algebra and Pixton's relations. In Section 3, we describe a family of topological recursion relations constructed from Pixton's relations, make their contributions away from the boundary explicit, and use them to prove Theorem 1.

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2. The tautological ring and Pixton's formula

The additive generators of the tautological ring are expressed in terms of the strata algebra on $\overline{\mathcal{M}}_{g,n}$. We recall the necessary definitions, referring the reader to [GraP03] and [Pi13] for further details.

2.1. **The strata algebra and the tautological ring of $\overline{\mathcal{M}}_{g,n}$.** A *stable graph* $\Gamma = (V, H, E, L, g, p, \iota, m)$ of genus g with n legs consists of the following data:

(1) a finite set of vertices V equipped with a genus function $g : V \to \mathbb{Z}_{\geq 0}$;
(2) a finite set of half-edges H equipped with a vertex assignment $p : H \to V$ and an involution $\iota : H \to H$;
(3) a set of edges E, which is the set of two-point orbits of ι;
(4) a set of legs L, which is the set of fixed points of ι, and which is marked by a bijection $m : \{1, \dots, n\} \to L$.

We require that the following properties are satisfied:

(i) The graph (V, E) is connected.
(ii) For every vertex $v \in V$, we have
$$2g(v) - 2 + n(v) > 0,$$
where $n(v) = \#p^{-1}(v)$ is the *valence* of the vertex v
(iii) The genus of the graph is g, in the sense that
$$g = h^1(\Gamma) + \sum_{v \in V} g(v),$$
where $h^1(\Gamma) = \#E - \#V + 1$.

An *automorphism* of a stable graph Γ consists of permutations of the sets V and H that commute with the maps g, p, ι and m (and hence preserve L and E). We denote by $\mathrm{Aut}(\Gamma)$ the group of automorphisms of Γ.

Given a stable curve C of genus g with n marked points, its dual graph is a stable graph of genus g with n legs. If Γ is such a stable graph, let
$$\overline{\mathcal{M}}_\Gamma := \prod_{v \in V} \overline{\mathcal{M}}_{g(v), n(v)}.$$

There is a canonical gluing morphism

$$\xi_\Gamma : \overline{\mathcal{M}}_\Gamma \to \overline{\mathcal{M}}_{g,n}, \tag{1}$$

whose image is the locus in $\overline{\mathcal{M}}_{g,n}$ having generic point corresponding to a curve with stable graph Γ. The degree of ξ_Γ, as a map of Deligne–Mumford stacks, is equal to $\#\operatorname{Aut}(\Gamma)$.

Additive generators of the tautological ring can be described in terms of certain decorations on stable graphs Γ. Namely, let

$$\gamma = (x_i : V \to \mathbb{Z}_{\geq 0}, y : H \to \mathbb{Z}_{\geq 0}) \tag{2}$$

be a collection of functions such that

$$d(\gamma_v) = \sum_{i>0} i x_i[v] + \sum_{h \in p^{-1}(v)} y[h] \leq 3g(v) - 3 + n(v) \tag{3}$$

for all $v \in V$. Then, for each v, define

$$\gamma_v = \prod_{i>0} \kappa_i^{x_i[v]} \prod_{h \in p^{-1}(v)} \psi_h^{y[h]} \in A^{d(\gamma_v)}(\overline{\mathcal{M}}_{g(v),n(v)}).$$

Associated to any such choice of decorations γ, there is a *basic class* on $\overline{\mathcal{M}}_\Gamma$, also denoted by γ, defined by

$$\gamma = \prod_{v \in V} \gamma_v \in A^{d(\gamma)}(\overline{\mathcal{M}}_\Gamma).$$

Here, the degree is $d(\gamma) := \sum_{v \in V} d(\gamma_v)$, and we abuse notation slightly by using a product over classes on the vertex moduli spaces to denote a class on $\overline{\mathcal{M}}_\Gamma$.

The *strata algebra*, by definition, is the finite-dimensional $\mathbb{Q}$-vector space spanned by isomorphism classes of pairs $[\Gamma, \gamma]$, where Γ is a stable graph of genus g with n legs and γ is a basic class on $\overline{\mathcal{M}}_\Gamma$. The product is defined by excess intersection theory (see [F99]). More precisely, for $[\Gamma_1, \gamma_1], [\Gamma_2, \gamma_2] \in \mathcal{S}_{g,n}$, the fiber product of ξ_{Γ_1} and ξ_{Γ_2} over $\overline{\mathcal{M}}_{g,n}$ is a disjoint union of ξ_Γ over all graphs Γ having edge set $E = E_1 \cup E_2$, such that Γ_1 is obtained by contracting all edges outside of E_1 and Γ_2 is obtained by contracting all edges outside of E_2. We then define

$$[\Gamma_1, \gamma_1] \cdot [\Gamma_2, \gamma_2] = \sum_\Gamma [\Gamma, \gamma_1 \gamma_2 \varepsilon_\Gamma], \tag{4}$$

where the excess class is

$$\varepsilon_\Gamma = \prod_{(h,h') \in E_1 \cap E_2} -(\psi_h + \psi_{h'}),$$

and we set $\gamma_1 \gamma_2 = 0$ whenever the degree condition (3) is violated.

Pushing forward elements of the strata algebra along the gluing maps (1) defines a ring homomorphism

$$q : \mathcal{S}_{g,n} \to A^*(\overline{\mathcal{M}}_{g,n})$$
$$q([\Gamma, \gamma]) = \xi_{\Gamma *}(\gamma),$$

and the image of q is precisely the tautological ring $R^*(\overline{\mathcal{M}}_{g,n})$. Elements of $\mathcal{S}_{g,n}$ in the kernel of q are referred to as *tautological relations.*

The strata algebra is filtered by degree, defined as

$$\deg[\Gamma, \gamma] = |E| + d(\gamma),$$

which corresponds under q to codimension in the Chow ring. The product (4) preserves the degree, so $\mathcal{S}_{g,n}$ is a graded ring:

$$\mathcal{S}_{g,n} = \bigoplus_{d=0}^{3g-3+n} \mathcal{S}_{g,n}^d.$$

Let $\partial\mathcal{S}_{g,n}$ denote the subalgebra of $\mathcal{S}_{g,n}$ spanned by classes $[\Gamma, \gamma]$ in which the graph Γ has at least one edge, and let $\partial^0\mathcal{S}_{g,n} \subset \partial\mathcal{S}_{g,n}$ denote the subalgebra spanned by such classes with no κ's in γ— that is, with $x_i[v] = 0$ for each i and v.

Definition 2. Let $\xi \in \mathcal{S}_{g,n}$. Then a *topological recursion relation*, or TRR, for ξ is a tautological relation

$$q(\xi + \omega) = 0$$

in which $\omega \in \partial^0\mathcal{S}_{g,n}$.

When Γ consists of a single vertex v, we will typically denote $[\Gamma, \gamma]$ simply by γ_v. In particular, Theorem 1 concerns TRR's for the case where ξ is a monomial in the ψ-classes.

2.2. **Pixton's formula.** Fix g and n, and fix a collection of integers $A = (a_1, \ldots, a_n)$ such that $\sum_j a_j = 0$. In this subsection, we recall the definition of Pixton's class, which is an inhomogeneous element of $\mathcal{S}_{g,n}$ depending on the choice of A.

To do so, one must first define auxiliary classes $\widetilde{\mathcal{D}}_{g,n}^r$, for an additional integer parameter $r > 0$, as follows. For a stable graph $\Gamma = (V, H, g, p, \iota)$ of genus g with n legs, a *weighting modulo r* on Γ is defined to be a map

$$w : H \to \{0, \ldots, r-1\}$$

satisfying three properties:

(1) For any $i \in \{1, \ldots, n\}$ corresponding to a leg ℓ_i of Γ, we have $w(\ell_i) \equiv a_i \pmod r$.

(2) For any edge $e \in E$ corresponding to two half-edges $h, h' \in H$, we have $w(h) + w(h') \equiv 0 \pmod r$.
(3) For any vertex $v \in V$, we have $\sum_{h \in p^{-1}(v)} w(h) \equiv 0 \pmod r$.

Define $\widetilde{\mathcal{D}}^r_{g,n}$ to be the class

$$\sum_{\Gamma} \frac{1}{\# \operatorname{Aut}(\Gamma)} \frac{1}{r^{h^1(\Gamma)}} \sum_{\substack{w \text{ weighting} \\ \text{mod } r \text{ on } \Gamma}} [\Gamma, \gamma_w] \in \mathcal{S}_{g,n},$$

where

$$\gamma_w = \prod_{i=1}^{n} e^{\frac{1}{2} a_i^2 \psi_i} \prod_{(h,h') \in E} \frac{1 - e^{-\frac{1}{2} w(h) w(h') (\psi_h + \psi_{h'})}}{\psi_h + \psi_{h'}}, \tag{5}$$

which can be viewed as a basic class with $x_i(v) = 0$ for all $i > 0$ and all vertices V.

The class $\widetilde{\mathcal{D}}^r_{g,n}$ is a polynomial in r for $r \gg 0$ (see [JPPZ17, Appendix]). Pixton's class, then, is defined as the constant term of this polynomial in r. Using the fact that

$$a_1 = -(a_2 + \cdots + a_n),$$

we can express Pixton's class in terms of the variables $a_2, \ldots, a_n$ alone, so we denote it by

$$\mathcal{D}_{g,n}(a_2, \ldots, a_n),$$

and we denote its component in degree d by $\mathcal{D}^d_{g,n}(a_2, \ldots, a_n)$.

The key point, conjectured by Pixton and proved by the first and second authors, is the following:

Theorem 3 ([CJ16]). *For each $d > g$, $\mathcal{D}^d_{g,n}(a_2, \ldots, a_n)$ is a tautological relation.*

We refer to these as "Pixton's relations" in what follows.

2.3. **Polynomiality properties.** In fact, Theorem 3 yields tautological relations in a simpler form than are initially apparent, as a result of the following crucial result of Pixton:

Theorem 4 (Pixton, [Pi17]). *The class $\mathcal{D}_{g,n}(a_2, \ldots, a_n)$ depends polynomially on $a_2, \ldots, a_n$.*

In particular, the coefficient of any monomial $a_2^{b_2} \cdots a_n^{b_n}$ in the class $\mathcal{D}^d_{g,n}(a_2, \ldots, a_n)$ yields a tautological relation, for each $d > g$:

$$q\left(\left[\mathcal{D}^d_{g,n}(a_2, \ldots, a_n) \right]_{a_2^{b_2} \cdots a_n^{b_n}} \right) = 0. \tag{6}$$

We will use this fact repeatedly in what follows.

Note that, by the definition of Pixton's class, none of the relations (6) involve κ classes. Furthermore, the powers of the ψ-classes that appear are controlled by the monomial in question:

Lemma 5. *The degree of Pixton's class in ψ_i is bounded by half the degree in a_i.*

More explicitly, suppose that the class

$$\left[\mathcal{D}^d_{g,n}(a_2,\ldots,a_n)\right]_{a_2^{b_2}\cdots a_n^{b_n}} \in \mathcal{S}^d_{g,n}$$

is expanded in the standard basis of the strata algebra, and let $[\Gamma,\gamma]$ be a basis element that appears with nonzero coefficient. For each leg l_i of Γ corresponding to a marked point $i \in \{2,\ldots,n\}$, if $\gamma = (\{x_j\},y)$ in the notation of (2), then

$$y(l_i) \leq \frac{a_i}{2}.$$

Proof. This is straightforward from the definition of $\mathcal{D}^d_{g,n}(a_2,\ldots,a_n)$ and Theorem 4. □

3. Main results

Given that Pixton's relations (6) do not involve any κ classes, they are a natural candidate for producing topological recursion relations (TRR's). Moreover, by multiplying the relations by appropriate ψ-classes, one can ensure that κ classes do not arise even after pushforward under forgetful maps, thereby yielding a large family of TRR's. The proof of Theorem 1, which we detail in this section, is an application of this idea.

3.1. A family of topological recursion relations.

Fix a genus g and a number of marked points n. Let M be a monomial of degree D in the variables $a_2,\ldots,a_n$, and let

$$N := n + 2g + 2 - D.$$

Define

$$\Omega^{\mathrm{pre}}_{g,M} \in \mathcal{S}^{g+1}_{g,N}$$

to be the coefficient of the monomial $M \cdot a_{n+1}\cdots a_N$ in Pixton's class $\mathcal{D}^{g+1}_{g,N}(a_2,\ldots,a_N)$. Note that we use here the polynomiality discussed in Section 2.3.

Let

$$\Pi : \overline{\mathcal{M}}_{g,N} \to \overline{\mathcal{M}}_{g,n+1}$$

be the forgetful map, and define

$$\Omega_{g,M} := \Pi_*(\Omega^{\mathrm{pre}}_{g,M}\cdot\psi_{n+2}\cdots\psi_N) \in \mathcal{S}^{g+1}_{g,n+1},$$

where we use Π_* to denote the induced map on strata algebras.

Lemma 6. *For any choice of monomial M, the class $\Omega_{g,M}$ is a TRR. Furthermore, if*

$$\pi : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$$

is the forgetful map and π_* denotes the induced map on strata algebras, then $\pi_*(\Omega_{g,M})$ is still a TRR.

Proof. By the definition of $\mathcal{D}^{g+1}_{g,N}$, there are no κ classes in $\Omega^{\text{pre}}_{g,M}$, and furthermore, by Lemma 5, $\Omega^{\text{pre}}_{g,M}$ has degree zero in $\psi_{n+1}, \ldots, \psi_N$. Multiplying $\Omega^{\text{pre}}_{g,M}$ by $\psi_{n+2} \cdots \psi_N$ kills the contribution from any dual graph containing a rational tail that is no longer stable after the pushforward Π_*. Thus, the only dual graphs occurring in $\Omega^{\text{pre}}_{g,M} \cdot \psi_{n+2} \cdots \psi_N$ are dual graphs from $\overline{\mathcal{M}}_{g,n+1}$ (with additional legs inserted at the vertices), and each such dual graph comes with degree one in $\psi_{n+2}, \ldots, \psi_N$. It follows from the dilaton equation that $\Omega_{g,M}$ has no κ classes.

The above discussion also implies that $\Omega_{g,M}$ has degree zero in ψ_{n+1}. Thus, by the string equation, pushing it forward under π_* creates no κ classes, and this proves the second claim. $\square$

The $n = 1$ case of Theorem 1 is an immediate consequence of Lemma 6:

Theorem 7. *There exists a TRR for $\psi_1^g \in \mathcal{S}^g_{g,1}$.*

Proof. Take $n = 1$ and $M = 1$ in the above. As observed in the proof of Lemma 6, the only dual graphs contributing to $\Omega^{\text{pre}}_{g,1}$ are dual graphs from $\overline{\mathcal{M}}_{g,2}$ with additional legs at the vertices. Of these, all contribute to the boundary part of $\pi_*(\Omega_{g,1})$ except for the trivial dual graph Γ_0 and the dual graph of the boundary divisor $\delta_{0,\{1,2\}}$ parameterizing curves on which both marked points lie on a rational tail. It is straightforward to compute that the contribution from the latter graph is zero, whereas the contribution from Γ_0 is a nonzero number γ. Thus, dividing $\pi_*(\Omega_{g,1})$ by γ gives the desired TRR. $\square$

The proof of Theorem 1 for $n > 1$ follows the same template, except that there are nonzero contributions to $\Omega_{g,M}$ both from the trivial dual graph and from the dual graph of each of the boundary divisors $\delta_{0,\{i,n+1\}}$ with $i \in \{2, \ldots, n\}$, all of which push forward to non-boundary components of $\pi_*(\Omega_{g,M})$. The next section computes these contributions explicitly.

3.2. **Graph contributions.** Throughout this subsection, we assume that $n \geq 2$, and we explicitly express the monomial M from the previous subsection as

$$M = \prod_{j=2}^{n} a_j^{b_j}.$$

Let Γ_0 denote the trivial dual graph in $\overline{\mathcal{M}}_{g,n+1}$, and for each $i \in \{2, \ldots, n\}$, let Γ_i denote the dual graph in $\overline{\mathcal{M}}_{g,n+1}$ on which marked points i and $n+1$ lie on a rational tail, which corresponds to the boundary divisor $\delta_{0,\{i,n+1\}}$.

Lemma 8. *The contribution of* Γ_0 *to* $\Omega_{g,M}$ *is*

$$\frac{(4g-1+n-\sum_j b_j)!}{(2g-2+n)!} \sum_{c_2,\ldots,c_n} \psi_1^{g+1-\sum_j c_j} \left(2g+1-2\sum_j c_j\right)!! \prod_{j=2}^{n} \frac{\psi_j^{c_j}}{2^{c_j} c_j! (b_j - 2c_j)!},$$

where the sum is over all $c_2, \ldots, c_n$ *such that* $2c_j \leq b_j$ *for each* j.

Proof. The contribution of Γ_0 to $\Omega_{g,M}^{\mathrm{pre}}$ is given by equation (5). Namely, if we set

$$M' := M \cdot a_{n+1} \cdots a_N,$$

then the contribution of Γ_0 to $\Omega_{g,M}^{\mathrm{pre}}$ is the degree-$(g+1)$ part of

$$\left[\exp\left(\frac{1}{2}\sum_{i=1}^{N} a_i^2 \psi_i\right)\right]_{M'} = \left[\exp\left(\frac{1}{2}\left(\sum_{i=2}^{N} a_i\right)^2 \psi_1 + \frac{1}{2}\sum_{i=2}^{n} a_i^2 \psi_i\right)\right]_{M'},$$

where we use that $a_1 = -(a_2 + \cdots + a_N)$, and the second sum is only up to n since M' is linear in the variables $a_{n+1} \ldots, a_N$. (Here, as usual $[\cdots]_{M'}$ denotes the coefficient of M' in a polynomial class.) One directly computes that this contribution equals

$$(7) \quad \frac{1}{2^{g+1}} \sum_{c_2,\ldots,c_n} \frac{\psi_1^{g+1-\sum_j c_j}}{(g+1-\sum_j c_j)! \prod_{j=2}^{n} c_j!} \cdot \frac{(2g+2-2\sum_j c_j)!}{\prod_j (b_j - 2c_j)!} \prod_{j=2}^{n} \psi_j^{c_j},$$

where sum is over all $c_2, \ldots, c_n$ such that $2c_j \leq b_j$ for each j.

Multiplying (7) by $\psi_{n+2} \cdots \psi_N$ and pushing forward, repeated application of the dilaton equation gives a coefficient of

$$\frac{(2g-2+N-1)!}{(2g-2+n)!} = \frac{(4g-1+n-\sum_j b_j)!}{(2g-2+n)!},$$

which proves the claim. □

In the next lemma, we denote by ψ' the ψ-class on the half-edge of Γ_i adjacent to the genus-g vertex.

Lemma 9. *The contribution of* Γ_i *to* $\Omega_{g,M}$ *is*

$$\frac{(2g+1-\sum_j b_j)!}{b_i!}\sum_{c_2,\ldots,c_n}(2k-1)!!(2c_i+1)!!\psi_1^k(\psi')^{c_i}\prod_{j\neq i}\frac{\psi_j^{c_j}}{2^{c_j}c_j!(b_j-2c_j)!}$$
$$\left(-\binom{4g+n-\sum_{j\neq i}b_j}{2g-2c_i-\sum_{j\neq i}b_j}+\sum_{d=0}^{b_i-2c_i+2}\binom{4g-1+n-\sum_j b_j}{2g-2c_i-\sum_{j\neq i}b_j-d}\binom{b_i+1}{d}\right),$$

where the sum is over all $c_2,\ldots,c_n$ *such that* $2c_j\leq b_j$ *for all* $j\neq i$ *(and there are no restrictions on* c_i*), and* k *denotes* $g-\sum_j c_j$*.*

Proof. For each choice of partition $\{n+2,\ldots,N\}=I_1\sqcup I_2$, there is a graph $\widetilde{\Gamma}_i^{I_1,I_2}$ on $\overline{\mathcal{M}}_{g,N}$ whose image under the forgetful map Π is Γ_i; namely, I_1 gives the additional legs on the genus-g vertex of Γ_i and I_2 gives the additional legs on the genus-0 vertex. Let $n_1=\#I_1$ and $n_2=\#I_2$, let ψ' denote the ψ-class on the genus-g vertex of $\widetilde{\Gamma}_i^{I_1,I_2}$, and let ψ'' denote the ψ-class on the genus-0 vertex.

When we multiply $\Omega_{g,M}^{\mathrm{pre}}$ by $\psi_{n+2}\cdots\psi_N$, any appearance of ψ'', ψ_i, or ψ_{n+1} in the contribution $\mathrm{Contr}(\widetilde{\Gamma}_i^{I_1,I_2})$ of $\widetilde{\Gamma}_i^{I_1,I_2}$ to $\Omega_{g,n}^{\mathrm{pre}}$ is killed for dimension reasons. Thus, it suffices to compute $\mathrm{Contr}(\widetilde{\Gamma}_i^{n_1,n_2})\Big|_{\psi'',\psi_i,\psi_{n+1}=0}$, which, from (5), equals the degree-$(g+1)$ part of

$$-\left[\exp\left(\frac{1}{2}\left(\sum_{i=2}^N a_i\right)^2\psi_1+\frac{1}{2}\sum_{\substack{j\in\{2,\ldots,n\}\\ j\neq i}}a_i^2\psi_j\right)\cdot\sum_{\ell\geq 0}\frac{(a_i+a_{n+1}+a_{I_2})^{2\ell+2}}{2^{\ell+1}(\ell+1)!}(\psi')^\ell\right]_{M'}. \tag{8}$$

Here, we denote $M'=M\cdot a_{n+1}\cdots a_N$ (as above), and

$$a_I:=\sum_{j\in I}a_j.$$

More explicitly, (8) equals

$$-\sum_{\substack{c_1+\cdots+c_n+\ell=g\\ c_i=0}} \frac{1}{\prod_{j\neq i} c_j!}\frac{\psi_1^{c_1}}{2^{c_1}}\prod_{j\neq i}\frac{\psi_j^{c_j}}{2^{c_j}}\cdot\frac{1}{(\ell+1)!}\frac{(\psi')^\ell}{2^{\ell+1}}\sum_{\substack{e_i+e_{n+1}\leq 2\ell+2\\ e_{n+1}\leq 1}}\binom{2\ell+2}{e_i,e_{n+1}}$$
$$\cdot\left[\left(\sum_{i=2}^N a_i\right)^{2c_1}\prod_{j\neq i} a_j^{2c_j}\cdot a_i^{e_i}a_{n+1}^{e_{n+1}}a_{I_2}^{2\ell+2-e_i-e_{n+1}}\right]_{M'}.$$

If we rename ℓ to c_i, put $c_1 = g-\sum c_j$, and expand the term $[\cdots]_{M'}$, we find

(9)
$$-\sum_{c_2,\ldots,c_n}\frac{\psi_1^{g-\sum c_j}}{2^{g-\sum c_j}(g-\sum c_j)!}\cdot\frac{(\psi')^{c_i}}{2^{c_i+1}(c_i+1)!}\prod_{j\neq i}\frac{\psi_j^{c_j}}{2^{c_j}c_j!}$$
$$\sum_{\substack{e_i+e_{n+1}\leq 2c_i+2\\ e_{n+1}\leq 1}}\frac{(2c_i+2)!}{e_i!}\binom{n_2}{2c_i+2-e_i-e_{n+1}}\frac{(2g-2\sum c_j)!}{(b_i-e_i)!\prod_{j\neq i}(b_j-2c_j)!}.$$

where the constraints on $c_2,\ldots,c_n$ are as stated in the lemma.

In particular, we note that this depends only on n_1 and n_2. It follows that the contribution of Γ_i to $\Omega_{g,M}$ equals

(10)
$$\sum_{n_1+n_2=2g+1-D}\binom{2g+1-D}{n_1}\cdot\frac{(2g-3+n+n_1)!}{(2g-3+n)!}\cdot\frac{n_2!}{0!}\cdot\mathrm{Contr}(\widetilde{\Gamma}_i^{n_1,n_2}),$$

in which $\mathrm{Contr}(\widetilde{\Gamma}_i^{n_1,n_2})$ denotes the expression from (9), and the two quotients of factorials come from the dilaton equation on the genus-g and genus-0 vertices, respectively.

Notice that

$$\sum_{n_1+n_2=2g+1-D}\binom{2g+1-D}{n_1}\frac{(2g-3+n+n_1)!}{(2g-3+n)!}\cdot n_2!\cdot\binom{n_2}{2c_i+2-e_i-e_{n+1}}$$
$$=\left(2g+2-\sum b_j-1\right)!\cdot\binom{4g-1+n-\sum b_j}{2g-1-\sum b_j-2c_i+e_i+e_{n+1}},$$

where we have applied the Chu–Vandermonde identity

$$\sum_{n_1+n_2=a}\binom{b+n_1}{n_1}\binom{n_2}{c}=\binom{a+b+1}{a-c}. \tag{11}$$

Combining this with (9) and (10), and abbreviating

$$\begin{aligned} d &= 2c_i + 2 - e_i - e_{n+1} \\ k &= g - \sum c_j, \end{aligned}$$

we have thus far expressed the contribution of Γ_i to $\Omega_{g,M}$ as

$$\begin{aligned} (12)\quad & -\Big(2g+1-\sum b_j\Big)! \sum_{c_2,\ldots,c_n} (2k-1)!! \cdot (2c_i+1)!! \cdot \psi_1^k (\psi')^{c_i} \\ & \cdot \prod_{j\neq i} \frac{\psi_j^{c_j}}{(2c_j)!!(b_j-2c_j)!} \cdot \\ & \cdot \sum_{d=0}^{2c_i+2} \binom{4g-1+n-\sum b_j}{2g+1-\sum b_j - d} \sum_{e_{n+1}=0}^{1} \frac{1}{b_i!} \binom{b_i}{2ci_i+2-d-e_{n+1}}. \end{aligned}$$

The sum in the third line can be re-expressed as

$$\begin{aligned} & \frac{1}{b_i!} \sum_{d=0}^{2c_i+2} \binom{4g-1+n-\sum b_j}{2g+1-\sum b_j - d} \binom{b_i+1}{b_i-2c_i-1+d} \\ = & \frac{1}{b_i!} \sum_{d'=0}^{b_i+1} \binom{4g-1+n-\sum b_j}{2g-2c_i-\sum_{j\neq i} b_j - d'} \binom{b_i+1}{d'} \\ & \qquad - \frac{1}{b_i!} \sum_{d'=0}^{b_i-2c_i-2} \binom{4g-1+n-\sum b_j}{2g-2c_i-\sum_{j\neq i} b_j - d'} \binom{b_i+1}{d'}, \end{aligned}$$

in which $d' = d - 2c_i - 1 + b_i$. Another application of the Chu–Vandermonde identity (11) re-writes the first of these two sums as

$$\frac{1}{b_i!} \binom{4g+n-\sum_{j\neq i} b_j}{2g-2c_i-\sum_{j\neq i} b_j},$$

and substituting this into (12) completes the proof. □

3.3. **Proof of Theorem 1.** The sum of the results of Lemma 8 and Lemma 9, after pushing forward under $\pi : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$, gives precisely the contribution of the trivial graph to a TRR in $A^g(\overline{\mathcal{M}}_{g,n})$. There is one such TRR for each choice of the monomial M, and our goal is to use these to deduce TRRs for any monomial

$$(13)\qquad \psi_1^k \prod_{j=2}^{n} \psi_i^{l_j}$$

of degree g in the ψ-classes on $\overline{\mathcal{M}}_{g,n}$, by descending induction on k.

It is natural, toward this end, to consider the TRR associated to

$$M = \prod_{j=2}^{n} a_j^{2l_j}.$$

The minimum power of ψ_1 in the contribution of Γ_0 to $\Omega_{g,M}$ (calculated in Lemma 8) is $k+1$, so after applying the string equation, the minimum power of ψ_1 in the pushforward of this contribution under π is k.

The problem, however, is that there is no constraint on the power c_i in the contribution of Γ_i calculated in Lemma 9, so the power of ψ_1 in that contribution can be as low as zero. In order to set up a descending induction on k, then, we must consider a linear combination of TRRs $\pi_*(\Omega_{g,M})$ for different monomials M, carefully chosen to cancel the contributions from the graphs Γ_i. This is the content of the following key lemma:

Lemma 10. *Fix $g > 0$ and $n \geq 2$. For any nonnegative integers $k, l_2, \ldots, l_n$ such that $k + \sum_j l_j = g$, let*

$$D_{\vec{l}} := \sum_{d_2,\ldots,d_n=0}^{1} \prod_{j=2}^{n} (-2l_j - 1)^{d_j} \frac{(2k+1)_{(\sum_j d_j)}}{(2g+n+2k-1)_{(\sum_j d_j)}},$$

where

$$a_{(b)} := \frac{a!}{(a-b)!}.$$

If $D_{\vec{l}} \neq 0$, then there exists a TRR for

$$\psi_1^k \prod_{j=2}^{n} \psi_i^{l_j} + \Psi_{\vec{l}}, \tag{14}$$

where $\Psi_{\vec{l}}$ is a linear combination of ψ-monomials in which the power of ψ_1 is greater than k.

Proof. Fix nonnegative integers $k, l_2, \ldots, l_n$ as in the statement of the lemma. For each choice of $d_2, \ldots, d_n$ such that $d_j \in \{0, 1\}$ for each j, let

$$M_{\vec{d}} = \prod_{j=2}^{n} a_j^{2l_j + d_j}.$$

We consider the TRR $\pi_*(\Omega_{g,M_{\vec{d}}})$.

By Lemma 8 and the string equation, the contribution of Γ_0 to $\Omega_{g,M_{\vec{d}}}$ pushes forward under π to

$$\frac{(2k+1)!!}{2^{\sum_j l_j} \prod_j c_j!} \frac{(4g - 1 + n - \sum_j (2l_j + d_j))!}{(2g-2+n)!} \psi_1^k \prod_{j=2}^{n} \psi_j^{l_j} + O(\psi_1^{k+1}).$$

Denote the coefficient of $\psi_1^k \prod_{j=2}^n \psi_j^{l_j}$ in this expression by $C_{0,\vec{d}}$.

The contribution of Γ_i, on the other hand, pushes forward to a linear combination of monomials

$$\psi_1^{k'} \prod_{j=2}^{n} \psi_j^{l'_j} \tag{15}$$

with

$$l'_j \leq l_j \text{ for } j \neq i \quad \text{and} \quad k' + \sum_j l'_j = g.$$

When $k' \leq k$, we have

$$2l_i + d_i - 2l'_i - 2 = 2(l - l') + d_i + 2\sum_{j \neq i}(l'_j - l_j) - 2 < 0,$$

so the second term in Lemma 9 does not appear, and the coefficient of the monomial (15) is simply

$$\frac{(2l'_i+1)!!(2l'-1)!!(2g+1-\sum_j(2l_j+d_j)!}{(2l_i+d_i)!\prod_{j\neq i}(2^{l'_j}l'_j!(2l_j+d_j-2l'_j)!)}\binom{4g+n-\sum_{j\neq i}(2l_j+d_j)}{2g-\sum_{j\neq i}(2l_j+d_j)-2l'_i}.$$

Denote this quantity by $C_{i,\vec{d}}^{\vec{l}}$.

It is straightforward to check that in the TRR

$$\sum_{d_2,\ldots,d_n=0}^{1}(-1)^{\sum_j d_j}\prod_{j=2}^{n}(2l_j+1)^{d_j}\frac{(2k+1)!}{(2g-\sum_j(2l_j+d_j)+1)!}\pi_*(\Omega_{g,M_{\vec{d}}}),$$

the coefficient of any monomial of the form (15) with $k' \leq k$ is zero, since the weighted sum of the coefficients $C_{i,\vec{d}}^{\vec{l}}$ vanishes. We thus obtain a TRR for a linear combination of ψ-monomials of the form (14) provided that the weighted sum of the coefficients $C_{0,\vec{d}}$ is nonzero. This is precisely the criterion stated in the lemma. □

From here, the proof of Theorem 1 is simply a computation of the coefficients $D_{\vec{l}}$.

Proof of Theorem 1. We have already proven the theorem in the case where $n = 1$. When $n = 2$, one can explicitly compute

$$D_{\vec{l}} = 1 - \frac{(2l_2+1)(2g-2l_2+1)}{2g+1+2k} = 2k\frac{1-2l_2}{2g+1+2k},$$

while for $n = 3$,

$$\begin{aligned} D_{\vec{l}} &= 1 - \frac{(2l_2+2l_3+2)(2g-2l_2-2l_3+1)}{2g+2+2k} \\ &\quad + \frac{(2l_2+1)(2l_3+1)(2g-2l_2-2l_3+1)(2g-2l_2-2l_3)}{(2g+2+2k)(2g+1+2k)} \\ &= 2k\frac{8kl_2l_3 - 4gl_2 - 4gl_3 + 4l_2l_3 + 2k + 1}{(2g+2+2k)(2g+1+2k)}. \end{aligned}$$

In both cases, the numerator of the fraction is always odd, so $D_{\vec{l}} \neq 0$ for every choice of $k, l_2, \ldots, l_n$ (for $k > 0$). Thus, applying Lemma 10 and descending induction on k, the theorem is proved for $n \leq 3$.

In general, it suffices to prove the theorem for $n \leq g$, since if $n > g$, a degree-g monomial in the ψ-classes on $\overline{\mathcal{M}}_{g,n}$ has a ψ-class with exponent zero, so a TRR can be obtained by pulling back a TRR on a moduli space with fewer marked points. Thus, for each genus g, one must simply check that $D_{\vec{l}} \neq 0$ for the finitely many possible choices of n, k, and $l_2, \ldots, l_n$.

We have carried this out by a computer program for $g \leq 26$. There is one case in which $D_{\vec{l}} = 0$, which is when $g = 7$, $n = 4$, $k = 3$, and $\vec{l} = (2,1,1)$. However, in this isolated case, the missing TRRs can be found by essentially the same methods, as we explain below. In all other cases $D_{\vec{l}} \neq 0$, so again, applying Lemma 10 and working by descending induction on k proves the theorem.

On $\overline{\mathcal{M}}_{7,4}$, this proof fails to produce TRRs for $\psi_1^3\psi_2^2\psi_3\psi_4$ and for $\psi_1^2\psi_2^2\psi_3^2\psi_4$. However, suppose we let $M = a_2^9a_3^3a_4$ and consider the tautological relation $\pi_*(\Omega_{g,M})$. It is clear that this relation does not involve $\psi_1^2\psi_2^2\psi_3^2\psi_4$, and that ψ-monomials with a positive power of ψ_4 can only occur in the contribution from Γ_4. Notice that the second line in Lemma 9 vanishes when $c_4 \geq 2$. Setting $c_3 = c_4 = 1$, we see that, after subtracting TRRs pulled back from $\overline{\mathcal{M}}_{7,3}$, we obtain a TRR for

$$-\frac{1}{2}\sum_{c_2=0}^{4} \frac{\psi_1^{5-c_2}\psi_2^{c_2}\psi_3\psi_4}{c_2!(4-c_2)!}.$$

So, by subtracting other known TRRs, we see that there needs to be a TRR for

$$-\frac{1}{8}\psi_1^3\psi_2^2\psi_3\psi_4 - \frac{1}{12}\psi_1^2\psi_2^3\psi_3\psi_4.$$

Because of the broken symmetry in ψ_1 and ψ_2, we obtain the missing relation for $\psi_1^3\psi_2^2\psi_3\psi_4$. The final missing relation for $\psi_1^2\psi_2^2\psi_3^2\psi_4$ follows from Lemma 10 and the fact that $D_{(2,2,1)} \neq 0$.

□

Remark 11. We have discovered one other case in which $D_{\vec{l}}$ vanishes: when $g = 35$, $n = 4$, $k = 22$, and $\vec{l} = (11, 1, 1)$. It is an interesting question whether this vanishing stops in sufficiently high genus, and if not, whether the requisite TRRs can be obtained by alternative methods.

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