POWERS OF THE THETA DIVISOR AND RELATIONS IN THE TAUTOLOGICAL RING

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Abstract. We show that the vanishing of the $(g+1)$-st power of the theta divisor on the universal abelian variety $\mathcal{X}_g$ implies, by pulling back along a collection of Abel–Jacobi maps, the vanishing results in the tautological ring of $\mathcal{M}_{g,n}$ of Looijenga, Ionel, Graber–Vakil, and Faber–Pandharipande. We also show that Pixton's double ramification cycle relations, which generalize the theta vanishing relations and were recently proved by the first and third authors, imply Theorem $\star$ of Graber and Vakil, and we provide an explicit algorithm for expressing any tautological class on $\overline{\mathcal{M}}_{g,n}$ of sufficiently high codimension as a boundary class.

1. Introduction

The tautological ring $R^*(\mathcal{M}_g)$ is the subring, of either the cohomology or the Chow ring of $\mathcal{M}_g$, generated by the Mumford–Morita–Miller κ-classes [Mu83, Mo84, Mi86] defined by

$$\kappa_i = \pi_*(\psi^{i+1}),$$

where $\pi : \mathcal{M}_{g,1} \to \mathcal{M}_g$ and $\psi = c_1(\omega_\pi)$. Faber and Pandharipande [FP05] gave an elegant extension of this definition to the Deligne-Mumford compactification: the rings $R^*(\overline{\mathcal{M}}_{g,n})$ are the smallest system of $\mathbb{Q}$-subalgebras (either of the cohomology or the Chow ring of $\overline{\mathcal{M}}_{g,n}$) closed under pushforward by the gluing morphisms

$$\overline{\mathcal{M}}_{g_1,n_1+1} \times \overline{\mathcal{M}}_{g_2,n_2+1} \to \overline{\mathcal{M}}_{g_1+g_2,n_1+n_2},$$

$$\overline{\mathcal{M}}_{g,n+2} \to \overline{\mathcal{M}}_{g+1,n}$$

and the forgetful morphisms

$$\overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}.$$

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Restricting $R^*(\overline{\mathcal{M}}_g)$ to $\mathcal{M}_g$ recovers the κ-ring $R^*(\mathcal{M}_g)$, while more generally, the restriction of the tautological ring to the open moduli space $\mathcal{M}_{g,n}$ is generated by the κ-classes together with the ψ-classes,

$$\psi_i = c_1(s_i^* \omega_\pi),$$

in which $\pi : \mathcal{M}_{g,n+1} \to \mathcal{M}_{g,n}$ and s_i is the section of π determined by the i-th marked point.

Faber initiated an extensive study of $R^*(\mathcal{M}_g)$ in [F99B], using classical methods. Based on his observations, he formulated a striking series of conjectures suggesting that the tautological ring possesses a rich and remarkably well-behaved structure. In particular, he proposed that $R^*(\mathcal{M}_g)$ vanishes in sufficiently high codimension, a result that was proved by Looijenga and Ionel:

Theorem 1 (Looijenga [L95], Ionel [I02])**.** *The tautological ring of $\mathcal{M}_{g,n}$ vanishes in degrees greater than or equal to g, as well as in degree $g-1$ when $n=0$.*

A key geometric insight into these and other properties of the tautological ring was provided by Graber and Vakil in [GraV05], in which Theorem 1 was shown to be a consequence of the following result:

Theorem 2 (Theorem $\star$ of [GraV05])**.** *Any tautological class on $\overline{\mathcal{M}}_{g,n}$ of codimension k can be represented by a class supported on the locus of curves having at least $k-g+1$ rational components.*

More generally, Theorem $\star$ also implies several other properties of the tautological rings, including the analogous vanishing statements to Theorem 1 for curves with rational tails and curves of compact type and the "socle" statements for these moduli spaces (that is, the one-dimensionality of the tautological ring in the smallest nonzero codimension); see [GraV05, Section 5].

A stronger form of Theorem $\star$ was proved by Faber and Pandharipande:

Theorem 3 ([FP05])**.** *There exists an expression for any codimension k tautological class in terms of tautological classes supported on curves with at least $k-g+1$ rational components.*

The point, here, is that the boundary expression is itself tautological. However, implementing the proof of [FP05] as an algorithm to compute this expression explicitly seems to be computationally impractical.

The main result of the current paper is a new proof of Theorem 1 from a family of tautological relations on $\mathcal{M}^{ct}_{g,n}$ that we call the Θ-relations. These relations arise by pulling back the universal theta divisor on the universal abelian variety to $\mathcal{M}_{g,n}^{ct}$ under suitable Abel-Jacobi maps, and observing that the its $(g+1)$-st power vanishes in the Chow ring. These were discussed in [H13, GZ14B], and we give the details in Section 2. Thus, we prove:

Theorem 4. *The Θ-relations on $\mathcal{M}_{g,n}^{ct}$ imply Theorem 1 in the Chow ring of $\mathcal{M}_{g,n}$.*

The key advantage of this new proof of Theorem 1 — aside from the fact that it proceeds by an entirely elementary argument from the Θ-relations — is that it leads naturally to a constructive proof of Theorem $\star$ (and hence also implies Theorem 3). Indeed, the Θ-relations can be viewed as the restriction to $\mathcal{M}_{g,n}^{ct}$ of a family of relations called the double ramification cycle relations (see Theorem 8 below), first conjectured by Pixton and later proved by the first and third authors [ClJ16]. These relations arise, as we discuss in Section 2.2, from a perspective on the theta divisor via the moduli space of relative stable maps to the projective line. By carefully tracking the boundary contributions to the double ramification cycle relations, the vanishing in the proof of Theorem 1 from the Θ-relations is upgraded to an explicit algorithm for computing tautological boundary expressions for tautological classes in degree at least g.

We summarize the preceding discussion in the following theorem:

Theorem 5. *Pixton's double ramification cycle relations on $\overline{\mathcal{M}}_{g,n}$ imply Theorem $\star$ (and its strengthening, Theorem 3), as well as an algorithm for computing explicit tautological boundary formulas in the Chow ring for any tautological class on $\overline{\mathcal{M}}_{g,n}$ of codimension at least g.*

The paper is organized as follows. We recall the definition of the Θ-relations and Pixton's double ramification cycle relations in Section 2, and in Section 3, we give the proofs of Theorems 4 and 5. We then exemplify our algorithm by using it to compute boundary expressions for ψ_1 and κ_1 on $\overline{\mathcal{M}}_{1,1}$ in Section 4. Throughout the paper, we work in the Chow ring with rational coefficients; all results also imply the analogous statements in the $\mathbb{Q}$-cohomology since, by definition, the cycle class map is surjective on the tautological ring.

2. The Θ-relations and Pixton's double ramification cycle relations

Fix a genus $g \geq 0$ and an integer $n > 0$ such that $2g - 2 + n > 0$, and let

$$A := (a_1, \ldots, a_n), \quad a_i \in \mathbb{Z}$$

be a vector satisfying the condition

$$\sum_{i=1}^{n} a_i = 0.$$

Define the locus $\mathcal{Z}_{g,A} \subset \mathcal{M}_{g,n}$ to consist of marked curves $(C, p_1, \ldots, p_n)$ satisfying

$$\mathcal{O}_C\left(\sum_{i=1}^{n} a_i p_i\right) \cong \mathcal{O}_C. \tag{1}$$

Then $\mathcal{Z}_{g,A}$ is of pure codimension g in $\mathcal{M}_{g,n}$. Eliashberg posed the question of computing the class $[\mathcal{Z}_{g,A}]$ of $\mathcal{Z}_{g,A}$, in the cohomology or the Chow ring of $\mathcal{M}_{g,n}$, and of defining and computing the extension of $[\mathcal{Z}_{g,A}]$ to $\overline{\mathcal{M}}_{g,n}$.

2.1. **The Θ-relations and moduli of abelian varieties.** One approach to Eliashberg's problem, introduced by Hain in [H13], is to consider (1) as a relation on the Jacobian variety of the curve C, and vary it in moduli. Let $\mathcal{A}_g$ denote the moduli space of principally polarized abelian varieties, and define the Abel–Jacobi map $s_A : \mathcal{M}_{g,n} \to \mathcal{X}_g$ to the universal abelian variety $\mathcal{X}_g = \{([A], z) \mid [A] \in \mathcal{A}_g, z \in A\}$ by the formula

$$s_A(C, p_1, \ldots, p_n) := (\mathrm{Jac}^0_C, \mathcal{O}_C(a_1p_1 + \ldots + a_np_n)). \tag{2}$$

The locus $\mathcal{Z}_{g,A}$ is then easily seen to be the preimage of the zero section of $\mathcal{X}_g$ under s_A:

$$\mathcal{Z}_{g,A} := s_A^{-1}\mathcal{Z}_g, \quad \text{where} \quad \mathcal{Z}_g := \{([A], 0) \mid [A] \in \mathcal{A}_g\} \subset \mathcal{X}_g.$$

A stable curve is of compact type if and only if its Jacobian is an abelian variety, and the Abel–Jacobi map s_A naturally extends to the moduli space of curves of compact type $\mathcal{M}^{ct}_{g,n} \subset \overline{\mathcal{M}}_{g,n}$ (see formula (14) in [GZ14A] or Section 0.2.3 in [JPPZ16]). To define this extension, let $(C, p_1, \ldots, p_n)$ be a stable marked curve of compact type, let $\widetilde{C}$ be the normalization, let $\widetilde{C} = C_1 \sqcup \ldots \sqcup C_N$ be the decomposition of C into irreducible components, and for each j, let $q_{jk} \in C_j$ be the preimages of the nodes. There is a unique way to assign weights a_{jk} to the nodes such that the weights at a pair of matching nodes sum to zero, and on each C_j, we have

$$\sum_{i\,:\,p_i \in C_j} a_i + \sum_k a_{jk} = 0.$$

Then

$$\mathrm{Jac}(C) = \prod_{j=1}^{N} \mathrm{Jac}(C_j),$$

and the extension $s_A : \mathcal{M}_{g,n}^{ct} \to \mathcal{X}_g$ is defined by the following formula:
(3)

$$s_A(C, p_1, \dots, p_n) = \left(\mathrm{Jac}(C), \prod_{j=1}^{N} \mathcal{O}_{C_j} \left(\sum_{i : p_i \in C_j}^{n} a_i p_i + \sum_{k} a_{jk} q_{jk} \right) \right).$$

The closure of $\mathcal{Z}_{g,A}$ in $\mathcal{M}_{g,n}^{ct}$ is contained in $s_A^{-1}(\mathcal{Z}_g)$, and

$$s_A^{-1}(\mathcal{Z}_g) \cap \mathcal{M}_{g,n} = \mathcal{Z}_{g,A}.$$

However, $s_A^{-1}(\mathcal{Z}_g)$ has components of excessive dimension, coming from curves having a rational tail. To account for these, we consider the pullback

$$R_{g,A}^{ct} := s_A^*[\mathcal{Z}_g], \tag{4}$$

either in the cohomology ring or in the Chow ring of $\mathcal{M}_{g,n}^{ct}$.

In [H13], Hain computed the class $R_{g,A}^{ct}$ in cohomology, using Hodge-theoretic techniques. Hain's calculations were simplified and extended to the Chow ring by the second and fourth authors in [GZ14B]. The main idea is to use the following result, which follows from the results of Deninger and Murre [DM91, Cor. 2.22], by applying the Fourier-Mukai transform to theta divisor, as explained in [EGM12, Exercise 13.2] or in [BL04, Sec. 16.4, Exercise 16.8.1]; this was also independently proven in cohomology by Hain [H13]:

Theorem 6. *Let $\Theta \in CH^1(\mathcal{X}_g)$ denote the universal symmetric theta divisor trivialized along the zero section. Then*

(1) $[\mathcal{Z}_g] = \dfrac{\Theta^g}{g!}$ *in* $CH^g(\mathcal{X}_g)$,
(2) $\Theta^{g+1} = 0$ *in* $CH^{g+1}(\mathcal{X}_g)$.

To compute $R_{g,A}^{ct}$, it thus suffices to compute the pullback of the divisor Θ, which is a standard calculation using test curves. In addition, for every A we get a relation $[s_A^*\Theta]^{g+1} = 0$ in $CH^{g+1}(\mathcal{M}_{g,n}^{ct})$. We summarize the results of [H13] and [GZ14B] in the following theorem, where from now we denote $[n]$ the set $\{1, \dots, n\}$:

Theorem 7 (Θ-relations)**.** *The pullback of Θ to $\mathcal{M}_{g,n}^{ct}$ along the Abel–Jacobi map (3) is equal to*

$$s_A^*\Theta = -\frac{1}{4}\sum_{h=0}^{g}\sum_{P\subset[n]} a_P^2\delta_h^P, \tag{5}$$

where $a_P = \sum_{i\in P} a_i$ for any $P\subset[n]$. Here, δ_h^P is the class of the closure of the locus of curves having an irreducible component of genus h containing the marked points indexed by P and an irreducible component of genus $g-h$ containing the remaining marked points. In the cases when the resulting curve is not stable, we set $\delta_0^{\{i\}} = \delta_g^{[n]\setminus\{i\}} = -\psi_i$ and $\delta_0^{\emptyset} = \delta_g^{[n]} = 0$.

Formula (5) and Theorem 6 imply that the pullback of the zero section is given by the following formula:

$$R_{g,A}^{ct} = \frac{1}{g!}[s_A^*\Theta]^g \in CH^g(\mathcal{M}_{g,n}^{ct}), \tag{6}$$

and that following relation holds:

$$[s_A^*\Theta]^{g+1} = 0 \in CH^{g+1}(\mathcal{M}_{g,n}^{ct}). \tag{7}$$

In [GZ14A], the second and fourth authors considered an extension of this method further into the boundary of $\overline{\mathcal{M}}_{g,n}$. It was shown that the Abel–Jacobi map s_A extends to a map $s_A : \overline{\mathcal{M}}_{g,n}^o \to \mathcal{X}_g^{part}$, where $\overline{\mathcal{M}}_{g,n}^o$ is the moduli space of stable marked curves having at most one non-separating node, and $\mathcal{X}_g^{part}$ is the universal family over Mumford's partial compactification of $\mathcal{A}_g$, parametrizing semiabelian varieties of torus rank at most one. The principal result of [GZ14A] is the extension of the formula $[\mathcal{Z}_g] = \Theta^g/g!$ for the zero section in $CH^g(\mathcal{X}_g)$ to an explicit polynomial relation in $CH^g(\mathcal{X}_g^{part})$, and the computation of its pullback to $CH^g(\overline{\mathcal{M}}_{g,n}^o)$.

Following this approach further would involve extending the Abel–Jacobi map to further intermediate compactifications of $\mathcal{M}_{g,n}$, and simultaneously adding boundary strata to $\mathcal{X}_g$ that parametrize higher torus-rank degenerations of principally-polarized abelian varieties. This leads to two technical difficulties. First, the extension of the formula $[\mathcal{Z}_g] = \Theta^g/g!$ to $\mathcal{X}_g^{part}$ obtained in [GZ14A] is already quite complicated, and the deeper boundary strata of $\mathcal{X}_g$ have an increasingly complex combinatorial structure, so the corresponding calculations do not seem combinatorially manageable at this point. In addition, on boundary strata not contained in $\overline{\mathcal{M}}_{g,n}^o$ the Abel–Jacobi map does not extend to a morphism, and its indeterminacy locus needs to be resolved. The question of extending the Abel–Jacobi map (3) to various compactifications of the universal Jacobian variety and resolving the singularities of the Abel–Jacobi map has been considered in a number of recent papers (see [Du15, Ho14, KPag15, Me11, MeRV14, Me16]).

2.2. **The double ramification cycle and Pixton's relations.** An alternative way to extend $\mathcal{Z}_{g,A}$ to $\overline{\mathcal{M}}_{g,n}$ is given by the double ramification cycle. To motivate the definition, we observe that the locus $\mathcal{Z}_{g,A} \subset \mathcal{M}_{g,n}$ of marked curves satisfying condition (1) has an equivalent definition as the locus of $(C, p_1, \ldots, p_n)$ admitting a ramified cover $f : C \to \mathbb{P}^1$ such that

- $f^{-1}(0) = \{p_i \mid a_i > 0\}$ and $f^{-1}(\infty) = \{p_i \mid a_i < 0\}$.
- The ramification profiles of f over 0 and ∞ are $\mu = \{a_i \mid a_i > 0\}$ and $\nu = \{|a_i| \mid a_i < 0\}$, respectively.

(No condition is imposed on the marked points p_i with $a_i = 0$.) To extend $\mathcal{Z}_{g,A}$ to $\overline{\mathcal{M}}_{g,n}$, then, one can compactify the space of such ramified covers, allowing both C and the target $\mathbb{P}^1$ to degenerate. The resulting object is what is called the moduli space of rubber relative stable maps to $\mathbb{P}^1$ and is denoted $\overline{\mathcal{M}}_{g,n_0}(\mathbb{P}^1; \mu, \nu)^\sim$, where $n_0 := \#\{i \mid a_i = 0\}$.

For more details on the moduli space of rubber relative stable maps, see [FP05]. The crucial property required here is that it admits a virtual fundamental class (constructed algebraically by Jun Li [Li01]):

$$[\overline{\mathcal{M}}_{g,n_0}(\mathbb{P}^1; \mu, \nu)^\sim]^{\mathrm{vir}} \in CH_{\mathrm{vdim}}(\overline{\mathcal{M}}_{g,n_0}(\mathbb{P}^1; \mu, \nu)^\sim).$$

From here, one defines the double ramification cycle by pushforward

$$R_{g,A} := \tau_*[\overline{\mathcal{M}}_{g,n_0}(\mathbb{P}^1; \mu, \nu)^\sim]^{\mathrm{vir}} \in CH^g(\overline{\mathcal{M}}_{g,n})$$

along the natural forgetful morphism

$$\tau : \overline{\mathcal{M}}_{g,n_0}(\mathbb{P}^1; \mu, \nu)^\sim \to \overline{\mathcal{M}}_{g,n}.$$

The restriction of $R_{g,A}$ to $\mathcal{M}_{g,n}$ is the class of $\mathcal{Z}_{g,A}$, essentially by definition. Marcus and Wise [MW13] proved, moreover, that the restriction of $R_{g,A}$ to $\mathcal{M}^{ct}_{g,n}$ is equal to the class $R^{ct}_{g,A}$ defined by (4), and in [CMW12], Cavalieri, Marcus, and Wise independently computed the restriction of $R_{g,A}$ to the moduli space $\mathcal{M}^{rt}_{g,n}$ of curves with rational tails, obtaining the same result as in Theorem 7.

Faber and Pandharipande proved in [FP05] that the class $R_{g,A}$ is tautological, and provided a method to calculate it in principle. The computational difficulties of their method, however, are too great to obtain an explicit formula. This situation was remedied by a conjecture of Pixton, which proposed not only a formula for $R_{g,A}$ (generalizing equation (6) of Theorem 7) but also a generalization of the Θ-relations (equation (7) of Theorem 7) to all of $\overline{\mathcal{M}}_{g,n}$.

The basic idea of Pixton's conjecture, which we explain further in Section 4.1 below, is to view the terms $[s_A^*\Theta]^g$ and $[s_A^*\Theta]^{g+1}$ appearing in Theorem 7 as appropriate multiples of the parts in degree g and $g+1$ of the mixed-degree class $\exp[s_A^*\Theta] \in CH^*(\mathcal{M}_{g,n}^{ct})$. The expression (5) can be packaged into an elegant formula for $\exp[s_A^*\Theta]$ as a graph sum, and Pixton used an ingenious modification of this graph sum to extend $\exp[s_A^*\Theta]$ to a class $\Omega_{g,A} \in CH^*(\overline{\mathcal{M}}_{g,n})$. Generalizing both statements of Theorem 7, then, he conjectured that $\Omega_{g,A}$ coincides with $R_{g,A}$ in codimension g and vanishes in higher codimension.

Both parts of Pixton's conjecture have recently been proven:

Theorem 8 ([JPPZ16, ClJ16])**.** *Let $[\bullet]_d$ denote the degree-d part of a mixed-degree class in $CH^*(\overline{\mathcal{M}}_{g,n})$. Then the class $\Omega_{g,A}$ satisfies the following:*

(1) $[\Omega_{g,A}]_g = R_{g,A}$.
(2) $[\Omega_{g,A}]_d = 0$ *for* $d > g$.

The second of these statements is what we refer to as "Pixton's double ramification cycle relations."

The proof of part (1), by the third author, Pandharipande, Pixton, and Zvonkine, uses localization on the moduli space of relative stable maps to an orbifold projective line in order to compare $\Omega_{g,A}$ to the virtual cycle of a simpler moduli space of orbifold stable maps. Part (2) was proved by the first and third authors by leveraging relations on the orbifold stable maps space previously observed in [Cl13]. In addition, it was shown in [ClJ16] that Pixton's relations follow from another collection of relations, also originally conjectured by Pixton, known as the 3-spin relations and proved in [J15] and [PPZ15].

Remark 9. The restriction of the class $\Omega_{g,A}$ to the moduli space $\overline{\mathcal{M}}_{g,n}^o$ of [GZ14A] is equal to the pullback of the zero section $\mathcal{Z}_g^{part}$ of the partial compactification of $\mathcal{X}_g^{part}$ under the Abel–Jacobi map s_A.

We give the explicit formula for $\Omega_{g,A}$ as Equation (28) in Section 4.1. The calculations of Section 3, in which we prove Theorems 4 and 5, do not require the full formula for $\Omega_{g,A}$ but only the existence of an extension of the Θ-relations.

3. Proofs of vanishing theorems

We now turn to the proof of the vanishing of the tautological ring $R^k(\mathcal{M}_{g,n})$ in degree $k \geq g - \delta_{0n}$, and of Graber–Vakil's Theorem $\star$, via explicit computation from the relations discussed in the previous section. More precisely, the results we need are the following:

Θ-relations, Theorem 7: Let $s_A^*\Theta$ be the pullback of the theta divisor (5). Then

$$[s_A^*\Theta]^{g+1} = 0 \in CH^{g+1}(\mathcal{M}_{g,n}^{ct}).$$

This relation is a polynomial of degree $2g+2$ in the variables a_i.

Pixton's relations, Theorem 8: For each A there exists an expression $[\Omega_{g,A}]_{g+1}$ in tautological classes of codimension $g+1$ that vanishes as an element of $CH^{g+1}(\overline{\mathcal{M}}_{g,n})$ and that restricts in $CH^{g+1}(\mathcal{M}_{g,n}^{ct})$ to the expression for $[s_A^*\Theta]^{g+1}/(g+1)!$ given by (5). In other words, one has the relation

$$[s_A^*\Theta]^{g+1} = D_{A,g+1} \in CH^{g+1}(\overline{\mathcal{M}}_{g,n}), \tag{8}$$

where $s_A^*\Theta$ on $\overline{\mathcal{M}}_{g,n}$ is defined by (5) and $D_{A,g+1}$ is a tautological class supported away from $\mathcal{M}_{g,n}^{ct}$.

It is conjectured (see [Pi16] or [ClJ16, Lemma 2.1]) that the classes $[\Omega_{g,A}]_{g+1}$, and hence $D_{A,g+1}$, are polynomial in the variables a_i. However, this fact has not yet been definitively established, and is not necessary for our results.

3.1. **Some properties of tautological classes.** Several formulas involving the tautological classes on $\overline{\mathcal{M}}_{g,n}$ will be useful later.

Let $\pi_{n+1} : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ be the map that forgets the last marked point. The pullbacks of ψ- and κ-classes under it are as follows (see [ACo96, Section 1]):

$$\pi_{n+1}^*\psi_i = \psi_i - \delta_0^{\{i,n+1\}}, \quad \pi_{n+1}^*\kappa_a = \kappa_a - \psi_{n+1}^a, \tag{9}$$

where $\delta_0^{\{i,n+1\}}$ is defined as in Theorem 7. The pushforward of a monomial in the ψ-classes is equal to

$$\pi_{n+1,*}[\psi_1^{k_1}\cdots\psi_n^{k_n}\psi_{n+1}^{k_{n+1}+1}] = \psi_1^{k_1}\cdots\psi_n^{k_n}\kappa_{k_{n+1}}.$$

More generally, suppose $n > m$. Then the pushforward of a monomial in the ψ-classes from $\overline{\mathcal{M}}_{g,n}$ to $\overline{\mathcal{M}}_{g,m}$ is given by the following formula, originally due to Faber:

$$(\pi_{m+1}\circ\cdots\circ\pi_n)_*[\psi_1^{k_1}\cdots\psi_m^{k_m}\psi_{m+1}^{k_{m+1}+1}\cdots\psi_n^{k_n+1}] = \psi_1^{k_1}\cdots\psi_m^{k_m}R(k_{m+1},\dots,k_n). \tag{10}$$

Here, $R(k_{m+1},\dots,k_n)$ is a polynomial in the κ-classes,

$$R(k_{m+1},\dots,k_n) = \sum_{\sigma\in S_{n-m}} \kappa_\sigma,$$

where κ_σ is defined as follows: Given a permutation $\sigma \in S_{n-m}$, write it as a product $\sigma = \alpha_1 \cdots \alpha_{\nu(\sigma)}$ of disjoint cycles, and for a cycle α, define $|\alpha| = \sum_{i \in \alpha} k_{m+i}$ to be the sum of those among $k_{m+1}, \ldots, k_n$ that are permuted by α. Then

$$\kappa_\sigma := \kappa_{|\alpha_1|} \cdots \kappa_{|\alpha_{\nu(\sigma)}|}.$$

Note that

$$R(k_{m+1}, \ldots, k_n) = \kappa_{k_{m+1}} \cdots \kappa_{k_n} + \text{(monomials in fewer than } n-m \ \kappa\text{-classes)}, \tag{11}$$

where the first term on the right-hand side corresponds to the trivial permutation in S_{n-m}. In particular, we note that $R(k, 0, \ldots, 0)$ is a nonzero multiple of κ_k and $R(0, \ldots, 0)$ is a nonzero constant (recall that $\kappa_0 = 2g - 2 + n$).

We will also use the following simple lemma:

Lemma 10. *Let $1 \le i_1, \ldots, i_k \le n$ be integers, and let $I \subset [n]$. Then*

$$\delta_0^I \psi_{i_1} \cdots \psi_{i_k} = 0 \tag{12}$$

on $\overline{\mathcal{M}}_{g,n}$ whenever $\#\{j | i_j \in I\} \ge \#I - 1$.

Proof. Indeed, δ_0^I is the class of the boundary divisor Δ_0^I, which is the image under the gluing map of the product $\overline{\mathcal{M}}_{0,\#I+1} \times \overline{\mathcal{M}}_{g,n-\#I+1}$. When $i_j \in I$, it is clear that the class ψ_{i_j} on $\overline{\mathcal{M}}_{g,n}$ restricts on Δ_0^I to the ψ-class of the corresponding point on $\overline{\mathcal{M}}_{0,\#I+1}$. Hence any class of the form (12) vanishes if $\#\{j | i_j \in I\} \ge \#I - 1 = \dim \overline{\mathcal{M}}_{0,\#I+1} + 1$. $\square$

Finally, we recall the low genus topological recursion relations, expressing the divisors ψ_i and κ_1 as boundary divisors (see Theorem 2.2 in [ACo98]):

$$\kappa_1 = \psi_1 = \frac{1}{12}\delta_{irr} \in CH^1(\overline{\mathcal{M}}_{1,1}), \tag{13}$$

$$\psi_i = \sum_{\substack{i \in I \subset [n] \\ \#I = n-2}} \delta_0^I \in CH^1(\overline{\mathcal{M}}_{0,n}). \tag{14}$$

Here, δ_{irr} is the class of the locus of curves that have a non-separating node.

3.2. **Additive generators of the tautological ring.** We recall, for future use in the proof of theorem $\star$ in subsection 3.5, the explicit set of additive generators for the tautological ring defined via the strata algebra of $\overline{\mathcal{M}}_{g,n}$; for references, see [GraP03] and [Pi13]. A stable graph $\Gamma = (V, H, g, p, \iota)$ consists of the following data:

(1) A set of vertices V equipped with a genus function $g : V \to \mathbb{Z}_{\geq 0}$.
(2) A set of half-edges H equipped with a vertex assignment $p : H \to V$ and an involution $\iota : H \to H$.

We define the set of edges E of Γ to be the set of orbits of ι that are of cardinality 2, and the set of legs L of Γ to be the set of fixed points of ι; the pair (V, E) is then an ordinary graph. The valence of a vertex $v \in V$ is defined as $n(v) = \#p^{-1}(v)$. We require that the following conditions be satisfied:

(1) The graph Γ is connected.
(2) For every vertex $v \in V$,
$$2g(v) - 2 + n(v) > 0.$$

We define the genus of a stable graph Γ to be

$$g(\Gamma) = h^1(\Gamma) + \sum_{v \in V} g(v), \tag{15}$$

where $h^1(\Gamma) = \#E - \#V + 1$. An automorphism of a stable graph is a permutation on both V and H, that preserves the incidence relations and acts as the identity on legs.

Given a stable curve C of genus g with n marked points, its dual graph is a stable graph of genus g with n legs. For a stable graph Γ of genus g with n legs, let

$$\overline{\mathcal{M}}_\Gamma := \prod_{v \in V} \overline{\mathcal{M}}_{g(v),n(v)}.$$

Then there is a canonical gluing morphism

$$\xi_\Gamma : \overline{\mathcal{M}}_\Gamma \to \overline{\mathcal{M}}_{g,n}, \tag{16}$$

whose image is the locus in $\overline{\mathcal{M}}_{g,n}$ in which the generic point corresponds to a curve with stable graph Γ.

Let Γ be a stable graph. For each $v \in V$, let $\{x_i[v]\}_{i>0}$ and $\{y[h]\}_{h \in p^{-1}(v)}$ be sets of positive integers. Associated to each such choice of integers, there is a basic class

$$\gamma_v = \prod_{i>0} \kappa_i^{x_i[v]} \prod_{h \in p^{-1}(v)} \psi_h^{y[h]} \in CH^{d(\gamma_v)}(\overline{\mathcal{M}}_\Gamma),$$

having degree

$$d(\gamma_v) = \sum_{i>0} ix_i[v] + \sum_{h\in p^{-1}(v)} y[h].$$

We define

$$\gamma = \prod_{v\in V} \gamma_v \in CH^{d(\gamma)}(\overline{\mathcal{M}}_\Gamma), \tag{17}$$

whose degree is $d(\gamma) = \sum_{v\in V} d(\gamma_v)$. The pair (Γ, γ) defines a tautological class

$$\xi_{\Gamma*}(\gamma) \in CH^{d(\gamma)+\#E}(\overline{\mathcal{M}}_{g,n}),$$

and the tautological ring of $\overline{\mathcal{M}}_{g,n}$ is spanned by classes of this form.

3.3. **A preliminary lemma.** As a starting point toward the proof of Theorem 4, and as an illustration of the methods we will use in the rest of the paper, we prove the following lemma:

Lemma 11. *Any degree-k monomial in the ψ-classes vanishes on $\mathcal{M}_{g,n}$ if $k \geq g+1$ and $n \geq 2g+3$.*

Proof. First, assume that $k = g+1$ and $n = 2g+3$.

Let $\iota : \mathcal{M}_{g,2g+3} \to \mathcal{M}^{ct}_{g,2g+3}$ be the inclusion map, and let $A = (a_1, \ldots, a_{2g+3})$ be such that $\sum a_i = 0$. The restriction of the Θ-relations to $CH^{g+1}(\mathcal{M}_{g,2g+3})$ give

$$\iota^*[s_A^*\Theta]^{g+1} = \left[\frac{1}{2}\sum_{i=1}^{2g+3} a_i^2\psi_i\right]^{g+1} = 0 \in CH^{g+1}(\mathcal{M}_{g,2g+3}),$$

where here and for the rest of the proof ψ_i denotes the ψ-class on $\mathcal{M}_{g,2g+3}$. Eliminating $a_{2g+3} = -(a_1 + \ldots + a_{2g+2})$ and dropping the $1/2$, we obtain

$$\left[\sum_{i=1}^{2g+2} a_i^2\psi_i + (a_1 + \ldots + a_{2g+2})^2\psi_{2g+3}\right]^{g+1} = 0 \tag{18}$$

for any $(a_1 \ldots, a_{2g+2}) \in \mathbb{Z}^{2g+2}$. This relation is a homogeneous polynomial in the integer variables a_i and the classes ψ_i, of degrees $2g+2$ and $g+1$, respectively, and we prove the lemma by alternatively considering (18) as a polynomial in one set of variables or the other.

First, we view (18) as a polynomial in the a-variables taking values in $CH^{g+1}(\mathcal{M}_{g,2g+3})$. It vanishes for all integer values of the a_i only if it is the zero polynomial— in other words, only if the coefficient in front of each monomial in the a_i is zero. Thus, any monomial in the a_i of degree $2g+2$ gives a relation in $CH^{g+1}(\mathcal{M}_{g,2g+3})$, which itself is a polynomial of degree $g+1$ in the ψ-classes. We need to check that there are enough such relations to ensure that every monomial in the ψ-classes of degree $g+1$ vanishes.

We now make a key observation. Consider (18) as a polynomial in the ψ-classes whose coefficients are polynomials in the a-variables. For every $1 \le j \le 2g+2$, ψ_j appears in (18) in the term $a_j^2\psi_j$, hence the coefficient of any ψ-monomial that is a multiple of ψ_j^k is an a-polynomial that is a multiple of a_j^{2k}.

If we now again view (18) as a polynomial in the a_i, then the observation of the preceding paragraph implies that, for every $1 \le j \le 2g+2$, the coefficient of any monomial that is not a multiple of a_j^{2k} is a polynomial in the ψ-classes that contains no monomials that are multiples of ψ_j^k. For example, the coefficient of the monomial $a_1 \cdots a_{2g+2}$ does not contain the classes $\psi_1, \ldots, \psi_{2g+2}$, and indeed, it is equal to $(2g+2)!\psi_{2g+3}^{g+1}$. We thus obtain our first vanishing,

$$\psi_{2g+3}^{g+1} = 0 \in CH^{g+1}(\mathcal{M}_{g,2g+3}) \tag{19}$$

which serves as the base of a descending induction on the power of ψ_{2g+3} from which we prove the claim.

Let $1 \le k \le g+1$, and suppose that we have shown that any monomial of degree $g+1$ in the ψ-classes that is a multiple of ψ_{2g+3}^k vanishes. Let $K = (k_1, \ldots, k_{2g+2})$ be non-negative integers satisfying

$$\sum_{i=1}^{2g+2} k_i = g+1-(k-1), \tag{20}$$

and denote

$$\Psi_K := \psi_1^{k_1} \cdots \psi_{2g+2}^{k_{2g+2}}.$$

Let j be the $2(k-1)$-st element, in increasing numerical order, in the set

$$I_K := \{i \mid k_i = 0\} \subset \{1, \ldots, 2g+2\}.$$

(The fact that $\#I_K \ge 2(k-1)$ is clear: indeed, otherwise we would have $g+1-(k-1) = \sum_{i=1}^{2g+2} k_i \ge (2g+2) - 2(k-1) + 1$, which would imply $k \ge g+3$.) For $1 \le i \le 2g+2$, define m_i by

$$m_i = \begin{cases} 2k_i, & i \notin I_K, \\ 1, & i \in I_K \text{ and } i \le j, \\ 0, & i \in I_K \text{ and } i > j. \end{cases}$$

Since $k_1 + \ldots + k_{2g+2} = g+1-(k-1)$, we have $m_1 + \ldots + m_{2g+2} = 2g+2$. To show that $\Psi_K \psi_{2g+3}^{k-1} = 0$, we consider the monomial $a_1^{m_1} \cdots a_{2g+2}^{m_{2g+2}}$.

The coefficient of this monomial in (18) is a polynomial of degree $g+1$ in the ψ-classes, and by the above observation, this polynomial contains no monomial that is a multiple of $\psi_j^{k_j+1}$ for any $1 \leq j \leq 2g+2$. But by (20), any ψ-monomial of degree $g+1$ that is not a multiple of any $\psi_j^{k_j+1}$ is a multiple of either Ψ_K or ψ_{2g+3}^k. Hence the monomial $a_1^{m_1} \cdots a_{2g+2}^{m_{2g+2}}$ imposes the following relation:

$$C_K \Psi_K \psi_{2g+3}^{k-1} + [\text{a multiple of } \psi_{2g+3}^k] = 0 \in CH^{g+1}(\mathcal{M}_{g,2g+3}),$$

where C_K is a positive multinomial coefficient depending on K. By induction, all ψ-monomials that are multiples of ψ_{2g+3}^k vanish, hence so does $\Psi_K \psi_{2g+3}^{k-1}$. This proves that any ψ-monomial of degree $g+1$ vanishes on $\mathcal{M}_{g,2g+3}$.

To finish the proof of the lemma, we note that, for $n > 2g+3$, any ψ-monomial of degree $g+1$ on $\mathcal{M}_{g,n}$ can be obtained using (9) by pulling back a ψ-monomial from $\mathcal{M}_{g,2g+3}$ along a forgetful map, and that the vanishing of all ψ-monomials in degree $g+1$ trivially implies vanishing in higher degrees. □

Remark 12. We have shown that the Θ-relations imply the vanishing of the monomials of degree $g+1$ in the ψ-classes on $\mathcal{M}_{g,2g+3}$, by what is really just a multidimensional Gaussian elimination. The same Gaussian elimination can be used to obtain boundary formulas for these classes on all of $\overline{\mathcal{M}}_{g,2g+3}$. For this, we use Pixton's double ramification cycle relations, as stated in equation (8). The left-hand side of (8) is a polynomial of degree $2g+2$ in the a_i, each coefficient of which is a polynomial of degree $g+1$ in the ψ-classes and the boundary divisors. The right-hand side is some tautological class supported on the divisor Δ_{irr} of curves with a non-separating node. Moving all boundary terms from the left to the right, we obtain an expression of the left-hand side of (18) as a boundary class on $\overline{\mathcal{M}}_{g,2g+3}$.

To obtain boundary formulas for degree $g+1$ monomials in the ψ-classes, we proceed from equation (18) as in the proof of Lemma 11, but we now keep track of the boundary on the right-hand side. If we assume that Pixton's class $[\Omega_{g,A}]_{g+1}$ is a polynomial in the a_i, then so is $D_{g,A}$, and the proof is identical: each monomial of degree $2g+2$ in the a_i imposes a relation in $\overline{\mathcal{M}}_{g,2g+3}$, and these relations imply the vanishing of all ψ-monomials on $\mathcal{M}_{g,2g+3}$, so the same Gaussian elimination produces boundary formulas for all such monomials on $\overline{\mathcal{M}}_{g,2g+3}$. Even without assuming polynomiality, one can construct a collection of finite difference operators[1] in the a_i that isolate the coefficients of any degree $2g+2$ polynomial in the a_i. We then apply these operators (which are

[1]For example, if $f(x,y) = ax^2+bxy+cy^2$, then $b = f(x+1,y+1) - f(x+1,y) - f(x,y+1) + f(x,y)$.

defined for any function of the a_i) to equation (8) and proceed as in the proof of Lemma 11.

3.4. **Proof of Theorem 1 from the Θ-relations.** In this subsection, we prove Theorem 4; that is, we deduce Theorem 1 from the Θ-relations. This also serves as the key technical step in our proof of Theorem $\star$.

Proof of Theorem 4. Recall that Theorem 1 states that $R^k(\mathcal{M}_{g,n})$ vanishes in degrees $k \geq g$ if $n > 0$ and in degrees $k \geq g-1$ if $n = 0$.

The proof of this statement is an elaboration of the technique of Lemma 11, which constitutes the special case in which we restrict attention to tautological classes involving only ψ-classes, assume that $n \geq 2g+3$, and, most importantly, prove vanishing starting in degree $g+1$ rather than g. In order to obtain relations in degree g, we consider the Θ-relations on $\mathcal{M}_{g,2g+3}$, multiply by appropriate classes, and push forward under a forgetful map. The major new complication is that we need too ensure that the forgetful map is proper. To this end, we have to pass to the moduli space of curves of rational type, which necessitates keeping track of those boundary terms in (5) that map onto the open part of the moduli space under the forgetful map.

Assume, first, that $n \leq g$; in particular, we must have $g > 0$. Choose non-negative integers $c_1, \ldots, c_{2g+3}$ such that $c_i \geq 1$ for each $n+1 \leq i \leq 2g+2$, and denote

$$\Psi_C := \psi_1^{c_1} \cdots \psi_{2g+3}^{c_{2g+3}}.$$

Let $c := \sum_{i=1}^{2g+3} c_i$.

We multiply the Θ-relation (5) by Ψ_C and pull back under the inclusion $\iota_{rt} : \mathcal{M}^{rt}_{g,2g+3} \to \mathcal{M}^{ct}_{g,2g+3}$. After substituting $a_{2g+3} = -a_1 - \ldots - a_{2g+2}$ as before, we obtain:

(21)

$$\iota_{rt}^*[2s_A^*\Theta]^{g+1}\Psi_C = \left[\sum_{i=1}^{2g+2} a_i^2\psi_i + \left(\sum_{i=1}^{2g+2} a_i\right)^2 \psi_{2g+3} + \sum_{I \subset [2g+2], \#I \geq 2} a_I^2 \delta_0^I \right.$$
$$\left. + \sum_{I \subset [2g+2], \#I \geq 1} a_{I^c}^2 \delta_0^{I \cup \{2g+3\}} \right]^{g+1} \Psi_C = 0 \in CH^{c+g+1}(\mathcal{M}^{rt}_{g,2g+3}).$$

Here, we denoted $a_{I^c} := \sum_{i \in [2g+2] \setminus I} a_i$, and we have explicitly separated out the boundary divisors parametrizing curves having the last marked point lying on the rational component. As in Lemma 11, this relation is a homogeneous polynomial of degree $2g+2$ in the variables a_i, and every monomial in the a_i gives a separate relation in the tautological ring, which is a polynomial of degree $c+g+1$ in the ψ-classes and boundary divisors on $\mathcal{M}^{rt}_{g,2g+3}$.

We now make the following important observation.

Claim 1. *The coefficient in* (21) *of any a-monomial that is a multiple of $a_1\cdots a_n$ is the sum of a polynomial in only the ψ-classes and a collection of terms supported on the boundary, and each boundary term that occurs is either a multiple of a divisor δ_0^I having $\#(I\cap[n])\geq 2$ or of a divisor $\delta_0^{I\cup\{2g+3\}}$ having $\#(I\cap[n])\geq 2$.*

Proof of Claim 1. First, we note that, according to Lemma 10, for any $I\subset[2g+2]$ we have

$$\delta_0^I\Psi_C=0 \text{ if } \#(I\cap[n])\leq 1, \tag{22}$$

$$\delta_0^{I\cup\{2g+3\}}\Psi_C=0 \text{ if } I\cap[n]=\emptyset. \tag{23}$$

Therefore, the only boundary divisors that appear in (21) are δ_0^I with $\#(I\cap[n])\geq 2$ and $\delta_0^{I\cup\{2g+3\}}$ with $\#(I\cap[n])\geq 1$.

Now, let $I\subset[2g+2]$ be such that $I\cap[n]=\{i\}$. Suppose that $\delta_0^{I\cup\{2g+3\}}$ appears in (21) in an a-monomial that is a multiple of a_i. Then, since $a^2_{I^c}$ does not involve a_i, this can only occur if $\delta_0^{I\cup\{2g+3\}}$ is multiplied by one of the classes ψ_i, ψ_{2g+3}, δ_0^J for $i\in J$, or $\delta_0^{J\cup\{2g+3\}}$ for $i\notin J$. In the first two cases, the product is zero by Lemma 10:

$$\delta_0^{I\cup\{2g+3\}}\psi_i\Psi_C=\delta_0^{I\cup\{2g+3\}}\psi_{2g+3}\Psi_C=0.$$

In the third case, the product is zero whenever $\#(J\cap[n])=1$, by the previous paragraph. The product is also zero in the fourth case whenever $\#(J\cap[n])=1$, because

$$\delta_0^{I\cup\{2g+3\}}\delta_0^{J\cup\{2g+3\}}=0$$

represents a geometrically empty intersection.

Now, consider the coefficient of any a-monomial in (21) that is a multiple of a_i. This coefficient consists of a polynomial only in the ψ-classes, as well as an expression supported on the boundary. The previous paragraph shows that any term in the boundary part containing $\delta_0^{I\cup\{2g+2\}}$ with $I\cap[n]=\{i\}$ also contains at least one boundary divisor δ_0^J or $\delta_0^{J\cup\{2g+3\}}$ with $\#(J\cap[n])\geq 2$. Hence, given an a-monomial in (21) that is a multiple of $a_1\cdots a_n$, applying the above reasoning for each a_i proves the claim. $\square$

The importance of Claim 1 is the following. Let $\Pi_n := \pi_{n+1} \circ \cdots \circ \pi_{2g+3}$ denote the map forgetting the points $p_{n+1}, \ldots, p_{2g+3}$:

$$\Pi_n : \mathcal{M}^{rt}_{g,2g+3} \to \mathcal{M}^{rt}_{g,n}.$$

If $(C, p_1, \ldots, p_{2g+3})$ lies on a boundary divisor Δ_0^I or $\Delta_0^{I \cup \{2g+3\}}$ having $\#(I \cap [n]) \geq 2$, then it remains singular after forgetting the points $p_{n+1}, \ldots, p_{2g+3}$ and stabilizing. In other words, the pushforward along Π_n of any tautological class on $\mathcal{M}^{rt}_{g,2g+3}$ that is a multiple of any of these divisors is a boundary stratum on $\mathcal{M}^{rt}_{g,n}$. Hence, if we take the pushforward of (21) to $\mathcal{M}^{rt}_{g,n}$, and only consider relations that come from a-monomials that are multiples of $a_1 \cdots a_n$, any terms involving boundary divisors on $\mathcal{M}^{rt}_{g,2g+3}$ vanish when restricted to $\mathcal{M}_{g,n}$.

We can get rid of all a-monomials that are not multiples of $a_1 \cdots a_n$ by formally differentiating (21) with respect to these variables. Thus, if $\iota_n : \mathcal{M}_{g,n} \to \mathcal{M}^{rt}_{g,n}$ denotes the inclusion, we have shown the following relation in $CH^{c-g-2+n}(\mathcal{M}_{g,n})$:

$$\frac{\partial^n}{\partial a_1 \cdots \partial a_n} \iota_n^* \Pi_{n*} \left[\sum_{i=1}^{2g+2} a_i^2 \psi_i + \left(\sum_{i=1}^{2g+2} a_i \right)^2 \psi_{2g+3} \right]^{g+1} \Psi_C = 0 \tag{24}$$

whenever $c_i \geq 1$ for $n+1 \leq i \leq 2g+2$.

This implies the vanishing of a collection of tautological classes on $\mathcal{M}_{g,n}$:

Claim 2. *Assume that $n \leq g$, and let $d_1, \ldots, d_{2g+3}$ be non-negative integers satisfying*

- $d := \sum_{i=1}^{2g+3} d_i \geq 3g + 3 - n$,
- $d_i \geq 1$ *for* $i \geq n+1$,
- $\# \left\{ i \mid d_i = 0 \right\} \leq d_{2g+3}$.

Then

$$\iota_n^* \Pi_{n*} [\psi_1^{d_1} \cdots \psi_{2g+3}^{d_{2g+3}}] = 0 \in CH^{d+n-2g-3}(\mathcal{M}_{g,n}). \tag{25}$$

Proof of Claim 2. We first rewrite the ψ-monomial in question in notation similar to what appears in (21). Namely, we claim that we can write

$$\psi_1^{d_1} \cdots \psi_{2g+3}^{d_{2g+3}} = \psi_1^{k_1} \cdots \psi_{2g+3}^{k_{2g+3}} \Psi_C,$$

where Ψ_C is a multiple of $\psi_{n+1} \cdots \psi_{2g+2}$, and the integers k_i satisfy

$$\sum_{i=1}^{2g+3} k_i = g + 1, \quad \min(1, \#\{i \mid 1 \leq i \leq n,\, k_i = 0\}) \leq k_{2g+3}. \tag{26}$$

Indeed, set

$$k'_i = \begin{cases} 1, & i \le n \text{ and } d_i > 0, \\ 0, & d_i = 0, \\ \min\left(1, \#\left\{i \mid d_i = 0\right\}\right), & i = 2g+3. \end{cases}$$

Then the integers k'_i satisfy the conditions

$$k'_i \le d_i \text{ for } 1 \le i \le n, \quad k'_i \le d_i - 1 \text{ for } n+1 \le i \le 2g+2,$$
$$\min(1, \#\{i \mid 1 \le i \le n,\, k'_i = 0\}) \le k'_{2g+3}.$$

Now let k_i be any collection of integers that satisfy the same inequalities as above, such that $k'_i \le k_i$, and such that they add to $g+1$.

To prove the claim, it is therefore enough to show that

$$\iota_n^* \Pi_{n*}[\psi_1^{k_1} \cdots \psi_{2g+3}^{k_{2g+3}} \Psi_C] = 0 \in CH^{c+n-g-2}(\mathcal{M}_{g,n}) \tag{27}$$

for any collection of integers $k_1, \ldots, k_{2g+3}$ satisfying (26) and any ψ-monomial Ψ_C of degree $c \ge 2g+2-n$ that is a multiple of $\psi_{n+1} \cdots \psi_{2g+2}$.

We prove (27) from the relations (24) by an argument essentially identical to the proof of Lemma 11, via descending induction on k_{2g+3}. First, we prove (27) for the case $(k_1, \ldots, k_{2g+3}) = (0, \ldots, 0, g+1)$, which satisfies (26) because $n \le g$. In this case we have

$$\iota_n^* \Pi_{n*}[\psi_{2g+3}^{g+1} \Psi_C] = 0,$$

since $(2g+2)! \iota_n^* \Pi_{n*}[\psi_{2g+3}^{g+1} \Psi_C]$ is the coefficient of $a_{n+1} \cdots a_{2g+2}$ in (24).

Now let $1 \le k \le g+1$, and assume that we have shown that equation (27) holds for any $(k_1, \ldots, k_{2g+3})$ satisfying condition (26) and such that $k_{2g+3} \ge k$. Let $K = (k_1, \ldots, k_{2g+2})$ be a collection of integers such that $(k_1, \ldots, k_{2g+2}, k-1)$ satisfies (26), and denote

$$\Psi_K := \psi_1^{k_1} \cdots \psi_{2g+2}^{k_{2g+2}}.$$

We need to show that $\iota_n^* \Pi_{n*}[\Psi_K \psi_{2g+3}^{k-1} \Psi_C] = 0$.

Let j be the $2(k-1)$-st element, in increasing numerical order, in the set

$$I_K := \{i \mid k_i = 0\} \subset \{1, \ldots, 2g+2\}.$$

Note that by assumption (26), we have $j \ge n$. For $1 \le i \le 2g+2$, define m_i by

$$m_i := \begin{cases} 2k_i, & i \notin I_K, \\ 1, & i \in I_K \text{ and } i \le j, \\ 0, & i \in I_K \text{ and } i > j. \end{cases}$$

Since $k_1 + \ldots + k_{2g+2} = g+1-(k-1)$, we see that $m_1 + \ldots + m_{2g+2} = 2g+2$. Furthermore, $m_i > 0$ if $i \le n$. Therefore, the monomial

$a_1^{m_1-1}\cdots a_n^{m_n-1}a_{n+1}^{m_{n+1}}\cdots a_{2g+2}^{m_{2g+2}}$ occurs in equation (24). A careful inspection shows that its coefficient is equal to

$$C_K\iota_n^*\Pi_{n*}[\Psi_K\psi_{2g+3}^{k-1}\Psi_C]+\iota_n^*\Pi_{n*}[\text{a multiple of }\psi_{2g+3}^{k}\Psi_C]=0,$$

where C_K is a nonzero coefficient, and each ψ-monomial that occurs in the second summand satisfies condition (26). By induction, all these summands vanish, hence $\iota_n^*\Pi_{n*}[\Psi_K\psi_{2g+3}^{k-1}\Psi_C]=0$. □

In fact, relations (25) imply the vanishing of all tautological classes of degree g on $\mathcal{M}_{g,n}$. To see this, recall that the tautological ring of $\mathcal{M}_g$ is generated by the κ-classes, while the tautological ring of $\mathcal{M}_{g,n}$ is generated by the κ- and the ψ-classes. We define the *length* of a monomial $\kappa_{b_1}\cdots\kappa_{b_l}$ in the κ-classes to be l.

Claim 3. *Let $g\geq 1$. Then any codimension-k monomial in the κ-classes vanishes on $\mathcal{M}_{g,1}$ for any $k\geq g$.*

Proof of Claim 3. The claim is proved by induction on the length. The only length one monomials are κ_k, and using (25) and (10), we see that

$$\iota_1^*\Pi_{1*}[\psi_2^{k+1}\psi_3\cdots\psi_{2g+2}\psi_{2g+3}]=C\cdot\kappa_k=0\text{ on }\mathcal{M}_{g,1},$$

where C is a nonzero constant. Hence, κ_k vanishes on $\mathcal{M}_{g,1}$.

Now, suppose that the claim has been proven for all κ-monomials of length less than l. Let $b_1,\ldots,b_l$ be positive integers. If $l>2g+2$, then the monomial $\kappa_{b_1}\cdots\kappa_{b_{l-1}}$ has degree at least $2g+2$ and length $l-1$, and thus vanishes by the induction assumption, hence $\kappa_{b_1}\cdots\kappa_{b_l}$ vanishes as well. If $l\leq 2g+2$, then using (25), (10), and (11), we see that

$$\begin{aligned}\iota_1^*\Pi_{1*}[\psi_2^{b_1+1}\cdots\psi_{l+1}^{b_l+1}\psi_{l+2}\cdots\psi_{2g+3}]&=C\cdot\kappa_{b_1}\cdots\kappa_{b_l}+\\&+(\kappa\text{-monomials of length less than }l)=0\text{ on }\mathcal{M}_{g,1},\end{aligned}$$

where C is a nonzero constant. Hence, $\kappa_{b_1}\cdots\kappa_{b_l}$ vanishes on $\mathcal{M}_{g,1}$ by induction. □

Claim 4. *Any codimension-k monomial in the ψ- and κ-classes that is a multiple of $\psi_1\cdots\psi_n$ vanishes on $\mathcal{M}_{g,n}$ for any $k\geq g$.*

Proof of Claim 4. First, assume that $n\leq g$. The monomials in question have the form

$$\psi_1^{d_1}\cdots\psi_n^{d_n}\kappa_{b_1}\cdots\kappa_{b_l},$$

where b_i and d_i are positive integers. We denote $m:=b_1+\ldots+b_l$ so that $d_1+\ldots+d_n=k-m$. With m fixed, we proceed by induction on the length l of the κ-monomial.

The only degree-m monomial in the κ classes of length one is κ_m, and using (25) and (10), we see that

$$\iota_n^*\Pi_{n*}[\psi_1^{d_1}\cdots\psi_n^{d_n}\psi_{n+1}\cdots\psi_{2g+2}\psi_{2g+3}^{m+1}] = C\cdot\psi_1^{d_1}\cdots\psi_n^{d_n}\kappa_m = 0 \text{ on } \mathcal{M}_{g,n},$$

where C is a nonzero constant, so $\psi_1^{d_1}\cdots\psi_n^{d_n}\kappa_m$ vanishes on $\mathcal{M}_{g,n}$. The above formula also holds when $m=0$ and $\kappa_0 = 2g-2+n$, showing that every monomial in only the ψ-classes vanishes.

Now suppose that we have shown the claim for every monomial of codimension m and length less than l. Then

$$\begin{aligned}&\iota_n^*\Pi_{n*}[\psi_1^{d_1}\cdots\psi_n^{d_n}\psi_{n+1}^{b_1+1}\cdots\psi_{n+l}^{b_l+1}\psi_{n+l+1}\cdots\psi_{2g+3}] = \\ &= C\psi_1^{d_1}\cdots\psi_n^{d_n}\cdot[\kappa_{b_1}\cdots\kappa_{b_l} + (\kappa\text{-monomials of length less than } l)] = 0\end{aligned}$$

on $\mathcal{M}_{g,n}$, where C is a nonzero constant. Hence, $\psi_1^{d_1}\cdots\psi_n^{d_n}\kappa_{b_1}\cdots\kappa_{b_l}$ vanishes on $\mathcal{M}_{g,n}$ by induction.

Finally, let $n > g$. If $g = 0$, then the claim is clearly true for dimension reasons. For $n > g$, the above shows that $\psi_1\cdots\psi_g$ vanishes on $\mathcal{M}_{g,g}$, so by (9) it vanishes on $\mathcal{M}_{g,n}$. It follows that $\psi_1\cdots\psi_n$ and any multiple of it vanishes as well. □

We are now ready to prove the main theorem: that any degree-k monomial in the ψ- and κ-classes vanishes on $\mathcal{M}_{g,n}$ for $k \geq g$, and additionally, for $k = g-1$ and $n = 0$.

For $n = 1$, this is immediate from the above: if the monomial contains ψ_1, then it vanishes by Claim 4, and if not, it vanishes by Claim 3.

To prove the claim for $n = 0$, note that the forgetful map $\pi : \mathcal{M}_{g,1} \to \mathcal{M}_g$ is proper. Since ψ_1^k vanishes on $\mathcal{M}_{g,1}$ for $k > g$, we have

$$\kappa_k = \pi_*\psi_1^{k+1} = 0 \text{ on } \mathcal{M}_g \text{ for } k \geq g-1.$$

More generally, suppose that we have shown that any κ-monomial of degree $k \geq g-1$ and length less than l vanishes on $\mathcal{M}_g$. Let $b_1,\ldots,b_l$ be positive integers such that $b_1+\ldots+b_l = k \geq g-1$. By equation (9), we have

$$\kappa_{b_1}\cdots\kappa_{b_l}\psi_1 = (\pi^*\kappa_{b_1}+\psi_1^{b_1})\cdots(\pi^*\kappa_{b_l}+\psi_1^{b_l})\psi_1$$

on $\mathcal{M}_{g,1}$. The left-hand side has degree at least g, hence it vanishes on $\mathcal{M}_{g,1}$ by Claim 4. Pushing forward the right-hand side under π, we get a multiple of $\kappa_{b_1}\cdots\kappa_{b_l}$ and a sum of κ-monomials of lengths less than l. Hence, $\kappa_{b_1}\cdots\kappa_{b_l}$ vanishes on $\mathcal{M}_g$, proving the claim for $n = 0$.

These cases provide the base for an induction on n, assuming $g > 0$; for $g = 0$, the obvious vanishing of the classes ψ_i and κ_i on $\mathcal{M}_{0,3}$ can be used as the base. Now, suppose that we have proven the claim for

$\mathcal{M}_{g,n}$, and consider a monomial $\Xi = \kappa_{b_1} \cdots \kappa_{b_l} \psi_1^{d_1} \cdots \psi_{n+1}^{d_{n+1}}$ on $\mathcal{M}_{g,n+1}$ of degree

$$b_1 + \ldots + b_l + d_1 + \ldots + d_{n+1} = k \geq g.$$

Proceed by descending induction on the number D of positive d_i. If $D = n+1$, then all of the d_i are positive, and Ξ vanishes by Claim 4. If not, then without loss of generality we can assume that $d_{n+1} = 0$. Using equation (9), we see that

$$\pi_{n+1}^*[\kappa_{b_1} \cdots \kappa_{b_l} \psi_1^{d_1} \cdots \psi_n^{d_n}] = (\kappa_{b_1} - \psi_{n+1}^{b_1}) \cdots (\kappa_{b_l} - \psi_{n+1}^{b_l}) \psi_1^{d_1} \cdots \psi_n^{d_n}.$$

The left-hand side is the pullback of a class from $\mathcal{M}_{g,n}$, which is zero by induction on n, and the expansion of the right-hand side is the sum of Ξ and a collection of classes that vanish by induction on D. Hence, Ξ vanishes on $\mathcal{M}_{g,n+1}$. This completes the proof of the theorem. $\square$

Remark 13. We have shown that the Θ-relations imply the vanishing of the tautological ring of $\mathcal{M}_{g,n}$ in degrees $g - \delta_{0n}$ and above. Just as in Remark 12, however, by upgrading the Θ-relations to Pixton's double ramification cycle relations in the form (8), and by keeping track of the boundary terms throughout the above computations, the same proof produces expressions for these classes as tautological classes supported on the boundary.

Thus, we have the following corollary of the proof of Theorem 4.

Corollary 14. *Any polynomial of degree k in the κ- and ψ-classes on $\overline{\mathcal{M}}_{g,n}$ is equivalent to an algorithmically computable tautological class supported on the boundary of $\overline{\mathcal{M}}_{g,n}$, where $k \geq g$ or $n = 0$ and $k = g-1$.*

3.5. **Proof of Theorem $\star$ from Pixton's double ramification cycle relations.** We are now ready to give a constructive proof of Theorem $\star$, thus completing the proof of Theorem 5.

Proof of Theorem 5. Let Γ be a stable graph of genus g with n legs. Let γ be a basic class on $\overline{\mathcal{M}}_\Gamma$ of the form (17), and let $\xi_{\Gamma *}(\gamma)$ be the corresponding tautological class on $\overline{\mathcal{M}}_{g,n}$, where ξ_Γ is the gluing map (16).

We apply Corollary 14 to every γ_v. If $d(\gamma_v) \geq g(v)$, then we can express γ_v as a boundary class using Pixton's double ramification cycle relations and the algorithm of Theorem 4. (If $g(v) = 0$ or $g(v) = 1$, we can alternatively use the divisorial relations (13) and (14).)

Therefore, possibly after replacing Γ by a graph representing a deeper boundary stratum, we can assume that

$$d(\gamma_v) \leq g(v) - 1 \text{ if } g(v) > 0, \quad d(\gamma_v) = 0 \text{ if } g(v) = 0.$$

Let $g_0 := \#\{v \in V \mid g(v) = 0\}$ denote the number of genus-zero vertices. Using (15), we see that

$$\begin{aligned}\deg \xi_{\Gamma*}(\gamma) &= \sum_{v\in V} d(\gamma_v) + \#E \leq \sum_{v\in V, g(v)>0} (g(v)-1) + \#E \\ &= g(\Gamma) - h^1(\Gamma_0) + g_0 - \#V + \#E = g(\Gamma) + g_0 - 1.\end{aligned}$$

Hence, $g_0 \geq \deg \xi_{\Gamma*}(\gamma) - g(\Gamma) + 1$, proving the theorem. □

Remark 15. The proofs of Theorem 4 and Theorem 5 together provide an explicit algorithm for expressing any tautological class of codimension $k \geq g$ on $\overline{\mathcal{M}}_{g,n}$ as a boundary class having $k - g + 1$ rational components. We provide an outline for this algorithm.

In the notation of Section 3.2, let (Γ, γ) be a marked stable graph defining a tautological class $\xi_{\Gamma*}(\gamma) \in R^k(\overline{\mathcal{M}}_{g,n})$ with a basic class γ_v at each vertex $v \in V$. We say that $\xi_{\Gamma*}(\gamma)$ has *property* $\star$ if

$$\deg(\gamma_v) \leq \max(g(v) - 1, 0) \text{ for every } v \in V.$$

By the proof of Theorem 5, this implies that Γ has at least $k - g + 1$ rational components. Our goal is to obtain an expression for every tautological class in $R^*(\overline{\mathcal{M}}_{g,n})$ as a linear combination of classes having property $\star$. We obtain such expressions by induction on the genus and the degree. Assume that we have already constructed such formulas for all classes in $R^*(\overline{\mathcal{M}}_{g',n})$ with $g' < g$ and all n, and for all classes in $R^{k'}(\overline{\mathcal{M}}_{g,n})$ with $k' < k$ and all n.

Then for each $v \in V$ such that $g(v) < g$ and $\deg(\gamma_v) \geq g(v)$, we use our database to express γ_v as a linear combination of classes having property $\star$. In addition, there may be at most one vertex u such that $g(u) = g$ and $\deg \gamma_u \geq g$. In this case, we use the algorithm of Theorem 4 to express γ_u as a linear combination of non-trivial boundary classes. The highest possible degree of a basic class on any such boundary class is $k - 1$, and by induction, our database expresses all such classes as linear combinations of classes having property $\star$. Hence, we obtain such an expression for γ_u as well. Gluing together these formulas, we obtain an expression for $\xi_{\Gamma*}(\gamma)$ in terms of classes having property $\star$. In this way, we obtain an expression for any given tautological class in terms of classes having property $\star$ in a finite number of steps.

4. Example

In this section, we exemplify our methods by reproving the divisorial formulas (13) expressing ψ_1 and κ_1 in terms of the boundary divisor δ_{irr}

on $\mathcal{M}_{1,1}$; note that this is not a circular argument, as these formulas were not used in the derivation of the main theorem

Before we begin, we note that the genus-zero divisorial formulas (14) follow from the pullback formulas (9) and from the relation $\psi_1 = 0$ on $\overline{\mathcal{M}}_{0,3}$, which can be formally obtained from relation (7) by substituting $a_3 = -a_1 - a_2$ and taking the coefficient of a_1a_2.

4.1. **Pixton's class.** We first recall the definition of Pixton's class $\Omega_{g,A}$.

Define auxiliary classes $\Omega^r_{g,A}$ depending on an additional integer parameter $r > 0$ as follows. Let $\Gamma = (V, H, g, p, \iota)$ be a stable graph of genus g with n legs (following the notation of Section 3.1), and let $A = (a_1, \ldots, a_n) \in \mathbb{Z}^n$. A *weighting modulo* r on Γ is a map

$$w : H \to \{0, \ldots, r-1\}$$

satisfying three properties:

(1) For any $i \in \{1, \ldots, n\}$ corresponding to a leg ℓ_i of Γ, we have $w(\ell_i) \equiv a_i \pmod r$.
(2) For any edge $e \in E$ corresponding to two half-edges $h, h' \in H$, we have $w(h) + w(h') \equiv 0 \pmod r$.
(3) For any vertex $v \in V$, we have $\sum_{h \in p^{-1}(v)} w(h) \equiv 0 \pmod r$.

(Cf. the discussion of weights in Section 2.1.) Define $\Omega^r_{g,A}$ to be the class

(28)
$$\sum_{\Gamma,w} \frac{1}{\#\operatorname{Aut}(\Gamma)} \frac{1}{r^{h^1(\Gamma)}} \xi_{\Gamma*}\left(\prod_{i=1}^n e^{\frac{1}{2}a_i^2\psi_i} \prod_{(h,h')\in E} \frac{1 - e^{-\frac{1}{2}w(h)w(h')(\psi_h+\psi_{h'})}}{\psi_h + \psi_{h'}}\right),$$

where the sum is over all isomorphism classes of stable graphs Γ together with a weighting w modulo r. Pixton has proven that the class $\Omega^r_{g,A}$ is a polynomial in r for $r \gg 0$ (see [JPPZ16, Appendix]). The class $\Omega_{g,A}$ is then defined as the constant term of this polynomial in r.

All stable graphs Γ corresponding to curves of compact type are trees. When Γ is a tree, then there exists a unique weighting modulo r, and when $r > \frac{1}{2}|\sum_{i=1}^n a_i|$, formula (28) is essentially obtained by expanding the formula for $\exp([s_A^*\Theta])$ and performing repeated intersections of divisors on $\mathcal{M}^{ct}_{g,n}$.

4.2. **Computing κ_1.** Equipped with Pixton's formula for $\Omega_{g,A}$, we proceed with the computation of κ_1 by computing the coefficient of $a_1a_2a_3a_4$ in

$$2\Pi_{1*}(\psi_2\psi_3\psi_4[\Omega_{1,A}]_2) = 0, \tag{29}$$

where the map $\Pi_1 : \overline{\mathcal{M}}_{1,5} \to \overline{\mathcal{M}}_{1,1}$ forgets all but the first marking.

Let us first consider the simpler question of computing κ_1 on $\mathcal{M}^{ct}_{1,1}$. In this case, we can replace $2[\Omega_{1,A}]_2$ by $[s_A^*\Theta]^2$. Since $\mathcal{M}^{rt}_{1,n} = \mathcal{M}^{ct}_{1,n}$, we can use (21) to compute $[s_A^*\Theta]^2$.

We claim that the coefficient of $a_1a_2a_3a_4$ in $\psi_2\psi_3\psi_4[s_A^*\Theta]^2$ is equal to

$$\frac{1}{4} \cdot 24\psi_2\psi_3\psi_4\psi_5^2. \tag{30}$$

To see this, first notice that we can remove the summands $a_i^2\psi_i$ for $i \in \{1, 2, 3, 4\}$ from (21), since they will not give a multiple of $a_1a_2a_3a_4$. Next, recall that multiplying with the class $\psi_2\psi_3\psi_4$ kills all boundary divisor classes δ_0^I for $I \subset \{1, \ldots, 5\}$ except when $\{1, 5\} \subset I$. Thus the coefficient of $a_1a_2a_3a_4$ in $\psi_2\psi_3\psi_4[s_A^*\Theta]^2$ equals the coefficient of $a_1a_2a_3a_4$ in

$$\frac{1}{4} \cdot \psi_2\psi_3\psi_4 \left((a_1 + a_2 + a_3 + a_4)^2\psi_5 - \sum_{I\subset\{1,2,3,4\},1\in I} \left(\sum_{i\notin I} a_i \right)^2 \delta_0^{I\cup\{5\}} \right)^2.$$

It remains to show that only the multiple of ψ_5^2 contributes. This is true since, on the one hand, $\psi_2\psi_3\psi_4\psi_5$ kills any of the boundary divisors by Lemma 10, and on the other hand, in the square of the boundary terms the variable a_1 does not appear. Since the coefficient of $a_1a_2a_3a_4$ in $(a_1 + a_2 + a_3 + a_4)^2$ is 24, we obtain formula (30).

Thus, the coefficient of $a_1a_2a_3a_4$ in $2\Pi_{1*}(\psi_2\psi_3\psi_4[\Omega_{1,A}]_2)$ is equal to

$$\frac{1}{4} \cdot 24^2\kappa_1 + C\delta_{irr} \tag{31}$$

for a constant C that we now need to determine.

To compute C, we only need to consider stable graphs Γ with $h^1(\Gamma) = 1$. We claim that there is only one such dual graph with non-zero coefficient of $a_1a_2a_3a_4$.

To see why this is the case, let us first look at the stable graph Γ with exactly one vertex and a loop $e = (h, h')$. There exist r weightings modulo r on Γ, which can be distinguished by the value of $w(h) \in \{0, \ldots, r-1\}$. Since we only need the degree-2 part of Pixton's class for (29), and since the edge term in (28) for Γ does not depend on the a_i, the summand in (28) for Γ is a non-homogeneous polynomial in the a_i of degree 2. Thus, it gives a zero coefficient of $a_1a_2a_3a_4$.

By similar arguments as in the compact-type case, the only remaining stable graphs Γ whose contribution will not be killed by $\psi_2\psi_3\psi_4$ have two vertices v_1, v_2 connected by a pair of edges e_1, e_2 such that the leg ℓ_5 associated to the fifth marked point lies on v_1 and the leg ℓ_1 associated to the first marked point lies on v_2. Let us write $e_i = (h_i, h_i')$, where h_i is the half-edge at vertex v_1 and h_i' is the half-edge at v_2. There are again r choices of weightings modulo r, indexed by $w(h_1)$. The locus in $\overline{\mathcal{M}}_{1,5}$ corresponding to Γ is of codimension 2, and therefore we need to take the constant term in the factors corresponding to the legs in (28). Notice that

$$w(h_1') \equiv -w(h_1), \qquad w(h_2) \equiv x - w(h_1), \qquad w(h_2') \equiv w(h_1) - x,$$

where

$$x = \sum_{i \text{ at } v_2} a_i,$$

and therefore the contribution of Γ to $[\Omega_{1,A}]_2$ depends on the a_i only in the quantity x. Thus, we can only have a non-zero coefficient of $a_1a_2a_3a_4$ when $\ell_1, \ell_2, \ell_3, \ell_4$ are on v_2; this is the unique stable graph Γ that contributes.

We now compute the contribution of Γ. As we have seen, it depends on the a_i only in the form $x = a_1 + a_2 + a_3 + a_4$. For the computation we can assume that x is positive. Let us also write $a := w(h_i)$. Since the edge term corresponding to e_1 in (28) vanishes when $a = 0$, we can assume that $a \in \{1, \ldots, r-1\}$. In the case that $a \leq x$, we can write

$$w(h_1) = a, \quad w(h_1') = r - a, \quad w(h_2) = x - a, \quad w(h_2') = r + a - x.$$

Otherwise,

$$w(h_1) = a, \quad w(h_1') = r - a, \quad w(h_2) = r + x - a, \quad w(h_2') = a - x.$$

By inclusion-exclusion, the contribution of Γ is therefore given by

$$\frac{1}{4}\frac{1}{r}\left(\sum_{a=1}^{r-1} a(r-a)(r+x-a)(a-x) \right.$$
$$\left. + \sum_{a=1}^{x} a(r-a)((x-a)(r+a-x) - (r+x-a)(a-x))\right).$$

The first sum gives only a polynomial of degree 2 in x and will therefore not lead to a coefficient of $a_1a_2a_3a_4$. From the remaining summand we obtain

$$\frac{1}{4}\sum_{a=1}^{x} a(r-a)(2(x-a)) \equiv -\frac{1}{4} \cdot 2 \sum_{a=1}^{x} a^2(x-a) \pmod{r},$$

which, by computing the power sum, equals

$$\frac{1}{4}\left(-\frac{1}{6}x^4 - \frac{2}{3}x^3 + \frac{7}{6}x^2 - \frac{1}{3}x\right).$$

The total contribution of Γ to the coefficient of $a_1a_2a_3a_4$ in (29) is therefore

$$-2\cdot\frac{1}{4}\cdot\frac{24}{6}\frac{1}{\#\operatorname{Aut}(\Gamma)}\Pi_{1*}(\psi_2\psi_3\psi_4\xi_{\Gamma*}[1]) = -\frac{1}{4}\cdot 48\delta_{irr}.$$

Plugging in this value of C into equation (31), we conclude the well-known formula

$$\kappa_1 = \frac{1}{12}\delta_{irr} \in CH^*(\overline{\mathcal{M}}_{1,1}).$$

4.3. **Computing ψ_1.** For the computation of ψ_1 in terms of κ_1 and δ_{irr}, we now consider the coefficient of $a_1^2a_2a_3$ in (29).

Similarly to the computation for κ_1, only the multiples of ψ_5^2 and $\psi_1\psi_5$ give coefficients that are divisible by a_1 and are not killed by the multiplication with $\psi_2\psi_3\psi_4$. Hence, the coefficient of $a_1^2a_2a_3$ in $2\Pi_{1*}(\psi_2\psi_3\psi_4[\Omega_{1,A}]_2)$ is equal to

$$0 = \frac{1}{4}\cdot 12\cdot 24\kappa_1 + \frac{1}{4}\cdot 4\cdot 24\psi_1 + C'\delta_{irr}$$

for some constant C'.

Similarly to the computation for κ_1, the only dual graphs Γ that can contribute to C' must have two vertices v_1, v_2 connected by a pair of edges e_1, e_2 with ℓ_5 on v_1 and ℓ_1 on v_2. Let us write $e_i = (h_i, h_i')$, where h_i is the half-edge at vertex v_1 and h_i' is the half-edge at v_2. Unlike for κ_1, not only the dual graph Γ for which $\ell_1, \ell_2, \ell_3, \ell_4$ are all on v_2 contributes to C', but also the dual graph Γ' whose set of legs at v_2 is $\{\ell_1, \ell_2, \ell_3\}$.

The computation of the contribution of Γ to C' is essentially the same as for κ_1. The result is

$$-2\cdot\frac{1}{4}\cdot\frac{12}{6}\frac{1}{\#\operatorname{Aut}(\Gamma)}\Pi_{1*}(\psi_2\psi_3\psi_4\xi_{\Gamma*}[1]) = -\frac{1}{4}\cdot 24\delta_{irr}.$$

The computation for Γ' is also very similar. The main difference is that we should define $x := a_1 + a_2 + a_3$. We find that the contribution of Γ' to C' is

$$-2\cdot\frac{1}{4}\cdot\frac{12}{6}\frac{1}{\#\operatorname{Aut}(\Gamma')}\Pi_{1*}(\psi_2\psi_3\psi_4\xi_{\Gamma'*}[1]) = -\frac{1}{4}\cdot 8\delta_{irr}.$$

Combining the contributions, we obtain the relation

$$9\kappa_1 + 3\psi_1 = \delta_{irr} \in CH^*(\overline{\mathcal{M}}_{1,1}),$$

and thus

$$\kappa_1 = \psi_1 = \frac{1}{12}\delta_{irr} \in CH^*(\overline{\mathcal{M}}_{1,1}).$$

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