# Isoperiodic deformations of the acoustic operator and periodic solutions of the Harry Dym equation

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## Abstract

We consider the problem of describing the possible spectra of an acoustic operator with a periodic finite-gap density. We construct flows on the moduli space of algebraic Riemann surfaces that preserve the periods of the corresponding operator. By a suitable extension of the phase space, these equations can be written with quadratic irrationalities.

## 1 Introduction

In the study of periodic potentials of the one-dimensional Schrödinger operator, we encounter the following problem. A given periodic potential $u(x)$ of the Schrödinger operator

$$L_S = -\frac{d^2}{dy^2} + u(y) \tag{1.1}$$

determines a Riemann surface, called the spectral curve, which is algebraic if the potential has a finite number of energy gaps. Conversely, given a hyperelliptic Riemann surface of genus $g$, there exists a potential $u(x)$ with the corresponding spectrum, which can be extended to a solution $u(x, \vec{t})$ of the KdV hierarchy. However, this potential is not in general periodic. We cannot extend the direct spectral transform to all quasi-periodic potentials. For

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instance, the spectrum of a quasi-periodic potential may have the structure of a Cantor set (see [1]), making it impossible to associate a Riemann surface to it. The problem of determining which Riemann surfaces correspond to periodic finite-gap potentials did not at first receive much attention.

The periods of a potential of the Schrödinger operator are expressed as certain Abelian integrals on the corresponding spectral curve. The potential has a given period if and only if these integrals are integer multiples of the period. Therefore, the subset of curves corresponding potentials with a given period forms a transcendental submanifold in the moduli space of hyperelliptic Riemann surfaces.

A description of the possible spectra of a periodic Scrödinger operator, not necessarily finite-gap, was given by Marchenko and Ostrovsky in [9] in terms of the properties of a conformal map of a certain type. However, their approach was difficult from a computational viewpoint. An effective solution was given by Grinevich and Schmidt in [6], called the method of isoperiodic deformations. Using an idea developed in [5], [8], the solution consists of finding a set of differential equations on the spectral data that do not change the periods of the solution. The submanifold corresponding to solutions of a given period is then preserved by these flows. By a suitable extension of the phase space, the equations of isoperiodic deformation can be written with a rational right-hand side, making them convenient for numerical simulation.

This paper extends this approach to the study of the periodic finite-gap densities acoustic operator

$$L_A = -r^2(x)\frac{d^2}{dx^2}. \tag{1.2}$$

Finite-gap densities of the acoustic operator and the corresponding periodic solutions of the Harry Dym equation

$$r_t = r^3 r_{xxx} \tag{1.3}$$

were first constructed by Dmitrieva in [2], [3], [4], using a Hopf transformation to the Schrödinger operator introduced in [12]. Periodic solutions of the Harry Dym equation were recently shown to be relevant to the Saffman-Taylor problem (see [13]), while finite-gap periodic densities of the acoustic operator are related to geodesics on the ellipsoid (see [10], [14], [15]).

In sections 2 and 3, we recall the spectral theory of the periodic Schrödinger operator and the method of isoperiodic deformations. In section 4, we derive

a spectral theory for the acoustic operator by relating it to the Schrödinger operator, and in section 5 we construct the equations of isoperiodic deformation for the acoustic operator. By extending the phase space, we write these equations with quadratic irrationalities. The principal result of this paper is Theorem 2, which gives the explicit form of the deformation equations.

## 2    Periodic finite-gap potentials of the Schrödinger operator

We first recall the spectral theory of the one-dimensional periodic finite-gap Schrödinger operator

$$L_S = -\frac{d^2}{dy^2} + u(y), \tag{2.1}$$

where $u(y)$ is a smooth periodic real-valued potential with period $\Pi$. For references, see [11].

We consider two spectral problems for $L_S$:

1. The standard problem in $L^2(\mathbb{R})$:

$$L_S\varphi = \lambda\varphi, \ \ |\varphi(y)| < \infty \ \text{ as } \ y \to \pm\infty,$$

2. The Dirichlet problem on a period:

$$L_S\varphi = \lambda\varphi, \ \ \varphi(y_0) = \varphi(y_0 + \Pi) = 0.$$

The spectrum of the first problem is continuous and consists of an infinite number of segments $[\Lambda_0, \Lambda_1], [\Lambda_2, \Lambda_3], \ldots$, with $\Lambda_{2j} < \Lambda_{2j+1} \leq \Lambda_{2j+2}$. The spaces between these segments $(\Lambda_{2j+1}, \Lambda_{2j+2})$, which may be of zero length, are called the energy gaps. The second problem has a purely discrete spectrum $d_j(y_0)$, with exactly one eigenvalue inside or on the boundary of each of the energy gaps, $d_j(y_0) \in [\Lambda_{2j-1}, \Lambda_{2j}]$, including the degenerate gaps.

The principal case of interest is when the potential $u(y)$ has only a finite number of energy gaps of non-zero length, such a potential is called finite-gap. Let $(-\infty, \lambda_0), (\lambda_1, \lambda_2), \ldots, (\lambda_{2g-1}, \lambda_{2g})$ denote the non-trivial energy gaps, and let $\gamma_j(y_0)$ denote the eigenvalue of the Dirichlet problem lying in the $j$-th non-trivial gap.

The direct spectral transform assigns to the finite-gap potential $u(y)$ the following data

1. A hyperelliptic Riemann surface $\Gamma$ of genus $g$ together with a two-sheeted covering $\lambda : \Gamma \to \mathbb{CP}^1$ ramified at $\lambda_0, \ldots, \lambda_{2g}$ and $\infty$

2. A meromorphic function $\varphi(y, y_0, P)$ on $\Gamma \backslash \lambda^{-1}(\infty)$, called the Bloch-Floquet function, that has $g$ simple poles $P_1(y_0), \ldots, P_g(y_0)$ on $\Gamma$ satisfying $\lambda(P_k(y_0)) = \gamma_k(y_0)$.

This function is a joint eigenfunction of the Schrödinger operator and the monodromy operator:

$$L_S \varphi(y, y_0, P) = \lambda(P) \varphi(y, y_0, P), \tag{2.2}$$

$$\varphi(y + \Pi, y_0, P) = \mu(P) \varphi(y, y_0, P), \tag{2.3}$$

and has the following high-energy expansion:

$$\varphi(y, y_0, P) = \exp(i(y - y_0)\sqrt{\lambda})(1 + o(1)). \tag{2.4}$$

The logarithmic derivative of the Bloch-Floquet function is equal to

$$\chi(y, P) = -i\frac{\varphi_y(y, y_0, P)}{\varphi(y, y_0, P)} = \frac{\sqrt{R(\lambda(P))}}{S(y, \lambda(P))} - \frac{i}{2}\frac{S_y(y, \lambda(P))}{S(y, \lambda(P))}, \tag{2.5}$$

where the functions $R(\lambda)$ and $S(y, \lambda)$ are defined as

$$R(\lambda) = \prod_{j=0}^{2g}(\lambda - \lambda_j), \quad S(y, \lambda) = \prod_{k=1}^{g}(\lambda - \gamma_k(y)). \tag{2.6}$$

The multi-valued function $p(P) = -\frac{i}{\Pi} \ln \mu(P)$ is called the *quasi-momentum*. Its differential

$$dp = -\frac{i}{\Pi}\frac{d\mu}{\mu}$$

is the unique meromorphic 1-form on $\Gamma$ satisfying the following properties:

1. $dp$ has a single pole of second order at infinity with the principal part

$$dp = \left(-\frac{1}{k^2} + O(1)\right)dk,$$

where $k = \lambda^{-1/2}$ is the local parameter, and

2. The periods of $dp$ over the $a$-cycles are equal to zero:

$$\oint_{a_k} dp = 0, \quad k = 1, \ldots, g \qquad (2.7)$$

Since the function $\mu(\lambda) = e^{i\Pi p(\lambda)}$ is single-valued, the $b$-periods of $\Omega_p$ are integral multiples of $2\pi/\Pi$:

$$\oint_{b_k} dp = \frac{2\pi n_k}{\Pi}, \quad n_k \in \mathbb{Z}, \quad k = 1, \ldots, g. \qquad (2.8)$$

Conversely, given a hyperelliptic Riemann surface $\Gamma$ together with a two-sheeted covering $\lambda : \Gamma \to \mathbb{CP}^1$ with real-valued ramification points $\lambda_0, \ldots, \lambda_{2g}$ and $\infty$, and a nonspecial divisor $D = P_1 + \cdots + P_g$ satisfying $\lambda(P_i) \in [\lambda_{2j-1}, \lambda_{2j}]$, there exists a smooth real-valued potential $u(y)$ of the Schrödinger operator with spectral data $\{\lambda_i, P_j(0)\}$. This potential is given in terms of the theta-function of $\Gamma$ by the Matveev-Its formula:

$$u(y) = -2\partial_y^2 \ln \theta(y\vec{U}_1 - \vec{A}(P_1) - \cdots - \vec{A}(P_g) - \vec{K}|B_{ij}) + C(\Gamma), \qquad (2.9)$$

where $\vec{A}$ is the Abel map, $\vec{K}$ is the vector of Riemann constants, $C(\Gamma)$ is a constant, and the vector $\vec{U}_1$ is the vector of $b$-periods of the unique meromorphic differential $dp$ on $\Gamma$ satisfying properties (1) and (2) above:

$$(\vec{U}_1)_k = \frac{1}{2\pi} \oint_{b_k} dp \quad k = 1, \ldots, g. \qquad (2.10)$$

This potential is periodic with period $\Pi$ only if the components of the vector $\vec{U}_1$ are integral multiples of $1/\Pi$. However, for generic spectral data the components of $\vec{U}_1$ are arbitrary real numbers, so the potential $u(y)$ is in general quasi-periodic.

Therefore, the problem of describing all finite-gap potentials of period $\Pi$ is reduced to the following: describe all hyperelliptic Riemann surfaces such that the 1-form $dp$, uniquely determined by conditions (1) and (2), has $b$-periods that are integral multiples of $2\pi/\Pi$. An effective solution of this problem was given in [6], which we now recall.

# 3 Isoperiodic deformations for the Schrödinger operator

Let $\Gamma_0$ be a hyperelliptic Riemann surface corresponding to a real-valued potential of the Schrödinger operator of period one. Consider a deformation $\Gamma(t)$ of $\Gamma_0$, i.e. let a continuously varying one-periodic family of hyperelliptic Riemann surfaces of genus $g$, equipped with two-sheeted covering maps $\lambda(t) : \Gamma(t) \to \mathbb{CP}^1$, branched at points $\lambda_0(t), \ldots, \lambda_{2g}(t)$ on the real axis and $\infty$, such that $\Gamma(0) = \Gamma_0$. If each of the curves $\Gamma(t)$ also corresponds to a potential of period one, then this deformation is called *isoperiodic*. The principal result of [6] consists of an effective description of all such deformations.

Suppose we have an isoperiodic deformation $\Gamma(t)$, then each of the curves has a unique meromorphic 1-form $dp(t)$ satisfying properties (1) and (2) above, such that its $b$-periods are integers. Consider the meromorphic 1-form

$$\omega = \frac{\partial p}{\partial t} d\lambda - \frac{\partial \lambda}{\partial t} dp \tag{3.1}$$

on $\Gamma_0$. This form has a double pole at infinity and no other singularities, such forms are called weakly meromorphic. If we choose the connection in such a way that $\frac{\partial p}{\partial t} = 0$, then the deformation is explicitly given in terms of the ramification points

$$\frac{\partial \lambda_j}{\partial t} = -\frac{\omega(\lambda_k)}{dp(\lambda_k)}. \tag{3.2}$$

Conversely, given a weakly meromorphic 1-form $\omega$, we can define a deformation of $\Gamma_0$ using the above formula. If we now choose the connection in such a way that $\frac{\partial \lambda}{\partial t} = 0$, then we see that

$$\frac{\partial p}{\partial t} = \frac{\omega}{d\lambda} \tag{3.3}$$

is a single-valued function on $\Gamma_0$, so therefore the periods of $dp(t)$ are constant. Therefore, if $\Gamma_0$ corresponds to a potential of period one, then so do the surfaces $\Gamma(t)$. Thus, isoperiodic deformations of the Schrödinger operator are described by meromorphic 1-forms with prescribed singularities, namely with a double pole at infinity. We now try to adapt this approach to obtain isoperiodic deformations for the acoustic operator.

# 4 Spectral theory of the acoustic operator

We now consider the acoustic operator

$$L_A = -r^2(x)\frac{d^2}{dx^2}, \qquad (4.1)$$

where the function $r(x)$, called the density, is smooth and positive. The spectral theory of the acoustic operator with a smooth periodic density has been widely studied. The $L^2(\mathbb{R})$ spectrum of the problem

$$L_A\psi(x,\lambda) = \lambda\psi(x,\lambda), \qquad (4.2)$$

like that of the periodic Schrödinger operator, has a zone structure and consists of an infinite number of bands $[\Lambda_0, \Lambda_1], [\Lambda_2, \Lambda_3], \ldots$, with $\Lambda_{2j} < \Lambda_{2j+1} \leq \Lambda_{2j+2}$. If the density $r(x)$ is smooth, then there is a constraint $\Lambda_0 = 0$. The finite-gap densities of $L_A$ can be constructed using the following well-known relation [12]:

**Proposition 1** *Suppose $r(x), \psi(x,\lambda)$ satisfy the acoustic equation*

$$-r^2(x)\frac{d^2}{dx^2}\psi(x,\lambda) = \lambda\psi(x,\lambda). \qquad (4.3)$$

*Perform a change of variables*

$$y(x) = \int_0^x \frac{dx'}{r(x')}. \qquad (4.4)$$

*Then the functions*

$$u(y) = \frac{1}{4}r_x^2(x) - \frac{1}{2}r(x)r_{xx}(x), \qquad (4.5)$$

$$\varphi(y,\lambda) = \psi(x,\lambda)r^{-1/2}(x), \qquad (4.6)$$

*satisfy the Schrödinger equation*

$$-\frac{d^2}{dx^2}\varphi(y,\lambda) + u(y)\varphi(y,\lambda) = \lambda\varphi(y,\lambda). \qquad (4.7)$$

*If $r(x)$ is a periodic density of $L_A$ with period $T$, then $u(y)$ is a periodic potential of $L_S$ with period*

$$\Pi = \int_{x_0}^{x_0+T} \frac{dx}{r'(x)}, \qquad (4.8)$$

*and the operators $L_A$ and $L_S$ have the same spectrum in $L^2(\mathbb{R})$.*

This proposition allows us to construct finite-gap densities of the acoustic operator by taking a finite-gap potential $u(y)$ of the Schrödinger operator, given by the Matveev-Its formula for some spectral data $\{\lambda_i, \gamma_k\}$ with $\lambda_0 = 0$, and solving the equations (4.4)-(4.5) for the density $r(x)$. This inverse transformation is only defined up to a parameter, as the acoustic equation has the gauge transformation

$$x \to \alpha x, \quad r(x) \to \alpha^{-1} r(\alpha x), \tag{4.9}$$

so that a periodic potential corresponds to a one-parameter family of periodic densities. The choice of this $\alpha$ should be considered as an additional spectral parameter of the problem.

Explicit formulas for $r(x)$ in terms of theta-functions were obtained by Dmitrieva in [2]-[4] by extending the Hopf transformation (4.4)-(4.6) to a transformation between the Harry Dym hierarchy and the Korteweg-de Vries hierarchy. Finite-gap periodic densities of the acoustic operator are then equivalent to $x$-periodic solutions of the Harry Dym equation

$$r_t = r^3 r_{xxx}. \tag{4.10}$$

Since periodic densities correspond to periodic potentials, and periodic potentials of the Schrödinger operator are described by the isoperiodic deformations, the problem of describing all periodic densities of the acoustic operator is in some sense solved. However, we would like to find a natural choice of the constant for the gauge transformation, depending explicitly on the spectral data, and a set of differential equations on the spectral data of the acoustic operator, such that the period of the unique density corresponding to the spectral data is preserved under these flows. We now turn to this problem.

Let $r(x)$ be a finite-gap density of period $T$, and let $\psi_\pm(x, x_0, \lambda)$ be the Bloch-Floquet function of the corresponding acoustic operator $L_A$, that is a joint eigenfunction of the acoustic and monodromy operators:

$$L_A \psi_\pm(x, x_0, \lambda) = \lambda \psi_\pm(x, x_0, P), \tag{4.11}$$

$$\psi_\pm(x + T, x_0, \lambda) = e^{\pm i T q(\lambda)} \psi_\pm(x, x_0, \lambda), \tag{4.12}$$

normalized by the relation

$$\psi_\pm(x, x_0, T)|_{x=x_0} = 1. \tag{4.13}$$

Let $L_S$ be the associated Schrödinger operator given by (4.4)-(4.6). Then the function $\psi(x, x_0, \lambda)$ can be expressed in terms of the Bloch-Floquet function (2.2)-(2.3) of $L_S$ as follows:

$$\psi_\pm(x, x_0, \lambda) = r^{-1/2}(x_0)r^{1/2}(x)\varphi(y, y_0, \lambda(P)). \tag{4.14}$$

Hence, we can consider the Bloch-Floquet as a meromorphic function on the spectral curve $\Gamma$ of $L_S$, i.e. we can set

$$\psi_\pm(x, x_0, \lambda) = \psi(x, x_0, \lambda(P)) \tag{4.15}$$

for some meromorphic function $\psi(x, x_0, P)$ on $\Gamma$ satisfying

$$L_A\psi(x, x_0, P) = \lambda(P)\psi(x, x_0, P), \tag{4.16}$$

$$\psi(x + T, x_0, P) = e^{iTq(P)}\psi(x, x_0, P), \tag{4.17}$$

where the multivalued function $q(P)$ is called the *quasi-momentum* of the acoustic operator $L_A$. It can be expressed in terms of the logarithmic derivative of the Bloch-Floquet function

$$\xi(x, P) = -i\frac{\psi_x(x, x_0, P)}{\psi(x, x_0, P)} \tag{4.18}$$

as follows:

$$q(P) = \frac{1}{T}\int_{x_0}^{x_0+T} \xi(x, P)dx. \tag{4.19}$$

To study the function $\xi(x, P)$, we first see that it can be expressed in terms of the logarithmic derivative $\chi(y, P)$ of $\varphi(y, y_0, P)$ as follows:

$$\xi(x, P) = \frac{1}{r(x)}\chi(y, P) - \frac{i}{2}\frac{r_x(x)}{r(x)}. \tag{4.20}$$

Substituting this in (2.5), we get

$$\xi(x, P) = \frac{\sqrt{R(\lambda)}}{r(x)S(x, \lambda)} - \frac{i}{2}\frac{r'(x)}{r(x)} - \frac{i}{2}\frac{S_x(x, \lambda)}{S(x, \lambda)}, \tag{4.21}$$

where $R(\lambda) = \lambda\prod_{j=1}^{2g}(\lambda - \lambda_j)$ and $S(x, \lambda) = \prod_{k=1}^{g}(\lambda - \gamma_k(y(x)))$. Therefore, the quasi-momentum $q(P)$ and its differential have the following high-energy expressions:

$$q(P) = \frac{1}{kT}\int_{x_0}^{x_0+T} \frac{dx}{r(x)} + O(1), \tag{4.22}$$

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$$dq(P) = \left(-\frac{1}{k^2 T}\int_{x_0}^{x_0+T}\frac{dx}{r(x)} + O(1)\right)dk, \tag{4.23}$$

where $k = \lambda^{-1/2}$.

A simple calculation also shows that $\xi(x, P)$ satisfies the Ricatti equation:

$$-i\xi'(x, P) + \xi^2(x, P) - \frac{\lambda(P)}{r^2(x)} = 0. \tag{4.24}$$

The spectrum of the acoustic operator always starts at $\lambda = 0$, i.e. the left-most branch point of $\Gamma$ is $\lambda = 0$. We consider the Ricatti equation in the neighborhood of this point. Let $k = \sqrt{\lambda}$ be the local parameter and consider the Taylor series for $\xi(x, P)$ at $k = 0$:

$$\xi(x, P) = \xi_0(x) + \xi_1(x)k + \xi_2(x)k^2 + O(k^3). \tag{4.25}$$

Substituting this into the Ricatti equation, we get that $\xi_0(x) = 0$, $\xi_1(x) = C$ for some constant $C$, and that $\xi_2(x)$ is a full derivative. Therefore, the quasi-momentum $q(P)$ and its differential near $k = 0$ are equal to

$$q(P) = Ck + O(k^3). \tag{4.26}$$

$$dq(P) = (C + O(k^2))dk. \tag{4.27}$$

On the other hand, comparing the obtained expression for $\xi(x, P)$ with (4.21), we see that

$$C = \frac{\sqrt{\lambda_1 \cdots \lambda_{2g}}}{(-1)^g r(x)\gamma_1(x)\cdots\gamma_g(x)}, \tag{4.28}$$

and hence $r(x)$ is expressed in terms of the spectral data and the additional constant $C$ as follows:

$$r(x) = \frac{\sqrt{\lambda_1 \cdots \lambda_{2g}}}{(-1)^g C\gamma_1(x)\cdots\gamma_g(x)}. \tag{4.29}$$

The constant $C$ in this formula corresponds to the gauge transformation (4.9) and should be considered in addition to the $\lambda_j$ and $\gamma_k$ as part of the spectral data of the problem.

To construct isoperiodic deformations of the acoustic operator, we need a rule of choosing the constant $C$. The natural choice seems to be $C = (-1)^g$. With this choice, the value of $dq$ at $\lambda = 0$ does not depend on the spectral curve $\Gamma$, which will be used to construct the isoperiodic deformations. Also,

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with this choice of $C$, a vanishingly small potential $u(x) \to 0$, or equivalently vanishingly small energy gaps $(\lambda_{2j-1} - \lambda_{2j} \to 0)$, correspond to a density $r(x) \to 1$ normalized at unity.

We summarize the results of this section in the following

**Theorem 1** *Let $\{\lambda_i, \gamma_k(y)\}$ be the spectral data of a finite-gap periodic Schrödinger operator $L_S$ with potential $u(y)$ such that $\lambda_0 = 0$. Then the density $r(x)$ of the associated acoustic operator $L_A$ related to $u(y)$ by (4.4)-(4.6) is given by the equation (4.29), where $C$ is an arbitrary constant. If we choose $C = (-1)^g$, then the differential of the quasi-momentum of $L_A$ has a pole of second order at $\lambda = \infty$ with principal part (4.23), and has fixed first and second order terms at $\lambda = 0$:*

$$dq(P) = ((-1)^g + O(k^2))dk. \tag{4.30}$$

Using this theorem, we construct isoperiodic deformations for the acoustic operator.

# 5 Isoperiodic deformations of the acoustic operator

We recall that a deformation of a Riemann surface is given by a meromorphic 1-form

$$\omega = \frac{\partial q}{\partial t} d\lambda. \tag{5.1}$$

To give an isoperiodic deformation for the acoustic operator, this 1-form must satisfy the following conditions:

1. The quasimomentum differential $dq$ has a second order pole at infinity, so $\frac{\partial q}{\partial t}$ has a first order pole at infinity. Since $d\lambda$ has a third order pole at infinity, $\omega$ has a pole of fourth order at infinity.

2. At zero $dq$ has fixed first and second order terms, therefore $\frac{\partial q}{\partial t}$ has a third order zero (with our choice of $C$). Since $d\lambda$ has a first order zero, $\omega$ has a zero of fourth order at $\lambda = 0$.

Therefore, a 1-form $\omega$ defines an isoperiodic deformation if and only if it has divisor $4 \cdot 0 - 4 \cdot \infty$. The space of such forms is $g$-dimensional. Since the total space of spectral curves has dimension $2g$, and there are $g$ conditions

for the period to be equal to one, these forms are a basis of isoperiodic deformations.

As before, the deformation is given explicitly in terms of the branch points as

$$\frac{\partial \lambda_j}{\partial t} = -\frac{\omega(\lambda_j)}{dq(\lambda_j)}, \quad j = 1, \ldots, 2g. \tag{5.2}$$

On a hyperelliptic surface, the 1-form $dq$ can be written explicitly as

$$dq = \frac{Q(\lambda)}{2\sqrt{R(\lambda)}} d\lambda, \tag{5.3}$$

where

$$Q(\lambda) = q_g \lambda^g + \cdots + q_0, \tag{5.4}$$

and $q_0 = (-1)^g \sqrt{\lambda_1 \cdots \lambda_{2g}}$. The coefficients of the polynomial $Q(\lambda)$ are determined by setting the $a$-periods of $dq$ to zero and are expressed in terms of certain hyperelliptic integrals.

An arbitrary 1-form $\omega$ with divisor $4 \cdot 0 - 4 \cdot \infty$ can be written as

$$\omega = \frac{f(\lambda)}{2\sqrt{R(\lambda)}} d\lambda, \tag{5.5}$$

where

$$f(\lambda) = f_{g+1} \lambda^{g+1} + \cdots + f_2 \lambda^2 = \lambda^2 (f_{g+1} \lambda^{g-1} + \cdots + f_2). \tag{5.6}$$

The deformation given by such an $\omega$ has the form

$$\frac{\partial \lambda_j}{\partial t} = -\frac{f(\lambda_j)}{Q(\lambda_j)}, \quad j = 1, \ldots, 2g. \tag{5.7}$$

It is possible to choose a basis in the space of meromorphic 1-forms with divisor $4 \cdot 0 - 4 \cdot \infty$ and construct a basis of deformations by this formula. However, the right-hand side of the deformation equations contains the coefficients of $Q(\lambda)$ that are in turn expressed as hyperelliptic integrals containing the ramification points. To avoid this difficulty, we use the approach of [6]. Factoring the polynomial $Q(\lambda)$

$$Q(\lambda) = (\beta_1 \lambda - \sqrt{\lambda_1 \lambda_2}) \cdots (\beta_g \lambda - \sqrt{\lambda_{2g-1} \lambda_{2g}}), \tag{5.8}$$

we consider the parameters $\beta_k$ as independent variables that are deformed along with the ramification points. This extension of the phase space greatly simplifies the deformation equations.

The deformations of $\beta_k$ are easily shown to be given by the following

**Lemma 1** *Let $\frac{\partial}{\partial t}$ be the flow given by the form $\omega$ with polynomial $f(\lambda)$. Then the deformations of $\lambda_i$ and $\beta_k$ have the form*

$$\frac{\partial \lambda_j}{\partial t} = -\frac{f(\lambda_j)}{Q(\lambda_j)} \tag{5.9}$$

$$\frac{\partial \beta_k}{\partial t} = \beta_k \left[ -\frac{1}{2\lambda_{2k-1}} \frac{f(\lambda_{2k-1})}{Q(\lambda_{2k-1})} - \frac{1}{2\lambda_{2k}} \frac{f(\lambda_{2k})}{Q(\lambda_{2k})} + \right.$$

$$\left. + \frac{1}{Q'(\lambda)} \cdot \left( f'(\lambda) - \frac{f(\lambda)R'(\lambda)}{2R(\lambda)} \right) \Bigg|_{\lambda = \frac{\sqrt{\lambda_{2k-1}\lambda_{2k}}}{\beta_k}} \right] \tag{5.10}$$

As we see, the equations no longer contain hyperelliptic integrals. This form makes the equations of isoperiodic deformation useful for numerical simulations.

To write the deformations explicitly, we choose a basis in the space of differentials with divisor $4 \cdot 0 - 4 \cdot \infty$ as follows:

$$\omega_k = \frac{c_k \lambda^2}{\beta_k \lambda - \sqrt{\lambda_{2k-1}\lambda_{2k}}} dq, \tag{5.11}$$

where $c_k$ are arbitrary constants. Then

$$\frac{f_k(\lambda)}{Q(\lambda)} = \frac{c_k \lambda^2}{\beta_k \lambda - \sqrt{\lambda_{2k-1}\lambda_{2k}}}, \tag{5.12}$$

and the deformation equations are given by our final theorem.

**Theorem 2** *Let $\Gamma$ be a hyperelliptic Riemann surface with ramification points $\lambda_0 = 0, \lambda_1, \ldots, \lambda_{2g}$, corresponding to a smooth density of period one. Let the zeroes of the quasimomentum be $\frac{\sqrt{\lambda_{2k-1}\lambda_{2k}}}{\beta_k}, k = 1, \ldots, g$. Consider the flows*

$$\frac{\partial \lambda_j}{\partial t_k} = -\frac{c_k \lambda_j^2}{\beta_k \lambda_j - \sqrt{\lambda_{2k-1}\lambda_{2k}}} \tag{5.13}$$

$$\frac{\partial \beta_l}{\partial t_k} = c_k \beta_k \left[ -\frac{1}{2\lambda_{2l-1}} \frac{\lambda_{2l-1}^2}{\beta_k \lambda_{2l-1} - \sqrt{\lambda_{2k-1}\lambda_{2k}}} - \frac{1}{2\lambda_{2l}} \frac{\lambda_{2l}^2}{\beta_k \lambda_{2l} - \sqrt{\lambda_{2k-1}\lambda_{2k}}} + \right.$$

$$\left. + \frac{\lambda_{2l-1}\lambda_{2l}}{\beta_l^2} \left( \beta_k \frac{\sqrt{\lambda_{2l-1}\lambda_{2l}}}{\beta_l} - \sqrt{\lambda_{2k-1}\lambda_{2k}} \right)^{-1} \right]; \quad l \neq k \tag{5.14}$$

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$$\frac{\partial \beta_k}{\partial t_k} = c_k \beta_k \left[ -\frac{1}{2\lambda_{2k-1}} \frac{\lambda_{2k-1}^2}{\beta_k \lambda_{2k-1} - \sqrt{\lambda_{2k-1}\lambda_{2k}}} - \frac{1}{2\lambda_{2l}} \frac{\lambda_{2k}^2}{\beta_k \lambda_{2k} - \sqrt{\lambda_{2k-1}\lambda_{2k}}} + \right.$$

$$+ \frac{1}{\beta_k} \left\{ \frac{\lambda_{2k-1}\lambda_{2k}}{\beta_k^2} \sum_{l \neq k} \beta_l \left( \beta_l \frac{\sqrt{\lambda_{2k-1}\lambda_{2k}}}{\beta_k} - \sqrt{\lambda_{2l-1}\lambda_{2l}} \right)^{-1} + 2\frac{\sqrt{\lambda_{2k-1}\lambda_{2k}}}{\beta_k} - \right.$$

$$\left. \left. \frac{\lambda_{2k-1}\lambda_{2k}}{\beta_k} \cdot \left( \frac{\beta_k}{\sqrt{\lambda_{2k-1}\lambda_{2k}}} - \sum_{j=1}^{2g} \frac{\beta_k}{\sqrt{\lambda_{2k-1}\lambda_{2k}} - \lambda_j \beta_k} \right) \right\} \right] \qquad (5.15)$$

*on the spectral data, where $\frac{\partial}{\partial t_k}$ is the flow given by $\omega_k$ defined by (5.11). Then any Riemann surface obtained by moving along these flows in the moduli space of hyperelliptic Riemann surfaces corresponds to a density $r(x)$ of period one of the acoustic operator. These flows form a basis in the space of isoperiodic deformations of the acoustic operator.*

We see that these extended equations do not involve hyperelliptic integrals.

# 6 Acknowledgements

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