# About linearization of Poisson-Nambu structures

AMS Sectional Meeting

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# Outline

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# The Poisson bracket

## Definition

A **Poisson bracket** on a smooth manifold $M$ is a bilinear map

$$\{\cdot,\cdot\}: \ C^\infty(M) \times C^\infty(M) \to C^\infty(M)$$

which is skew-symmetric and satisfies

$$\{f, g_1 g_2\} = \{f, g_1\} g_2 + g_1 \{f, g_2\}$$
$$\{f, \{g_1, g_2\}\} = \{\{f, g_1\}, g_2\} + \{g_1, \{f, g_2\}\}$$

Equivalently, $\pi \in \mathfrak{X}^2(M)$ is a **Poisson bivector** if:

$$[\pi, \pi]_{SN} = 0 \quad \text{or} \quad \forall f \in C^\infty(M): \ \mathcal{L}_{X_f} \pi = 0$$

# Properties of Poisson structures

Let $(M, \pi)$ be a Poisson manifold. Then $(M, \pi)$ admits

- a foliation $\mathcal{F}$ via
$$T\mathcal{F} = \mathrm{Im}(\pi^\sharp : T^*M \to TM)$$
Leaves are even dimensional and symplectic

- a Lie algebroid structure $T^*_\pi M := (T^*M, [\cdot,\cdot]_\pi, \pi^\sharp)$ where
$$[\alpha,\beta]_\pi := \mathcal{L}_{\pi^\sharp\alpha}\,\beta - \mathcal{L}_{\pi^\sharp\beta}\,\alpha - \mathrm{d}\,\pi(\alpha,\beta)$$

- Poisson cohomology via
$$\mathrm{d}_\pi := [\pi,\cdot] : \mathfrak{X}^\star(M) \to \mathfrak{X}^{\star+1}(M)$$

# Examples

0. **trivial**: $\pi = 0$
1. **symplectic**: $(\pi^\sharp)^{-1} = \omega^\flat : TM \to T^*M$
2. **linear**: On a finite dimensional vector space $\mathfrak{g}$:

$$\left\{ \begin{matrix} \text{Lie algebra structures} \\ [\cdot,\cdot] \text{ on } \mathfrak{g} \end{matrix} \right\} \xleftrightarrow{1:1} \left\{ \begin{matrix} \text{linear Poisson structures} \\ \pi_{\mathfrak{g}} \text{ on } \mathfrak{g}^* \end{matrix} \right\}$$

The correspondence is given by:

$$\pi_{\mathfrak{g}}(\xi)(X_1, X_2) = \langle \xi, [X_1, X_2] \rangle \qquad \text{where } \xi \in \mathfrak{g}^*, X_1, X_2 \in \mathfrak{g}$$

# The question of local normal forms

## Weinstein splitting theorem ('83)

Locally around $p \in M$:

$$(M, \pi, p) = (S, \pi_S = \omega_s^{-1}, p_S) \times (T, \pi_T, p_T) \qquad \text{with} \quad \pi_T(p_T) = 0$$

Let $p \in M$ with $\pi(p) = 0$

$$[\mathrm{d}_p f_1, \mathrm{d}_p f_2] := \mathrm{d}_p \pi(\mathrm{d} f_1, \mathrm{d} f_2) \quad \text{for} \quad f_1, f_2 \in C^\infty(M)$$

## Definition

$(\mathfrak{g}_p^* = T_p M, \pi_{\mathfrak{g}_p})$ is the **linear approximation** of $\pi$ at $p$

## Question

Is $\pi$ linearizable at $p \in M$, i.e. does there exist a local isomorphism

$$\Phi : (M, \pi, p) \to (T_p M, \pi_{\mathfrak{g}_p}, 0)?$$

# Linearization

**Definition**

- A Lie algebra $\mathfrak{g}$ is called **Poisson non-degenerate**, if the answer is yes whenever $\mathfrak{g}_x \simeq \mathfrak{g}$
- otherwise $\mathfrak{g}$ is called **Poisson degenerate**

**Theorem (Weinstein '83):**

Any semisimple Lie algebra $\mathfrak{g}$ is formally Poisson non-degenerate.

**Theorem (Conn '84):**

Any semisimple Lie algebra $\mathfrak{g}$ is analytically Poisson non-degenerate.

# Linearization - $C^\infty$ case

Let $\mathfrak{g}$ be a semisimple Lie algebra $\rightsquigarrow$ Iwasawa decomposition of $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$$

| Lie algebra $\mathfrak{g}$ | $\dim \mathfrak{a}$ | | |
|---|---|---|---|
| $\mathfrak{sl}(2,\mathbb{R}) \simeq \mathfrak{su}(1,1)$ | 1 | ✗ | Weinstein '83 |
| $\mathfrak{g} = \mathfrak{k}$, e.g. $\mathfrak{so}(n)$, $\mathfrak{su}(n)$ | 0 | ✓ | Conn '85 ,<br>Crainic & Fernandes '11 |
| $2 \leq \dim \mathfrak{a}$,<br>e.g. $\mathfrak{so}(p,q)$ with $2 \leq q \leq p$ | q | ✗ | Weinstein '87 |
| $\dim \mathfrak{a} = 1$ & $\mathfrak{k}$ not semisimple<br>e.g. $\mathfrak{su}(n,1)$ | 1 | ✗ | Monnier & Zung '05 |
| $\mathfrak{so}(3,1) \simeq \mathfrak{sl}(2,\mathbb{C})$ | 1 | ✓ | Marcut & Z. '21 |
| $\dim \mathfrak{a} = 1$ & $\mathfrak{k}$ semisimple<br>e.g. $\mathfrak{so}(p,1)$ with $4 \leq p$ | 1 | ? | |

# Poisson degeneracy for $\mathfrak{sl}_2(\mathbb{R})$

For $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R})$ :

$$\pi_{\mathfrak{sl}_2(\mathbb{R})} = -x_3 \partial_{x_1} \wedge \partial_{x_2} + x_1 \partial_{x_2} \wedge \partial_{x_3} + x_2 \partial_{x_3} \wedge \partial_{x_1}$$

The associated foliation is described by

$$f(x_1, x_2, x_3) := \frac{1}{2}(x_1^2 + x_2^2 - x_3^2)$$

The level sets of $f$

**Weinstein '83**

$$\pi = \pi_{\mathfrak{sl}_2(\mathbb{R})} + \frac{g(f)}{r^2} \partial_r \wedge \partial_{x_3}$$

is not linearizable for $g \in C^\infty(\mathbb{R})$:

$$g(x) \begin{cases} > 0 & \text{if } x > 0 \\ = 0 & \text{else.} \end{cases}$$

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# The Nambu bracket

**Definition**

A **Nambu bracket of order** $q$ on a smooth manifold $M$ is a q-linear map

$$\{\cdot, \ldots, \cdot\}: \ C^\infty(M) \times \cdots \times C^\infty(M) \to C^\infty(M)$$

which is skew-symmetric and satisfies

$$\{f_1, \ldots, f_{q-1}, g_1 g_2\} = \{f_1, \ldots, f_{q-1}, g_1\} g_2 + g_1 \{f_1, \ldots, f_{q-1}, g_2\}$$

$$\{f_1, \ldots, f_{q-1}, \{g_1, \ldots, g_q\}\} = \sum_{i=1}^{q} \{g_1, \ldots, \{f_1, \ldots, f_{q-1}, g_i\}, \ldots, g_q\}$$

Equivalently, $\Pi \in \mathfrak{X}^q(M)$ is a **Nambu vector** if:

$$\forall f_1, \ldots, f_{q-1} \in C^\infty(M) : \ \mathcal{L}_{X_{f_1, \ldots, f_{q-1}}} \Pi = 0$$

# Properties of Nambu structures

Let $\Pi \in \mathfrak{X}^q(M)$ be a Nambu structure. Then $(M, \Pi)$ admits

- a foliation $\mathcal{F}$ via

$$T\mathcal{F} = \mathrm{Im}(\Pi^\sharp : \wedge^{q-1} T^*M \to TM)$$

  Leaves are zero or $q$ dimensional

- a Leibniz algebroid structure $(\wedge^{q-1} T^*M, [\cdot, \cdot]_\Pi, \Pi^\sharp)$ where

$$[\alpha, \beta]_\Pi := \mathcal{L}_{\pi^\sharp \alpha} \beta + (-1)^n \iota_\Pi(\mathrm{d}\,\alpha)\beta$$

- Leibniz cohomology

# Examples

0. **trivial**: $\Pi = 0$
1. **orientation**: $0 \neq \Pi \in \mathfrak{X}^m(M)$
2. **linear**: On a finite dimensional vector space $\mathfrak{F}$:

$$\left\{ \begin{matrix} \text{Filipov algebra structure} \\ [\cdot, \dots, \cdot] \text{ on } \mathfrak{F} \end{matrix} \right\} \hookleftarrow \left\{ \begin{matrix} \text{linear Nambu structures} \\ \Pi_{\mathfrak{F}} \text{ on } \mathfrak{F}^* \end{matrix} \right\}$$

given by:

$$\Pi(\xi)(X_1, \dots, X_q) = \langle \xi, [X_1, \dots, X_q] \rangle$$

where $\xi \in \mathfrak{F}^*$ and $X_1, \dots, X_q \in \mathfrak{F}$.

# The question of local normal forms

**Alekseevsky & Guha '96, Gautheron '96, Nakanishi '98**

For $q \geq 3$, $\Pi \in \mathfrak{X}^q(M)$ is Nambu iff for all $p \in M$ with $\Pi(p) \neq 0$ there are coordinates $(x_1, \dots, x_m)$ such that

$$\Pi = \partial_{x_1} \wedge \cdots \wedge \partial_{x_q}$$

Let $p \in M$ with $\Pi(p) = 0$

$$[\mathrm{d}_p f_1, \dots, \mathrm{d}_p f_q] := \mathrm{d}_p \Pi(\mathrm{d} f_1, \dots, \mathrm{d} f_q) \quad \text{for} \quad f_1, \dots, f_q \in C^\infty(M)$$

**Definition**

$(\mathfrak{F}_p^* = T_p M, \Pi_{\mathfrak{F}_p})$ is the **linear approximation** of $\Pi$ at $p$

**Question**

Is $\Pi$ linearizable at $p \in M$, i.e. does there exist a local isomorphism

$$\Phi : (M, \Pi, p) \to (T_p M, \Pi_{\mathfrak{F}_p}, 0)?$$

# Nambu structures via forms

## Dufour & Zung '99

Let $(M^m, \Omega)$ be an oriented manifold. Then $\Pi \in \mathfrak{X}^q(M)$ is Nambu iff $\omega := \iota_\Pi \Omega$ satisfies for all $P \in \mathfrak{X}^{m-q-1}(M)$:

$$\iota_P \omega \wedge \omega = 0 \qquad \text{and} \qquad \iota_P \omega \wedge \mathrm{d}\,\omega = 0 \tag{1}$$

## Definition

We call $\omega \in \Omega^k(M)$ an **integrable differential form (IDF)** if (1) holds.

## Remark

- IDFs induce a singular foliation.
- IDFs have been extensively studied in the literature (Reeb '52, Kupka '64, Medeiros '77, Moussu '76 & '83, ..)

# Classification of linear Nambu structures

## Dufour & Zung '99, Medeiros '00

Let $V^m$ be a vector space and $\omega \in \Omega^p(V)$ a linear IDF. Then there are linear coordinates on $V$ such that $\omega$ is of one of the following two types:

- **Type 1:**

$$\omega = \mathrm{d}\, x_1 \wedge \cdots \wedge \mathrm{d}\, x_{p-1} \wedge \mathrm{d}\left(\sum_{j=p}^{p+r} \pm x_j^2 + \sum_{i=1}^{s} x_i x_{p+r+i}\right)$$

  with $-1 \leq r \leq m-p$ and $0 \leq s \leq m-p-r$;

- **Type 2:**

$$\omega = \sum_{i=1}^{p+1} a_i \,\mathrm{d}\, x_1 \wedge \cdots \wedge \widehat{\mathrm{d}\, x_i} \wedge \cdots \wedge \mathrm{d}\, x_{p+1} \quad \text{where } a_i = \sum_{j=1}^{p+1} a_i^j x_j$$

  and $(a_i^j)$ in Jordan normal form.

# Linearization

## Dufour & Zung '99

Type 2 singularities are linearizable under a non-resonancy condition.

## Definition

We call $p \in M$ with $\Pi(p) = 0$ a **nondegenerate** singularity if an associated linear IDF is of Type 1 with quadratic function

$$f = \sum_{j=p}^{m} \pm x_j^2.$$

## Dufour & Zung '99, Zung '13

Let $p \in M^m$ be a nondegenerate singularity of $\Pi \in \mathfrak{X}^q(M)$, then:

- $\Pi$ is formally linearizable at $p$.
- $\Pi$ is smoothly linearizable at $p$ if the signature of $f$ is not $(2, \star)$ or $(\star, 2)$.

# Non-Linearization

## Moussu '76

Let $f$ be the Morse function on $\mathbb{R}^m$ given by

$$f = x_1^2 + x_2^2 - \sum_{i=3}^{m} x_i^2.$$

Then the integrable differential form

$$\alpha := \mathrm{d}f + \frac{g(f)}{x_1^2 + x_2^2}(x_2 \mathrm{d}x_1 - x_1 \mathrm{d}x_2)$$

is not linearizable for $g \in C^\infty(\mathbb{R})$ with

$$g(x) \begin{cases} > 0 & \text{if } x > 0 \\ = 0 & \text{else.} \end{cases}$$

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# The statement

**Theorem (Z.)**

Let $\Pi$ be a Nambu structure on $M^m$ of order $m-1$ with $m \geq 3$ and $p \in M$ such that $\Pi(p) = 0$. Assume $p$ is nondegenerate and there exists a volume form $\Omega$ around $p \in M$ such that

$$\mathrm{d}\iota_{\Pi}\Omega = 0. \tag{2}$$

Then $\Pi$ is linearizable at $p$.

**Corollary**

Let $\pi$ be Poisson structure on $M^3$ which is unimodular locally around a zero $p \in M$ with $\mathfrak{g}_p \simeq \mathfrak{sl}_2(\mathbb{R})$. Then $\pi$ is linearizable around $p$.

# The proof

**Colin de Verdiere & Vey '79**

Let $\Omega$ be a volume form on $\mathbb{R}^m$. Given a smooth function $g$ and a Morse function $f$ on $\mathbb{R}^m$ with

$$j_0^2 g = j_0^2 f,$$

there exists a smooth function $h$ and a diffeomorphism $\phi$ locally around 0:

$$(f, h(f)\Omega_{std}) = \phi^*(g, \Omega).$$

Strategy of the proof:

1. By (2) there exists $g$ smooth with

$$\mathrm{d}\, g = \iota_\Pi \Omega$$

2. Applying the result above implies

$$\Pi = \widetilde{h}(f)\Pi_{\mathfrak{F}}.$$

   Thus we can use a Moser argument.

# Questions

- Can we obtain similar results for singularities of Nambu structures on $M^m$ of order different from $m-1$?
- Can we find similar results for linear Poisson structures of semisimple Lie algebras with real rank 1 and non-semisimple compact part?

*Thank you for your attention!*