Isomorphisms of linear symplectic torus quotients

Christopher Seaton
Joint work with Hans-Christian Herbig and Gerald Schwarz

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Skidmore College
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Real linear symplectic quotients

Real symplectic quotients

Let K be a compact Lie group and $V \cong \mathbb{C}^n$ a finite-dimensional unitary K-module.

Let $\rho\colon V \to \mathfrak{k}^*$ denote the homogeneous quadratic moment map

$$(\rho(v), \xi) := \frac{\sqrt{-1}}{2}\langle v, \xi.v\rangle, \quad v \in V, \xi \in \mathfrak{k}.$$

Let $Z := \rho^{-1}(0)$, the **real shell**, a K-invariant subset of V.

Note that 0 is (usually) a singular value of ρ so that Z is a singular real algebraic K-variety.

The **real linear symplectic quotient** is Z/K.

Structures of a real linear symplectic quotient

Differentiable space: $\mathcal{C}^\infty(Z/K) := \mathcal{C}^\infty(V)^K/\mathcal{I}_Z^K$ where $\mathcal{I}_Z$ is the vanishing ideal of Z and $\mathcal{I}_Z^K := \mathcal{I}_Z \cap \mathcal{C}^\infty(V)^K$.

Graded algebra of real regular functions: $\mathbb{R}[Z/K] := \mathbb{R}[V]^K/I_Z^K$ where $I_Z = \sqrt[\mathbb{R}]{(\rho)}$ is the vanishing ideal of Z in $\mathbb{R}[V]$ and $I_Z^K := I_Z \cap \mathbb{R}[V]^K$.

Poisson bracket on smooth and regular functions.

Stratified symplectic space (Sjamaar–Lerman, 1991): The stratification of V/K by isotropy types restricts to a stratification of Z/K into smooth symplectic manifolds.

Choosing a generating set of $\mathbb{R}[V]^K$ (a **Hilbert basis**) yields an embedding $V/K \to \mathbb{R}^k$, the **Hilbert embedding**. This realizes Z/K as a semialgebraic set.

(Schwarz 1976, Mather 1977) The smooth structure above coincides with the induced smooth structure as a subset of $\mathbb{R}^k$.

When K is finite: Linear symplectic orbifolds

When K is a finite group, Z/K is a **linear symplectic orbifold**.

- The moment map is $J = 0$.
- $Z = J^{-1}(0) = V$.
- $\mathcal{C}^\infty(V_0) = \mathcal{C}^\infty(V)^K$.
- $\mathbb{R}[V_0] = \mathbb{R}[V]^K$.

When K is finite: Linear symplectic orbifolds

Example

Let $K = \{\pm 1\}$ act on $\mathbb{C}$ by multiplication.

Using coordinates $(z, \overline{z})$ for $\mathbb{C}$, the real invariants $\mathbb{R}[V]^K$ of the action are generated by

$$u = z^2, \quad v = \overline{z}^2, \quad w = z\overline{z}.$$

They satisfy the relation: $$w^2 - uv = 0.$$

$$\mathbb{R}[V_0] = \mathbb{R}[V]^K \quad \cong \quad \mathbb{R}[u, v, w]/\langle w^2 - uv \rangle, \qquad \text{with } w \geq 0.$$

$K = \mathbb{T}^\ell$

When $K = \mathbb{T}^\ell$ is a torus, the action is described by a weight matrix $A = (a_{i,j}) \in \mathbb{Z}^{\ell\times n}$ in coordinates $(z_1, \ldots, z_n)$ for $V \cong \mathbb{C}^n$:

$$(t_1, \ldots, t_\ell)\cdot(z_1, \ldots, z_n) = (t_1^{a_{1,1}} \cdots t_\ell^{a_{\ell,1}} z_1, \ldots, t_1^{a_{1,n}} \cdots t_\ell^{a_{\ell,n}} z_n).$$

Identifying $\mathfrak{g}^*$ with $\mathbb{R}^\ell$, the moment map is given by

$$\rho_i(z_1, \ldots, z_n) = -\frac{1}{2}\sum_{j=1}^{n} a_{i,j}|z_i|^2, \qquad i = 1, \ldots, \ell.$$

Then $Z = \{(z_1, \ldots, z_n) \in V \mid \sum_{j=1}^{n} a_{i,j}|z_i|^2 = 0 \quad \forall i\}$.

If $\sqrt[\mathbb{R}]{(\rho)} = (\rho)$ then $I_Z^{\mathbb{T}^\ell} = I_Z$ so that $\mathbb{R}[Z/\mathbb{T}^\ell] = \mathbb{R}[V]^{\mathbb{T}^\ell}/I_Z$.

Example: $K = \mathbb{T}^1$

Example

Let $K = \mathbb{T}^1$ act on $\mathbb{C}^2$ with weight matrix $(-1, 1)$,

$$t(z_1, z_2) = (t^{-1}z_1, tz_2).$$

Then $Z = \{(z_1, z_2) : |z_1|^2 = |z_2|^2\}$ is homeomorphic to the cone on $\mathbb{T}^2$.

The real invariants of the action are generated by

$$p_1 = z_1\overline{z_1}, \quad p_2 = z_2\overline{z_2}, \quad p_3 = z_1z_2, \quad p_4 = \overline{z_1}\,\overline{z_2}.$$

(in real coordinates $(z_1, z_2, \overline{z_1}, \overline{z_2})$, the weight matrix is $(-1, 1, 1, -1)$).

The ideal $I_Z^{\mathbb{T}^1} = \langle p_1 - p_2 \rangle$, so $\mathbb{R}[Z/\mathbb{T}^1] = \mathbb{R}[V]^{\mathbb{T}^1}/I_Z^{\mathbb{T}^1}$ is generated by the quadratics p_1, p_3, p_4 with relation $p_1^2 - p_3p_4$.

The only inequality is $p_1 \geq 0$.

Example: $K = \mathbb{T}^1$ (cont.)

Example

Let $K = \mathbb{T}^1$ act on $\mathbb{C}^2$ with weight matrix $(-1, 1)$,

$$t(z_1, z_2) = (t^{-1} z_1, t z_2).$$

$$p_1 = z_1 \overline{z_1}, \quad p_2 = z_2 \overline{z_2}, \quad p_3 = z_1 z_2, \quad p_4 = \overline{z_1}\,\overline{z_2}.$$

$$\mathbb{R}[Z/\mathbb{T}^1] = \mathbb{R}[V]^{\mathbb{T}^1}/I_Z^{\mathbb{T}^1} = \langle p_1, p_3, p_4 : p_1^2 - p_3 p_4 \rangle, \qquad p_1 \geq 0.$$

$\{\cdot,\cdot\}$	p_1	p_3	p_4
p_1	0	$c\, p_3$	$-c\, p_4$
p_3		0	$-2c\, p_1$
p_4			$0,$

$$c = 2\sqrt{-1}.$$

(Lerman-Montgomery-Sjamaar, 1993) The resulting symplectic quotient is the same as the orbifold $\mathbb{C}/\pm 1$:

$$\mathbb{R}[\mathbb{C}/\pm 1] = \mathbb{R}[\mathbb{C}]^{\pm 1} = \langle z^2, \overline{z}^2, z\overline{z} \rangle \cong \mathbb{R}[u, v, w]/\langle w^2 - uv \rangle, \quad w \geq 0.$$

Example: $K = \mathbb{T}^2$

Example

Let $K = \mathbb{T}^2$ act on $\mathbb{C}^4$ with weight matrix $\begin{pmatrix} -1 & 0 & 1 & 1 \\ 0 & -1 & 2 & 2 \end{pmatrix}$.

$$Z = \{(z_1, z_2, z_3, z_4) : |z_1|^2 = |z_3|^2 + |z_4|^2, \quad |z_2|^2 = 2|z_3|^2 + 2|z_4|^2\}.$$

The real invariants of the action are generated by

$$m_1 = z_1\overline{z_1},\ \ m_2 = z_2\overline{z_2},\ \ m_3 = z_3\overline{z_3},\ \ m_4 = z_4\overline{z_4},\ \ p_1 = z_3\overline{z_4},\ \ p_2 = z_4\overline{z_3},$$
$$p_3 = z_1 z_2^2 z_3,\ \ p_4 = \overline{z_1}\,\overline{z_2}^2\overline{z_3},\ \ p_5 = z_1 z_2^2 z_4,\ \ p_6 = \overline{z_1}\,\overline{z_2}^2\overline{z_4}.$$

On Z, $m_1 = m_3 + m_4$ and $m_2 = 2m_3 + 2m_4$, and the remaining relations are

$$m_3 m_4 - p_1 p_2,\ p_1 p_4 - m_3 p_6,\ m_4 p_4 - p_2 p_6,\ p_2 p_3 - m_3 p_5,\ m_4 p_3 - p_1 p_5,$$
$$4m_3^3 p_2 + 4m_4^3 p_2 + 12 m_3 p_1 p_2^2 + 12 m_4 p_1 p_2^2 - p_4 p_5,\ 4m_3^3 p_1 + 4m_4^3 p_1 + 12 m_3 p_1^2 p_2 + 12 m_4 p_1^2 p_2 - p_3 p_6,$$
$$4m_4^4 + 4m_3^2 p_1 p_2 + 12 m_4^2 p_1 p_2 + 12 p_1^2 p_2^2 - p_5 p_6,\ 4m_3^4 + 12 m_3^2 p_1 p_2 + 4 m_4^2 p_1 p_2 + 12 p_1^2 p_2^2 - p_3 p_4$$

The resulting symplectic quotient is the same as the symplectic quotient corresponding to the $\mathbb{T}^1$-action with weight matrix $(-1, 3, 3)$.

Motivating question

Motivating question

For $i = 1, 2$:

- K_i a compact Lie group,
- V_i a unitary K_i-module,
- Z_i the real shell in V_i,
- Z_i/K_i the real symplectic quotient.

A **diffeomorphism** $\Phi\colon Z_1/K_1 \to Z_2/K_2$ is a homeomorphism such that $\Phi^*\colon \mathcal{C}^\infty(Z_2/K_2) \to \mathcal{C}^\infty(Z_1/K_1)$ is an isomorphism.

A **symplectomorphism** $\Phi\colon Z_1/K_1 \to Z_2/K_2$ is a diffeomorphism such that $\Phi^*\colon \mathcal{C}^\infty(Z_2/K_2) \to \mathcal{C}^\infty(Z_1/K_1)$ is a Poisson isomorphism.

A diffeo/symplectomorphism is **regular** if Φ^* restricts to an isomorphism $\mathbb{R}[Z_2/K_2] \to \mathbb{R}[Z_1/K_1]$ and **graded regular** if this isomorphism preserves the grading.

The above two examples are graded regular symplectomorphisms.

Question: When are two real linear symplectic quotients (graded) regularly symplectomorphic?

(Graded) regular symplectomorphisms with orbifolds are rare

Theorem (Herbig-Schwarz-S., 2015)

Let $\mathbb{T}^1$ act on $V = \mathbb{C}^n$ such that the corresponding symplectic quotient has real dimension greater than 2. Then there does not exist a regular diffeomorphism between the corresponding symplectic quotient and a linear symplectic orbifold.

For weight matrices of the form $(-a_1, a_2)$ with $a_i > 0$, the symplectic quotient is graded regularly symplectomorphic to the linear symplectic orbifold $\mathbb{C}/\big(\mathbb{Z}/\langle a_1 + a_2\rangle\big)$.

For any weight matrix $(\pm a_1, \ldots, \pm a_n)$, $a_i > 0$, containing positive and negative weights with $n > 2$, there is no graded regular symplectomorphism with a linear symplectic orbifold.

There are lots of graded regular symplectomorphisms

(Herbig-Lawler-S., 2020) Let $\mathbb{T}^\ell$ act on $\mathbb{C}^{\ell+k}$ with weight matrix

$$A = \begin{pmatrix} \mathbf{D} & \vert & c_1\mathbf{n} & c_2\mathbf{n} & \cdots & c_k\mathbf{n} \end{pmatrix}.$$

- $\mathbf{D} = \operatorname{diag}(-a_1, \ldots, -a_\ell)$ with each $a_i > 0$,
- $\mathbf{n} = (n_1, \ldots, n_\ell)^T$ with each $n_i > 0$,
- each $c_j > 0$.

Define

$$\alpha(A) = \operatorname{lcm}(a_1, \ldots, a_\ell), \qquad m_i(A) = \frac{n_i \alpha(A)}{a_i},\ i = 1, \ldots, \ell, \qquad \beta(A) = \sum_{i=1}^{\ell} m_i(A).$$

The real symplectic quotient associated to A is graded regularly symplectomorphic to the real symplectic quotient associated to the $\mathbb{T}^1$-representation on $\mathbb{C}^{k+1}$ with weight matrix

$$\begin{pmatrix} -\alpha(A) & c_1\beta(A) & c_2\beta(A) & \cdots & c_k\beta(A) \end{pmatrix}.$$

■

Examples of graded regular symplectomorphisms

Example

- The real symplectic quotients associated to the $\mathbb{T}^2$ action with weight matrix $\begin{pmatrix} -1 & 0 & 1 & 1 & 1 & 1 \\ 0 & -1 & 2 & 2 & 2 & 2 \end{pmatrix}$ and the $\mathbb{T}^1$-action $(-1, 3, 3, 3, 3)$ are graded regularly symplectomorphic.

- The real symplectic quotients associated to the $\mathbb{T}^2$ action with weight matrix $\begin{pmatrix} -1 & 0 & 1 & 3 & 5 & 7 \\ 0 & -1 & 2 & 6 & 10 & 14 \end{pmatrix}$ and the $\mathbb{T}^1$-action $(-1, 3, 9, 15, 21)$ are graded regularly symplectomorphic.

- The real symplectic quotients associated to the $\mathbb{T}^3$ action with weight matrix $\begin{pmatrix} -1 & 0 & 0 & 3 & 6 & 9 \\ 0 & -2 & 0 & 1 & 2 & 3 \\ 0 & 0 & -3 & 2 & 4 & 6 \end{pmatrix}$ and the $\mathbb{T}^1$-action $(-6, 25, 50, 75)$ are graded regularly symplectomorphic.

Complex linear symplectic quotients

Complex symplectic quotients

Let $G = K_{\mathbb{C}}$ denote the complexification of K.

The action of K on V extends to an action of G on V.

(Kempf-Ness, 1979) The inclusion of Z into V induces a homeomorphism between the symplectic quotient Z/K and the categorical quotient $V /\!\!/ G := \operatorname{Spec} \mathbb{C}[V]^G$.

The complex moment map is $\mu = \rho \otimes_{\mathbb{R}} \mathbb{C} \colon\ V \oplus V^* \to \mathfrak{g}^*$.

The **complex shell** $N := \mu^{-1}(0)$ is the subscheme of $V \oplus V^*$ associated to (μ).

The **complex linear symplectic quotient** is Spec $\big(\mathbb{C}[V \oplus V^*]^G/(\mu)^G\big)$.

If $\sqrt[\mathbb{R}]{(\rho)} = (\rho)$, the complex symplectic quotient is equal to $N /\!\!/ G$, the affine GIT quotient parameterizing the closed G-orbits in N, and

$$\operatorname{Spec}\big(\mathbb{C}[V \oplus V^*]^G/(\mu)^G\big) = \operatorname{Spec}(\mathbb{R}[Z/K] \otimes_{\mathbb{R}} \mathbb{C}).$$

In general,

$$\mathbb{R}[Z/K] \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{C}(V \oplus V^*)^G / \big(\sqrt[\mathbb{R}]{(\rho)} \otimes_{\mathbb{R}} \mathbb{C}\big)^G.$$

Question: When are two (nice enough) complex linear symplectic quotients isomorphic as complex Poisson varieties?

Stable G-modules

Definition

The G-representation V is **stable** if V contains an open dense subset consisting of closed orbits.

Example

$K = \mathbb{T}^1$, so $G = K_{\mathbb{C}} = \mathbb{C}^\times$.

Let K act on $V = \mathbb{C}$ with weight matrix (1):

$$tz = z \qquad\qquad (t \in K, z \in \mathbb{C}),$$

extends to an action of $\mathbb{C}^\times$.

Two orbits: $\mathbb{C}^\times \subset \mathbb{C}$ and $\{0\}$. Only $\{0\}$ is closed.

V is not stable as a $\mathbb{C}^\times$-module.

There are no nonconstant invariant polynomials: $\mathbb{C}[V]^G = \mathbb{C}$.

The quotient $V/\!\!/G$ is a point.

Stable G-modules

Definition

The G-representation V is **stable** if V contains an open dense subset consisting of closed orbits.

Example

$K = \mathbb{T}^1$, so $G = K_{\mathbb{C}} = \mathbb{C}^\times$.

Let K act on $V = \mathbb{C}$ with weight matrix $(1, -1)$:

$$t(z_1, z_2) = (t^{-1} z_1, t z_2) \qquad (t \in K, (z_1, z_2) \in V),$$

extends to an action of $\mathbb{C}^\times$.

The orbit of $(z_1, z_2) \in \mathbb{C}$ is closed unless $z_1 = 0$ xor $z_2 = 0$.

V is stable as a $\mathbb{C}^\times$-module.

$\mathbb{C}[V]^K = \mathbb{C}[z_1 z_2]$.

The quotient $V /\!\!/ G \simeq \mathbb{C}$.

FPIG and TPIG

Let $\pi\colon V \to V/\!\!/G$ denote the orbit map.

The variety $V/\!\!/G$ is stratified by orbit types of closed orbits.

There is a unique open stratum $(V/\!\!/G)_{\mathrm{pr}}$, the **principal orbit type**.

Stable is equivalent to $V_{\mathrm{pr}} := \pi^{-1}\big((V/\!\!/G)_{\mathrm{pr}}\big)$ consisting of closed orbits.

Definition

(V, G) has **FPIG** if closed orbits in $\pi^{-1}\big((V/\!\!/G)_{\mathrm{pr}}\big)$ have finite isotropy and **TPIG** if they have trivial isotropy.

Example

- For the $K = \mathbb{T}^1$-action on $V = \mathbb{C}$ with weight matrix (1), $tz = z$, V does not have FPIG.

  $V_{\mathrm{pr}} = \mathbb{C}$ and the closed orbit $\{0\}$ has isotropy $\mathbb{T}^1$.
- For the $K = \mathbb{T}^1$-action on $V = \mathbb{C}$ with weight matrix $(1, -1)$, $t(z_1, z_2) = (t^{-1}z_1, tz_2)$, V has TPIG.

  $V_{\mathrm{pr}} = \{(z_1, z_2) \in \mathbb{C}^2 : z_1, z_2 \neq 0\}$ has trivial isotropy. ▪

k-principal G-modules

Definition

(V, G) is k-**principal** if codim $V \smallsetminus V_{\mathrm{pr}} \geq k$.

Example

- For the $K = \mathbb{T}^1$-action on $V = \mathbb{C}$ with weight matrix (1), $tz = z$, V is k-principal for all k.

  $V_{\mathrm{pr}} = V$.
- For the $K = \mathbb{T}^1$-action on $V = \mathbb{C}$ with weight matrix $(1, -1)$, $t(z_1, z_2) = (t^{-1}z_1, tz_2)$, V is 1-principal but not 2-principal.

  $V \smallsetminus V_{\mathrm{pr}}$ contains $\mathbb{C} \times \{0\}$ and $\{0\} \times \mathbb{C}$.

k-modular G-modules

For $r = 0, 1, \ldots, \dim G$, let

$$V_{(r)} = \{x \in V : \dim G_x = r\}.$$

The irreducible components of the $V_{(r)}$ are called **sheets**.

Definition

(V, G) is k-**modular** if $\operatorname{codim} V_{(r)} \geq r + k$ for $r = 1, 2, \ldots, \dim G$.

Example

- For the $K = \mathbb{T}^1$-action on $V = \mathbb{C}$ with weight matrix (1), $tz = z$, V is 0-modular but not 1-modular.

  $V_{(1)} = \{0\}$.
- For the $K = \mathbb{T}^1$-action on $V = \mathbb{C}$ with weight matrix $(1, -1)$, $t(z_1, z_2) = (t^{-1}z_1, tz_2)$, V is 1-modular but not 2-modular.

  $V_{(1)} = \{(0, 0)\}$.

k-large G-modules

Definition

(V, G) is k-**large** if it has FPIG, is k-principal, and is k-modular.

Example

- The $\mathbb{C}^\times$-representation with weights $(1, 1)$ does not have FPIG so is not k-large for any k.
- The $\mathbb{C}^\times$-representation with weights $(1, -1)$ has TPIG, is 1-principal but not 2-principal, and is 1-modular but not 2-modular; hence it is 1-large but not 2-large.

k-large G-modules

"Most" representations are k-large.

(Schwarz, 1995; Herbig-Schwarz-S., 2020)

1. If G is connected and simple, for any k, all but finitely many G-modules with $V^G = \{0\}$ are k-large.
2. If G is connected and semisimple, all but finitely many G-modules with $V^G = \{0\}$ whose irreducible subrepresentations have finite kernels are k-large.

k-large $(\mathbb{C}^\times)^\ell$-modules

Theorem (Herbig-Schwarz, 2013)

Let $G = (\mathbb{C}^\times)^\ell$ and V be a faithful G-module. Then V is stable if and only if it is 1-large.

Theorem (Wehlau, 1992)

If V is a $G = (\mathbb{C}^\times)^\ell$-module, then there is a subtorus G' and stable G'-submodule V' such that $\mathbb{C}[V']^{G'} = \mathbb{C}[V]^G$.

Restricting to the stable sub-G'-module V' does not change the real symplectic quotient (but can change the complex symplectic quotient).

k-large $(\mathbb{C}^\times)^\ell$-modules

Theorem

If V is a 1-modular and faithful $G = (\mathbb{C}^\times)^\ell$-module, then there is a Lagrangian submodule V' of $V \oplus V^*$ such that V' is stable. The complex symplectic quotients of V and V' coincide.

Example

The $\mathbb{C}^\times$-representation V with weight matrix $(1,1)$ is not stable but is 1-modular. The weight matrix of $V \oplus V^*$ is $(1,1,-1,-1)$, so we can replace V by V' with weights $(1,-1)$.

If $G^\circ = (\mathbb{C}^\times)^\ell$ and V is a faithful G-module of dimension n, then V is k-modular if and only if every $\ell \times (n-k)$ submatrix of the weight matrix has rank ℓ.

Hence k-modularity is generic among $(\mathbb{C}^\times)^\ell$-representations of dimension at least $\ell + k$.

Large G-modules have good shells

(Herbig-Schwarz, 2013)

- N is a complete intersection, i.e. the μ_i form a regular sequence, if and only if V is 0-modular.
- If V is 0-modular, then N is reduced and irreducible if and only if V is 1-modular.
- If V is 2-modular, then N is normal.
- If V is 1-large, then the ideal $(\rho) \subset \mathbb{R}[V]$ of the real moment map ρ is a real ideal:
$$(\rho) = \sqrt[\mathbb{R}]{(\rho)}.$$
In particular, if (V, G) is 1-large, then the complex symplectic quotient is
$$N /\!\!/ G = \operatorname{Spec}(\mathbb{R}[Z/K] \otimes_{\mathbb{R}} \mathbb{C}).$$

Isomorphisms of linear symplectic torus quotients

Minimal representations

- K a compact Lie group with $K^\circ = \mathbb{T}^\ell$, $G = K_\mathbb{C}$.
- V a faithful 1-modular G-module.
- $N = \mu^{-1}(0)$ the complex shell and N_{sing} the set of singular points in N.
- A the weight matrix for the K°-action on V.

Theorem

$\operatorname{codim}_N N_{\text{sing}} \geq 3$ *with equality if and only if there is an* r *with* $1 \leq r \leq \ell$ *and* $n - r - 1$ *columns of* A *of rank* $\ell - r$*.*

Definition

The G-module V (or the shell N) is **minimal** if $\operatorname{codim}_N N_{\text{sing}} \geq 4$.

Example

$\begin{pmatrix} -1 & 0 & 1 & 1 \\ 0 & -1 & 1 & 1 \end{pmatrix}$ is not minimal; removing the first 2 columns yields rank 1.

$\begin{pmatrix} -1 & 0 & 1 & 1 \\ 0 & -1 & 1 & 2 \end{pmatrix}$ is minimal; removing any 2 columns does not reduce the rank.

Replacing V with a minimal representation

Every complex symplectic torus quotient is the symplectic quotient of a minimal representation.

Theorem

V a faithful 1-modular G-module as above.
There is a linear subspace $V' \subset V$ such that, if $G' \leq G$ is the stabilizer of V':

(1) *V' is a 1-modular faithful G'-module.*

(2) *There is a G'-equivariant inclusion $N_{V'} \to N$ inducing a Poisson isomorphism $N' /\!\!/ G' \simeq N /\!\!/ G$.*

(3) *N' is minimal.*

If V is a stable G-module, then V' is a stable G'-module. ∎

Replacing V with a minimal representation

Example

Let $G = (\mathbb{C}^\times)^2$ act on $V = \mathbb{C}^4$ with weight matrix

$$\begin{pmatrix} -a & b & 0 & 0 \\ 0 & 0 & -c & d \end{pmatrix}, \qquad a, b, c, d > 0, \quad \gcd(a, b) = \gcd(c, d) = 1.$$

V is not minimal. Removing the first two columns reduces the rank by $r = 1$.

Let

$$V' = \operatorname{span}(\sqrt{b}e_1 + \sqrt{a}e_2, \sqrt{d}e_3 + \sqrt{c}e_4)$$

and

$$G' = \mathbb{Z}/\langle a + b\rangle \times \mathbb{Z}/\langle c + d\rangle \leq G.$$

Then V' is minimal and $N/\!\!/G \simeq N'/\!\!/G' = \mathbb{C}^2/\big(\mathbb{Z}/\langle a + b\rangle \times \mathbb{Z}/\langle c + d\rangle\big)$. ∎

Replacing V with a minimal representation

Example

Let $G = (\mathbb{C}^\times)^2$ act on $V = \mathbb{C}^4$ with weight matrix

$$\begin{pmatrix} 3 & 0 & -4 & 6 \\ 1 & -3 & 0 & 0 \end{pmatrix}.$$

V is not minimal.

Let

$$V' = \operatorname{span}(\sqrt{3}e_1 + e_2, e_3, e_4)$$

and

$$G' = \{(t^4, t^{-3}) : t \in \mathbb{C}^\times\}.$$

Then V' is minimal and $N' /\!\!/ G' \simeq N /\!\!/ G$.

V' is isomorphic to the circle-representation with weights $(9, -16, 24)$.

Classifying complex linear symplectic quotients

Minimal representations classify complex symplectic quotients up to changing the Lagrangian submodule.

Theorem

For $i = 1, 2$, assume:

- K_i *a compact Lie group with* K_i° *a torus,* $G_i = (K_i)_{\mathbb{C}}$*,*
- V_i *a faithful* 1*-modular* G_i*-module,*
- $N_i = \mu^{-1}(0)$ *the complex shell.*

If

$$N_1 /\!\!/ G_1 \simeq N_2 /\!\!/ G_2$$

as affine varieties, then there is a linear isomorphism

$$\Gamma\colon\ V_1 \oplus V_1^* \xrightarrow{\simeq} V_2 \oplus V_2^*$$

inducing isomorphisms $N_1 \simeq N_2$ *and* $G_1 \simeq G_2$*.*

Real linear symplectic quotients

Example (Herbig-Lawler-S., 2020)

- $K = \mathbb{S}^1$
- $V_1 = \mathbb{C}^3$ with weight vector $(-2, 3, 6)$
- $V_2 = \mathbb{C}^3$ with weight vector $(-3, 2, 6)$

As $G = \mathbb{C}^\times$-modules, $V_1 \oplus V_1^* \simeq V_2 \oplus V_2^*$ so the corresponding complex symplectic quotients are isomorphic as Poisson varieties.

If Z_i, $i = 1, 2$, denote the real shells, then there are isomorphisms

$$\mathbb{R}[Z_1]^K \simeq \mathbb{R}[Z_2]^K.$$

These are isomorphisms of the Zariski closures of Z_1/K and Z_2/K as real algebraic varieties.

However, no such isomorphism preserves the inequalities defining Z_1/K and Z_2/K.

The real symplectic quotients are not regularly diffeomorphic. ▪

Replacing V with a minimal representation

Definition

The unitary K-module V is **minimal** if V is minimal as a $G = K_{\mathbb{C}}$-module.

Theorem

- *K a compact Lie group with K° a torus, $G = K_{\mathbb{C}}$,*
- *V a faithful unitary K-module that is stable as a G-module,*
- *$Z = \rho^{-1}(0)$ the real shell and $N = \mu^{-1}(0)$ the complex shell.*

There is a complex linear subspace $V' \subset V$ such that, if $K' \leq K$ is the stabilizer of V' and Z' is the real shell of V':

(1) *V' is a stable faithful G'-module.*

(2) *There is a K'-equivariant inclusion $Z' \to Z$ inducing a graded regular symplectomorphism $Z'/K' \simeq Z/K$.*

(3) *V' is minimal.*

Classifying real linear symplectic quotients

Theorem

- *For $i = 1, 2$, K_i a compact Lie group with K_i° a torus, $G_i = (K_i)_\mathbb{C}$,*
- *V_i a faithful unitary K_i-module that is stable as a G_i-module,*
- *$Z_i = \rho^{-1}(0)$ the real shells and $N_i = \mu^{-1}(0)$ the complex shell.*

The following are equivalent:

(1) *There is a regular isomorphism $\varphi\colon Z_1/K_1 \to Z_2/K_2$.*

(2) *There is a real isomorphism $\Phi\colon N_1/\!\!/G_1 \to N_2/\!\!/G_2$*

(3) *There is a real linear isomorphism*

$$\Gamma\colon V_1 \oplus V_1^* \xrightarrow{\simeq} V_2 \oplus V_2^*$$

inducing (necessarily real) isomorphisms $N_1 \simeq N_2$ and $G_1 \simeq G_2$.

(4) *There is a linear isomorphism*

$$\Gamma'\colon V_1 \to V_2$$

inducing isomorphisms $Z_1 \simeq Z_2$ and $K_1 \simeq K_2$.

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Classifying real linear symplectic quotients

Corollary

For $i = 1, 2$, assume:

- K_i *a compact Lie group with* K_i° *a torus,* $G_i = (K_i)_{\mathbb{C}}$,
- V_i *a faithful unitary* K_i*-module that is stable as a* G_i*-module,*
- $Z_i = \rho^{-1}(0)$ *the real shells and* $N_i = \mu^{-1}(0)$ *the complex shell.*

If there is a regular isomorphism $Z_1/K_1 \to Z_2/K_2$*, then there is a graded regular symplectomorphism* $Z_1/K_1 \to Z_2/K_2$.

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Thank you!