

Symmetries of bundle gerbes and exact Courant algebroids

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Outline

- 1 motivation: prequantization of symplectic manifolds
- 2 Lie 2-algebras and 2-plectic manifolds
- 3 bundle gerbes and their infinitesimal symmetries

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There is a Lie algebra isomorphism $C^\infty(M) \rightarrow Q(P, \gamma)$, given by

$$f \mapsto \text{Hor}_\gamma(X_f) + (\pi^*f) \frac{\partial}{\partial \theta}$$

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Here: $Q(P, \gamma) = \{Y \in \mathfrak{X}(P) \mid L_Y \gamma = 0\}$, sometimes called *infinitesimal quantomorphisms*.

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and when ω is symplectic, a commutative diagram

$$\begin{array}{ccc} C^\infty(M) & \longrightarrow & A(M, \omega) \\ \downarrow & & \downarrow \\ Q(P, \gamma) & \longrightarrow & \mathfrak{X}(P)^{S^1} \end{array}$$

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These form (part of) a Lie 2-algebra (Rogers).

Lie 2-algebras and 2-plectic manifolds

Definition

A Lie 2-algebra is a 2-term chain complex $V_1 \xrightarrow{d} V_0$ equipped with a skew-symmetric chain map $[-, -] : V_\bullet \otimes V_\bullet \rightarrow V_\bullet$.

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satisfying a coherence condition on $V_\bullet^{\otimes 4}$.

Example: Poisson Lie 2-algebra of observables (Rogers)

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Note: Can drop non-degeneracy condition, and use pairs instead:

$$C^\infty(M) \rightarrow \{(X, \beta) \in \Gamma(TM \oplus T^*M) \mid \iota_X \omega = -d\beta\}$$

Example: Atiyah Lie 2-algebra (Fiorenza-Rogers-Schreiber)

Let $\omega \in \Omega^3(M)$ be closed. Let $A_\bullet(M, \omega)$ be

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$$[(u, \alpha), (v, \beta)] = ([u, v], L_u\beta - L_v\alpha - \frac{1}{2}d(\iota_u\beta - \iota_v\alpha) - \iota_v\iota_u\omega)$$

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The Jacobiator is

$$J(u_1, \alpha_1; u_2, \alpha_2; u_3, \alpha_3) = -\frac{1}{6} \left(\langle [(u_1, \alpha_1), (u_2, \alpha_2)], (u_3, \alpha_3) \rangle^+ + \text{c. p.} \right).$$

where $\langle -, - \rangle^+$ is the standard symmetric pairing on $TM \oplus T^*M$.

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A multiplicative vector field is a functor $X = (X_0, X_1) : G \rightarrow TG$ such that $\pi_G \circ X = \text{id}_G$.

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Theorem (Berwick-Evans – Lerman, Ortiz – Waldron)

Multiplicative vector fields $\mathbb{X}(G_1 \rightrightarrows G_0)$ form part of a strict Lie 2-algebra,

$$\Gamma(A_G) \rightarrow \mathbb{X}(G), \quad a \mapsto (dt(a), \vec{a} + \overleftarrow{a}).$$

The bracket in degree 0 is Lie bracket, while in mixed degree it is

$$[(X_0, X_1), a] = [X_1, \vec{a}]|_{G_0}.$$

bundle gerbes / prequantization

Example: bundle gerbes from Čech data (Hitchin)

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Groupoid multiplication: $(x_{ij}, \zeta)(x_{jk}, \eta) = (x_{ik}, g_{ijk}(x)\zeta\eta)$, where $g_{ijk} : U_i \cap U_j \cap U_k \rightarrow S^1$.

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Associativity $\Leftrightarrow g$ is a 2-cocycle, hence $[g] \in H^2(M; \underline{S^1}) \cong H^3(M; \mathbb{Z})$.

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The diagram shows a commutative square. At the top right is P , with a downward arrow to $X \times_M X$. On the left is X , with a downward arrow labeled π to M . Two horizontal arrows point from $X \times_M X$ to X , one above the other.

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- $\pi : X \rightarrow M$, surjective submersion,
- an S^1 -bundle $P \rightarrow X \times_M X$,
- a Lie groupoid structure on P covering the one on $X \times_M X$,
- S^1 -action on P is compatible: $(p \cdot \zeta)q = p(q \cdot \zeta) = (pq) \cdot \zeta$ for all $\zeta \in S^1$ and $p, q \in P$ that make sense.

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Get $\mathcal{L}\mathcal{A}$ -groupoid, whose Lie 2-algebra of multiplicative sections is $\mathbb{X}(P)$.

connection data on bundle gerbes

Let $P \rightarrow X \times_M X \rightrightarrows X \xrightarrow{\pi} M$ be a bundle gerbe over M .

Definition

A connection is a *multiplicative* connection form $\gamma \in \Omega^1(P)$,
i.e., $\text{mult}_P^* \gamma = \text{pr}_1^* \gamma + \text{pr}_2^* \gamma$.

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Given (γ, B) , $\exists! \omega \in \Omega^3(M)$ such that $\pi^* \omega = dB$. Call ω the 3-curvature of the connection data (γ, B) .

Can be helpful to view connection data (γ, B) in terms of the Bott-Shulman-Stasheff complex associated to $P \rightrightarrows X$.

$$\begin{array}{ccccccc}
 & \vdots & & \vdots & & \vdots & \\
 & \uparrow & & \uparrow & & \uparrow & \\
 \Omega^2(X) & \longrightarrow & \Omega^2(P) & \longrightarrow & \Omega^2(P \times_X P) & \longrightarrow & \dots \\
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 \Omega^2(X) & \longrightarrow & \Omega^2(P) & \longrightarrow & \Omega^2(P \times_X P) & \longrightarrow & \dots \\
 & \uparrow & & \uparrow & & \uparrow & \\
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 & \uparrow & & \uparrow & & \uparrow & \\
 \Omega^0(X) & \longrightarrow & \Omega^0(P) & \longrightarrow & \Omega^0(P \times_X P) & \longrightarrow & \dots \\
 & & & \xrightarrow{\partial} & & & \\
 \end{array}$$

$\begin{array}{ccc} & \epsilon & \\ \gamma \vdash & \longrightarrow & 0 \\ & \epsilon & \end{array}$

with $\partial = \sum \pm \partial_i^*$.

Can be helpful to view connection data (γ, B) in terms of the Bott-Shulman-Stasheff complex associated to $P \rightrightarrows X$.

$$\begin{array}{ccccccc}
 & \vdots & & \vdots & & \vdots & \\
 & \uparrow & & \uparrow & & \uparrow & \\
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 & \uparrow & & \uparrow & & \uparrow & \\
 \epsilon & \longleftarrow & & \epsilon & & & \\
 B & \longleftarrow & \bullet & & & & \\
 \uparrow & & \uparrow & & & & \\
 \pm d & & \gamma & & & & \\
 & & \longleftarrow & & & & \\
 & & 0 & & \epsilon & & \\
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Note: U descends to a vector field u on M , and $L_u \omega = 0$.

(Motivation) Say (U, V) is multiplicative vector field, and that $L_U \omega = 0$.

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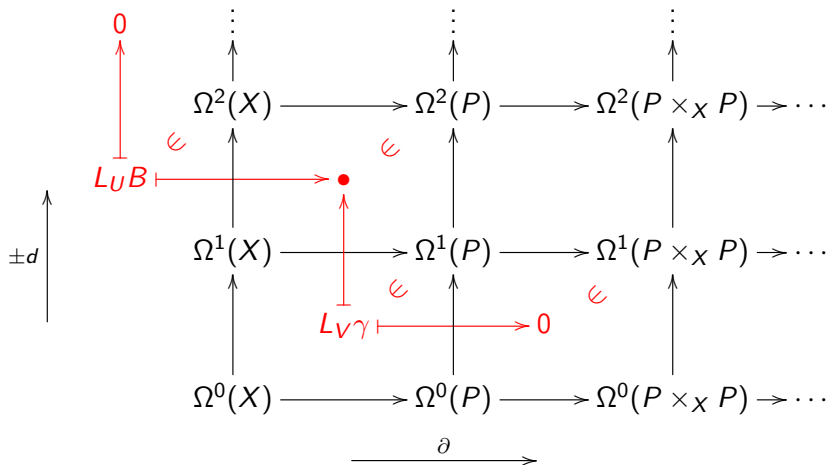
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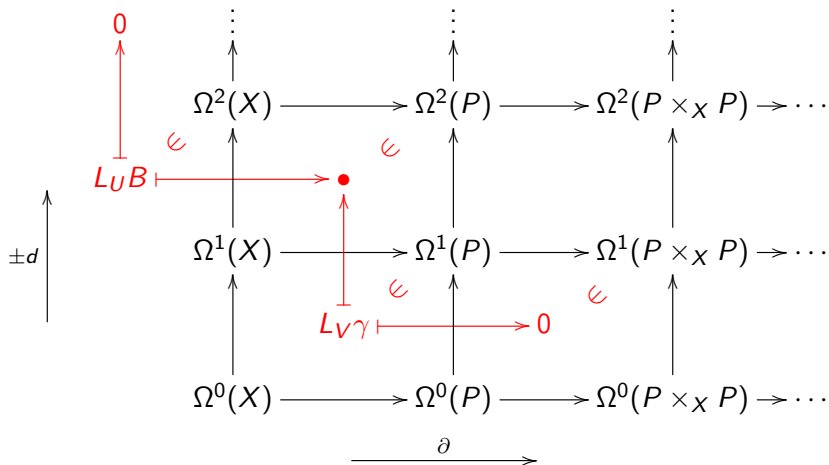
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i.e., $D(L_U B + L_V \gamma) = 0$, where $D = \partial \pm d$ is total differential.

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Definition/Proposition [K-Vaughan]

Connection preserving multiplicative vector fields on P form a Lie 2-algebra,

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Denote this Lie 2-algebra by $Q_\bullet(P; \gamma, B)$.

Example: the trivial gerbe with curving $B \in \Omega^2(M)$

Let $X = M \xrightarrow{\text{id}} M$ and $P = M \times S^1$. Let $B \in \Omega^2(M)$, viewed as the curving of the trivial connection.

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$$C^\infty(M) \rightarrow \{(u, A) \in \Gamma(TM \oplus T^*M) \mid L_u B = dA\}, \quad f \mapsto (0, -df)$$

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Note: if $(u, A) \in Q_0$, then $d(\iota_u B - A) = d\iota_u B - L_u B = -\iota_u dB$. Hence $(u, A) \mapsto (u, \iota_u B - A)$ defines an isomorphism of Lie 2-algebras $Q_\bullet \rightarrow L_\bullet(M, dB)$.

About morphisms of Lie 2-algebras

A morphism of Lie 2-algebras $F : V_{\bullet} \rightarrow W_{\bullet}$ consists of a chain map $F_{\bullet} : V_{\bullet} \rightarrow W_{\bullet}$ together with a chain homotopy $\varphi : V_{\bullet} \otimes V_{\bullet} \rightarrow W_{\bullet}$, from the chain map

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satisfying

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(There are also 2-morphisms, so Lie 2-algebras form a 2-category.)

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$$\varphi_B : \mathbb{X}(X \times_M X \rightrightarrows X)^{\otimes 2} \rightarrow \Gamma(A_P), \quad (U, V) \mapsto (\iota_V \iota_U B) \frac{\partial}{\partial \theta} \Big|_X,$$

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Proposition

The map F defined above is a Lie 2-algebra morphism if and only if the 3-curvature ω vanishes.

Generalized morphisms of Lie 2-algebras

We localize the 2-category of Lie 2-algebras with respect to quasi-isomorphisms, using Noohi's 'butterflies.'

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A butterfly $E : V_{\bullet} \dashrightarrow W_{\bullet}$ of Lie 2-algebras is a vector space E equipped with a skew-symmetric bracket $[-, -]$, together with a commutative diagram

$$\begin{array}{ccccc} V_1 & & & & W_1 \\ & \searrow \kappa & & \swarrow \lambda & \\ & & E & & \\ & \swarrow \sigma & & \searrow \rho & \\ V_0 & & & & W_0 \end{array}$$

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The diagram shows a central node E with four arrows pointing towards it: κ from V_1 , λ from W_1 , σ from V_0 , and ρ from W_0 . Additionally, there are vertical arrows: $V_1 \rightarrow V_0$ and $W_1 \rightarrow W_0$.

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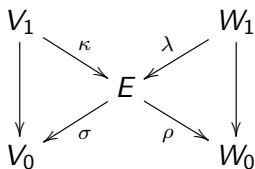
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- for every $a, b, c \in E$,

$$\lambda J(\rho(a), \rho(b), \rho(c)) + \kappa J(\sigma(a), \sigma(b), \sigma(c)) = [a, [b, c]] + \text{cyc. perm.}$$

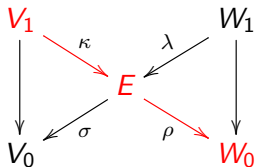
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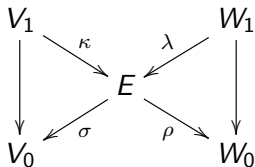
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There are 2-morphisms of butterflies, and we can compose butterflies, obtaining a 2-category.

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Theorem (K-Vaughan, Djounvouna-K)

There are invertible butterflies E, F , and G of Lie 2-algebras that fit in a 2-commutative diagram,

$$\begin{array}{ccccc}
 L_{\bullet}(M, \omega) & \xrightarrow{R} & C_{\bullet}(M, \omega) & \xrightarrow{S} & A_{\bullet}(M, \omega) \\
 \downarrow E & & \downarrow F & & \downarrow G \\
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- cf. similar results of Fiorenza-Rogers-Schreiber, and Sevestre-Wurzbacher

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But $\alpha - \iota_U B$ is not necessarily basic:

$$\begin{aligned} \partial(\alpha - \iota_U B) &= L_V \gamma - \iota_V \partial B \\ &= L_V \gamma - \iota_V d\gamma \\ &= d\iota_V \gamma \end{aligned}$$

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We know U descends to a vector field u on M , and

$$\begin{aligned} \pi^* \iota_u \omega &= \iota_U dB \\ &= L_U B - d\iota_U B \\ &= d(\alpha - \iota_U B) \end{aligned}$$

But $\alpha - \iota_U B$ is not necessarily basic:

$$\begin{aligned} \partial(\alpha - \iota_U B) &= L_V \gamma - \iota_V \partial B \\ &= L_V \gamma - \iota_V d\gamma \\ &= d\iota_V \gamma \end{aligned}$$

γ multiplicative $\Rightarrow \partial\iota_V \gamma = 0$;

Idea of proof:

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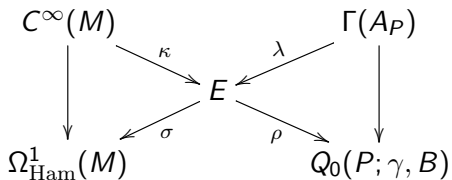
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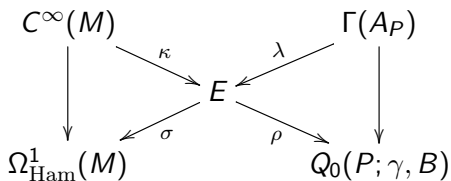
$$\begin{aligned} \partial(\alpha - \iota_U B) &= L_V \gamma - \iota_V \partial B \\ &= L_V \gamma - \iota_V d\gamma \\ &= d\iota_V \gamma \end{aligned}$$

γ multiplicative $\Rightarrow \partial\iota_V \gamma = 0$; so there is a function $g \in C^\infty(X)$ with $\partial g = \iota_V \gamma$, and then $\alpha - \iota_U B - dg$ is basic and descends to a Hamiltonian 1-form on M .

The butterfly E holds all the choices of g .

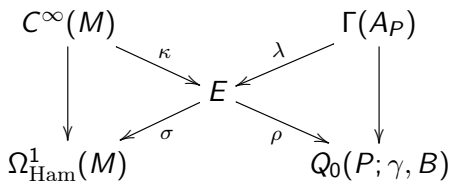


The butterfly E holds all the choices of g .



Set $E = \{(U, V, \alpha, g) \in Q_0 \times C^\infty(X) \mid \partial g = \iota_V \gamma\}$.

The butterfly E holds all the choices of g .



Set $E = \{(U, V, \alpha, g) \in Q_0 \times C^\infty(X) \mid \partial g = \iota_V \gamma\}$.

Then define bracket on E , and check this works...

F, G obtained similarly.

Thank you.

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