Symmetries of bundle gerbes and exact Courant algebroids

Derek Krepski

University of Manitoba

~~AMS Session on~~ Poisson Geometry, Diffeology and Singular Spaces
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Outline

1. motivation: prequantization of symplectic manifolds

2. Lie 2-algebras and 2-plectic manifolds

3. bundle gerbes and their infinitesimal symmetries

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There is a Lie algebra isomorphism $C^\infty(M) \longrightarrow Q(P, \gamma)$, *given by*

$$f \longmapsto \mathrm{Hor}_\gamma(X_f) + (\pi^* f)\frac{\partial}{\partial \theta}$$

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Here: $Q(P, \gamma) = \{Y \in \mathfrak{X}(P) \,|\, L_Y\gamma = 0\}$, sometimes called *infinitesimal quantomorphisms*.

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and when ω is symplectic, a commutative diagram

$$\begin{tikzcd}
C^\infty(M) \arrow[r] \arrow[d] & A(M,\omega) \arrow[d] \\
Q(P,\gamma) \arrow[r] & \mathfrak{X}(P)^{S^1}
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These form (part of) a Lie 2-algebra (Rogers).

Lie 2-algebras and 2-plectic manifolds

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satisfying a coherence condition on ${V_\bullet}^{\otimes 4}$.

Example: Poisson Lie 2-algebra of observables (Rogers)

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where $\iota_{X_i}\omega = -d\beta_i$.

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Note: Can drop non-degeneracy condition, and use pairs instead:

$$C^\infty(M) \to \{(X, \beta) \in \Gamma(TM \oplus T^*M) \,\big|\, \iota_X\omega = -d\beta\}$$

Example: Atiyah Lie 2-algebra (Fiorenza-Rogers-Schreiber)

Let $\omega \in \Omega^3(M)$ be closed. Let $A_\bullet(M, \omega)$ be

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The Jacobiator is

$$J(u_1, \alpha_1; u_2, \alpha_2; u_3, \alpha_3) = -\frac{1}{6}\left(\langle[(u_1, \alpha_1), (u_2, \alpha_2)], (u_3, \alpha_3)\rangle^+ + \text{c. p.}\right)\ .$$

where $\langle -, - \rangle^+$ is the standard symmetric pairing on $TM \oplus T^*M$.

multiplicative vector fields on Lie groupoids

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A multiplicative vector field is a functor $X = (X_0, X_1) : \mathsf{G} \to T\mathsf{G}$ such that $\pi_{\mathsf{G}} \circ X = \mathrm{id}_{\mathsf{G}}$.

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Theorem (Berwick-Evans – Lerman, Ortiz – Waldron)

Multiplicative vector fields $\mathbb{X}(G_1 \rightrightarrows G_0)$ form part of a strict Lie 2-algebra,

$$\Gamma(A_G) \to \mathbb{X}(G), \quad a \mapsto (dt(a), \overrightarrow{a} + \overleftarrow{a}).$$

The bracket in degree 0 is Lie bracket, while in mixed degree it is

$$[(X_0, X_1), a] = [X_1, \overrightarrow{a}]\big|_{G_0}.$$

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Example: bundle gerbes from Čech data (Hitchin)

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Groupoid multiplication: $(x_{ij}, \zeta)(x_{jk}, \eta) = (x_{ik}, g_{ijk}(x)\zeta\eta)$, where $g_{ijk} : U_i \cap U_j \cap U_k \to S^1$.

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Associativity $\Leftrightarrow$ g is a 2-cocycle, hence $[g] \in H^2(M; \underline{S^1}) \cong H^3(M; \mathbb{Z})$.

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- a Lie groupoid structure on P covering the one on $X \times_M X$,
- S^1-action on P is compatible: $(p \cdot \zeta)q = p(q \cdot \zeta) = (pq) \cdot \zeta$ for all $\zeta \in S^1$ and $p, q \in P$ that make sense.

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Get $\mathcal{LA}$-groupoid, whose Lie 2-algebra of multiplicative sections is $\mathbb{X}(P)$.

connection data on bundle gerbes

Let $P \to X \times_M X \rightrightarrows X \xrightarrow{\pi} M$ be a bundle gerbe over M.

Definition

A connection is a *multiplicative* connection form $\gamma \in \Omega^1(P)$, i.e., $\mathrm{mult}_P^*\gamma = \mathrm{pr}_1^*\gamma + \mathrm{pr}_2^*\gamma$.

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A curving for γ is a 2-form $B \in \Omega^2(X)$ such that $s^*B - t^*B = d\gamma$.

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A curving for γ is a 2-form $B \in \Omega^2(X)$ such that $s^* B - t^* B = d\gamma$.
Given (γ, B), $\exists! \, \omega \in \Omega^3(M)$ such that $\pi^* \omega = dB$. Call ω the 3-curvature of the connection data (γ, B).

Can be helpful to view connection data (γ, B) in terms of the Bott-Shulman-Stasheff complex associated to $P \rightrightarrows X$.

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
 & \Omega^2(X) \arrow[r] \arrow[u] & \Omega^2(P) \arrow[r] \arrow[u] & \Omega^2(P \times_X P) \arrow[r] \arrow[u] & \cdots \\
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with $\partial = \sum \pm \partial_i^*$.

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with $\partial = \sum \pm \partial_i^*$.

Can be helpful to view connection data (γ, B) in terms of the Bott-Shulman-Stasheff complex associated to $P \rightrightarrows X$.

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A multiplicative vector field (U, V) on $P \rightrightarrows X$ (weakly) preserves (γ, B) if $\exists\, \alpha \in \Omega^1(X)$ with

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Note: U descends to a vector field u on M, and $L_u \omega = 0$.

(Motivation) Say (U, V) is multiplicative vector field, and that $L_u\omega = 0$.

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i.e., $D(L_U B + L_V\gamma) = 0$, where $D = \partial \pm d$ is total differential.

$\Rightarrow$ (U, V) weakly preserves (B, γ) whenever $L_U B + L_V\gamma$ is exact.

Let $P \to X \times_M X \rightrightarrows X \xrightarrow{\pi} M$ be a bundle gerbe over M, with connection data (γ, B).

Definition/Proposition [K-Vaughan]

Connection preserving multiplicative vector fields on P form a Lie 2-algebra,

$$\Gamma(A_P) \longrightarrow \{(U, V, \alpha) \in \mathbb{X}(P) \times \Omega^1(X) \,\big|\, L_U B = d\alpha, L_V \gamma = \partial\alpha\},$$

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Denote this Lie 2-algebra by $Q_\bullet(P; \gamma, B)$.

Example: the trivial gerbe with curving $B \in \Omega^2(M)$

Let $X = M \xrightarrow{\mathrm{id}} M$ and $P = M \times S^1$. Let $B \in \Omega^2(M)$, viewed as the curving of the trivial connection.

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$$C^\infty(M) \to \{(u, A) \in \Gamma(TM \oplus T^*M) \mid L_u B = dA\}, \quad f \mapsto (0, -df)$$

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<u>Note</u>: if $(u, A) \in Q_0$, then $d(\iota_u B - A) = d\iota_u B - L_u B = -\iota_u dB$. Hence $(u, A) \mapsto (u, \iota_u B - A)$ defines an isomorphism of Lie 2-algebras $Q_\bullet \to L_\bullet(M, dB)$.

About morphisms of Lie 2-algebras

A morphism of Lie 2-algebras $\mathsf{F} : V_\bullet \to W_\bullet$ consists of a chain map $\mathsf{F}_\bullet : V_\bullet \to W_\bullet$ together with a chain homotopy $\varphi : V_\bullet \otimes V_\bullet \to W_\bullet$, from the chain map

$$V_\bullet \otimes V_\bullet \to W_\bullet \quad x \otimes y \mapsto \mathsf{F}([x, y])$$

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(There are also 2-morphisms, so Lie 2-algebras form a 2-category.)

Example: Horizontal lift and 3-curvature

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and the curving B induces a chain homotopy,

$$\varphi_B : \mathbb{X}(X \times_M X \rightrightarrows X)^{\otimes 2} \to \Gamma(A_P), \quad (U, V) \mapsto (\iota_V \iota_U B) \frac{\partial}{\partial \theta}\Big|_X,$$

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measuring failure to preserve brackets. But φ_B doesn't satisfy the compatibility condition for F to be a Lie 2-algebra morphism, unless:

Proposition

The map F defined above is a Lie 2-algebra morphism if and only if the 3-curvature ω vanishes.

Generalized morphisms of Lie 2-algebras

We localize the 2-category of Lie 2-algebras with respect to quasi-isomorphisms, using Noohi's 'butterflies.'

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A butterfly $E : V_\bullet \dashrightarrow W_\bullet$ of Lie 2-algebras is a vector space E equipped with a skew-symmetric bracket $[-,-]$, together with a commutative diagram

$$\begin{tikzcd}
V_1 \arrow[dd] \arrow[dr, "\kappa"] & & W_1 \arrow[dd] \arrow[dl, "\lambda"'] \\
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- ρ, σ, κ, λ compatible with brackets;
- for every $a, b, c \in E$,

$$\lambda J(\rho(a), \rho(b), \rho(c)) + \kappa J(\sigma(a), \sigma(b), \sigma(c)) = [a, [b, c]] + \text{cyc. perm.}$$

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Let E be a butterfly.

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E is invertible if the other diagonal sequence $0 \to V_1 \to E \to W_0 \to 0$ is short exact. The inverse is obtained by switching the wings.

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There are 2-morphisms of butterflies, and we can compose butterflies, obtaining a 2-category.

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Theorem (K-Vaughan, Djounvouna-K)

There are invertible butterflies E, F, and G of Lie 2-algebras that fit in a 2-commutative diagram,

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- *cf*. similar results of Fiorenza-Rogers-Schreiber, and Sevestre-Wurzbacher

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Suppose we seek a map $Q_0(P;\gamma,B)\to L_0(M,\omega)$. Take (U,V,α) with (U,V) multiplicative on $P\rightrightarrows X$ and $\alpha\in\Omega^1(X)$ satisfying

$$L_UB=d\alpha,\quad L_V\gamma=\partial\alpha.$$

We know U descends to a vector field u on M, and

$$\begin{aligned}\pi^*\iota_u\omega&=\iota_UdB\\&=L_UB-d\iota_UB\\&=d(\alpha-\iota_UB)\end{aligned}$$

But $\alpha-\iota_UB$ is not necessarily basic:

$$\begin{aligned}\partial(\alpha-\iota_UB)&=L_V\gamma-\iota_V\partial B\\&=L_V\gamma-\iota_Vd\gamma\\&=d\iota_V\gamma\end{aligned}$$

γ multiplicative $\Rightarrow\partial\iota_V\gamma=0$; so there is a function $g\in C^\infty(X)$ with $\partial g=\iota_V\gamma$, and then $\alpha-\iota_UB-dg$ is basic and descends to a Hamiltonian 1-form on M.

The butterfly E holds all the choices of g.

$$\begin{tikzcd}
C^\infty(M) \arrow[dd] \arrow[dr, "\kappa"] & & \Gamma(A_P) \arrow[dd] \arrow[dl, "\lambda"'] \\
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Then define bracket on E, and check this works...

F, G obtained similarly.

Thank you.

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