Proof of the Ginzburg–Kazhdan Conjecture

K – a compact Lie group

$K \curvearrowright M$ – Hamiltonian K-action

In 2001, Guillemin–Jeffrey–Sjamaar introduced a procedure called symplectic implosion $K \curvearrowright M \rightsquigarrow M_{impl} \curvearrowleft U(1)^n$

Thm (GJS '01) (1) $\overline{G/N} = (T^*K)_{impl}$

(2) $M_{impl} \cong (M \times \overline{G/N})/\!/\!/K$

$G = K_{\mathbb{C}}$, N max'l unipotent subgroup

Ex $G \in \{\underbrace{SL_n, GL_n}, SO_n, Sp_n, \dots\}$

N – strictly upper triangular matrices

Ex $SL_2/N = SL_2/\begin{pmatrix}1 & * \\ & 1\end{pmatrix} \cong \mathbb{C}^2 \setminus 0$

$\mathcal{O}(\mathbb{C}^2 \setminus 0) \cong \mathbb{C}[x,y]$.

$SL_2/N \subseteq \overline{SL_2/N} = \mathrm{Spec}(\mathbb{C}[x,y]) = \mathbb{C}^2$

Ex $SL_3/N \subseteq \overline{SL_3/N} = \left\{ \begin{matrix} (a,b,c,x,y,z): \\ ax+by+cz=0 \end{matrix} \right\}$

Prop $\overline{G/N} = \bigcup_{\Theta} G/[P_\Theta, P_\Theta]$

Ex $\overline{SL_2/N} = SL_2/N \cup SL_2/[SL_2, SL_2]$

$\overline{SL_3/N} = SL_3/N \cup (\mathbb{C}^3 \setminus 0) \cup (\mathbb{C}^3 \setminus 0) \cup \{0\}$

$G \curvearrowright \overline{G/N} \curvearrowleft \overset{=(T^*K)_{impl}}{N \backslash B} \cong (\mathbb{C}^*)^n$

Rmk Sometimes, $\overline{G/N}$ can be viewed as a functor $\{\text{Hamiltonian } K\text{-spaces}\} \to \{\text{Hamiltonian } U(1)^n\text{-spaces}\}$

Def The universal symplectic implosion of a compact Lie group K is the space $\overline{G/N}$.

Def The universal hyperkähler implosion is $\overline{T^*(G/N)}$.

Thm (Dancer–Kirwan–Swann '13)

$\overline{T^*(SL_n/N)}$ admits a description as a quiver variety, in particular, $\overline{T^*(SL_n/N)}$ is stratified by finitely many holomorphic symplectic submanifolds.

Ex $\overline{T^*(SL_2/N)} \cong \mathbb{C}^4$

Thm (Jia '21) $\overline{T^*(SL_3/N)}$ is isomorphic to the minimal (nonzero) nilpotent orbit closure of $\mathfrak{so}_8$.

Thm (Ginzburg–Riche '14)

$\mathcal{O}(T^*(G/N))$ is finitely generated as a $\mathbb{C}$-algebra.

Thm (Gelfand–Graev, Ginzburg–Riche, Ginzburg–Kazhdan)

$\overline{T^*(G/N)} \curvearrowleft T \rtimes W$ where $W = N_G(T)/T$

Ex $G = SL_n$ or $G = GL_n$, $N_G(T)/T \cong S_n$

Ex $\overline{T^*(SL_2/N)} = T^*(\mathbb{C}^2) = \mathbb{C}^2 \times \mathbb{C}^2$ and this an action of $\mathbb{Z}/2\mathbb{Z}$ which is, up to sign, given by swapping the two factor.

Conj (Ginzburg–Kazhdan '18)

$\overline{T^*(G/N)}$ has symplectic singularities

$T^*G_{hkerimpl} = \overline{T^*(G/N)}$

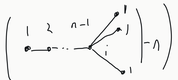
$T^*K_{impl} \cong \overline{G/N}$

Thm (Jia '21 for $G = SL_n$, G. '23)

$\overline{T^*(G/N)}$ has symplectic singularities.

Cor (G. '23) $\overline{T^*(G/N)}$ is $\mathbb{Q}$-factorial terminal.

Thm (G.W.)

$\overline{T^*(SL_n/N)} \cong \mathcal{M}_C$ 

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