Poisson Sigma Models, the Symplectic Category and 2-Segal Sets

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Special Session on Poisson geometry, Diffeology and Singular Spaces
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Goals

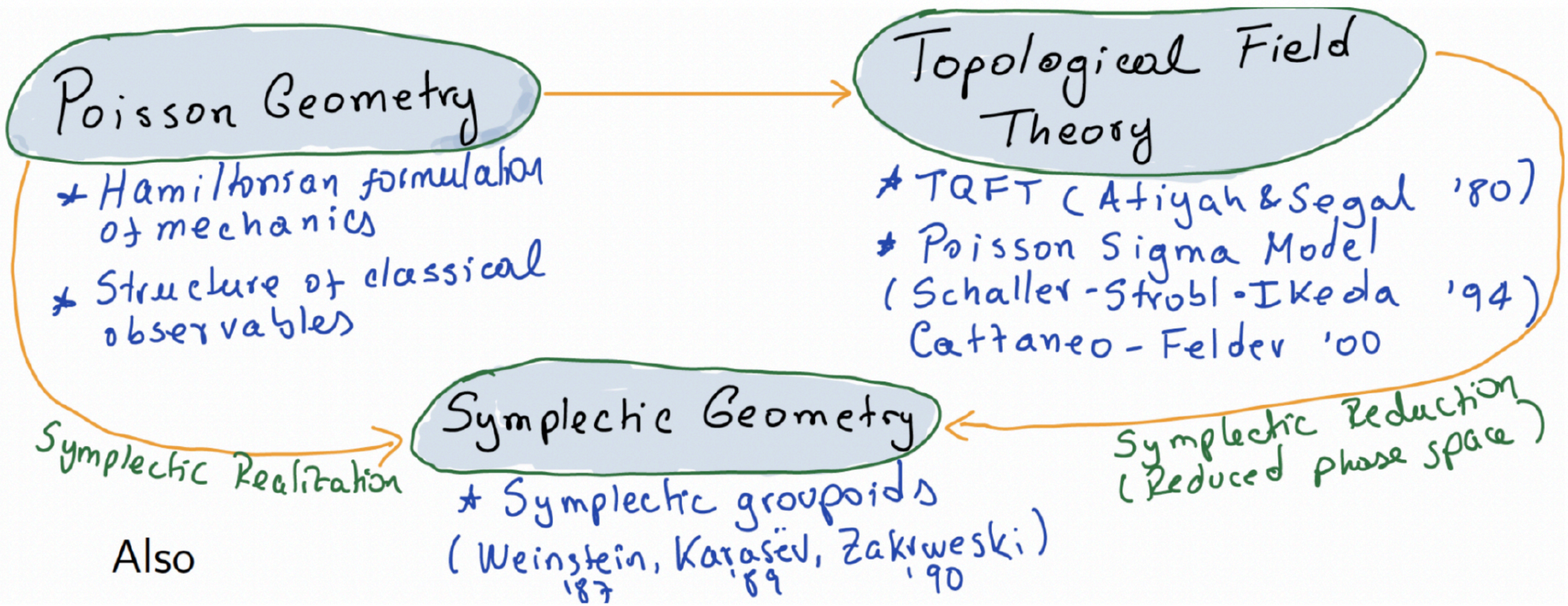
Based on joint work with:

- Rajan Mehta and Molly Keller (*Rev. in Math. Phys (34) 10 (2022)*)
- Mehta, Adele Long and Sophia Marx (*Contemp. Math, AMS (2024)*)
- Mehta and Walker Stern (*Journal of Geometry and Physics* (2024))

Objectives of the Talk

1. Frobenius algebras & Poisson Sigma Model: 2D TQFT and symplectic geometry
2. A toy model of the Wehrheim-Woodward construction
3. The 2-Segal picture: Frobenius and commutative pseudomonoids in $\mathbf{Span}_2$ as paracyclic and Γ-structures on 2-Segal sets.

Motivation



Also

1. Correspondence between 2D TQFT and commutative Frobenius algebras
2. An intermediate step in quantization:

The Poisson sigma model

The ingredients...

- (X,Π) Poisson manifold, $(\Sigma,\partial\Sigma)$ surface (poss. with boundary)
- Space of Fields:
  - Bulk: $\mathcal{F}_\Sigma$: $\mathrm{Map}(T\Sigma, T^*X) \ni (x,\eta)$
  - Boun $\mathcal{F}_{\partial\Sigma}$: $\mathrm{Map}(T\partial\Sigma, T^*X) \cong T^*PX$
- Action Functional: $S_{PSM}(x,\eta) = \int_\Sigma \eta \wedge dx + \frac{1}{2}\Pi^{ij}(x)\eta_i \wedge \eta_j$

Theorem (Cattaneo-Felder, 00) The Reduced phase space of the PSM (when Σ = disk), if smooth, is the source-simply connected symplectic groupoid $(G,\omega) \rightrightarrows (X,\pi)$ that integrates (X,π).

The symplectic category

Definition

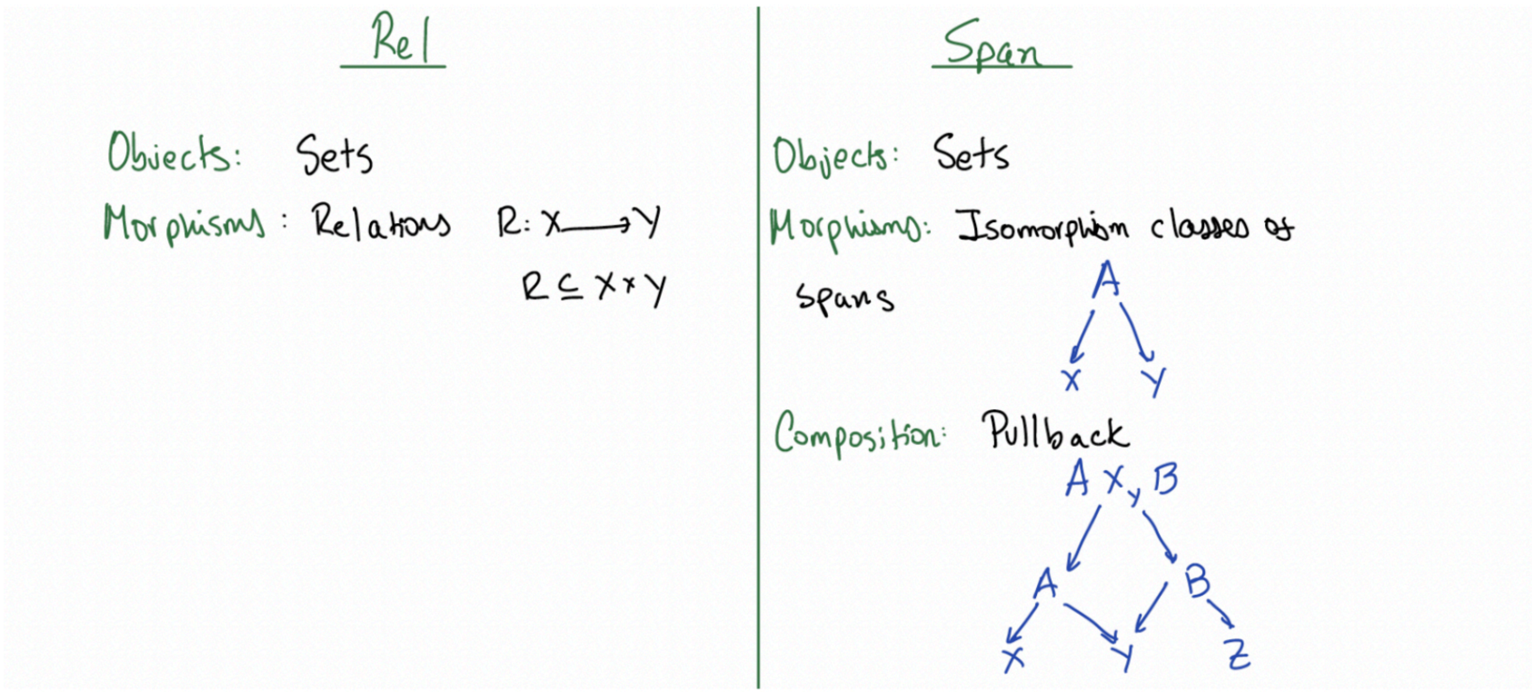
- Objects: Symplectic Manifolds (M, ω)
- Morphisms: Lagrangian Relations / Correspondences
  $L : M \longrightarrow N \quad (L \subseteq M \times \overline{N})$

- Issue: Composition only partially defined (strong transv)
- Possible solution: Wehrheim-Woodward '07

Wehrheim-Woodward's Construction '07

- Objects: Symplectic Manifolds
- Morphisms: (Formal) sequences of Lagrangian relations / strongly transversal compositions

What happens in Set?



Theorem (Li-Bland, Weinstein '14)

$$WW(Rel) = Span$$

- Idea: **Span** is a good set-theoretic model for **Symp**.
- Question: Can we study TQFTs with values in Span?

Frobenius Objects in $\mathcal{C}$

Let $\mathcal{C}$ be a monoidal category.

Definition

A Frobenius object in $\mathcal{C}$ is an object $X \in Ob(\mathcal{C})$ and the following morphisms:

- $\eta : 1 \longrightarrow X$ (unit)
- $\mu : X \otimes X \longrightarrow X$ (multiplication)
- $\varepsilon : X \longrightarrow 1$ (counit)

s.t

(1)

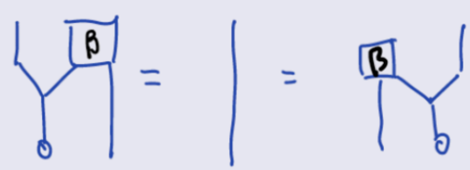
(Unitality)

(2)

(Associativity)

(3) $\exists \beta : 1 \longrightarrow X \otimes X$ s.t
(trace)

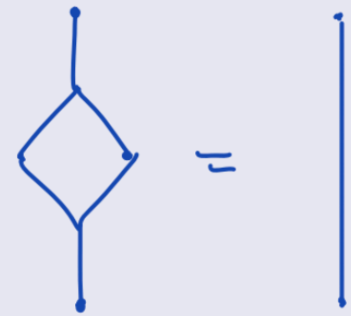
(Non-degeneracy)



Frobenius Objects II

Definition

A Frobenius object is special if



Some results about Frobenius objects

- β is unique:

- (comultiplication):

- The natural co-unitality/co-associativity follows.

Classification of Frobenius objects

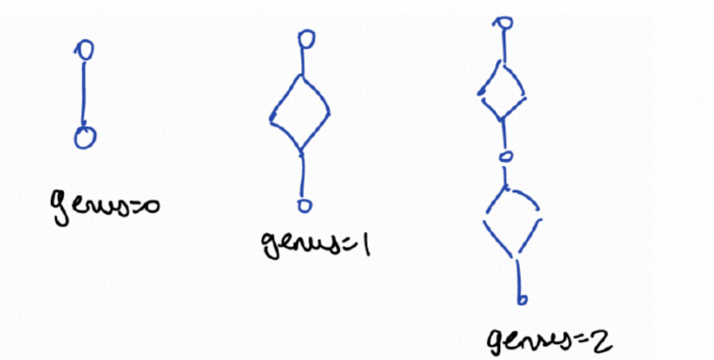
Theorem (Dijkgraaf '89–Abrams '96)

Commutative Frobenius objects in $\mathcal{C} \longleftrightarrow \mathcal{C}$*-valued 2D TQFTs where*
$\mathcal{C}$*-valued 2D TQFTs=symmetric monoidal functors* $2Cob \longrightarrow \mathcal{C}$

Theorem (Cattaneo,C-,Heunen '13)

Special Frobenius objects in Rel $\longrightarrow$ *Groupoid objects in Set*

Here, Rel is considered as a dagger symmetric monoidal category.
Also, one can recover topological invariants of surfaces via $Hom_{\mathcal{C}}(\mathbb{1}, \mathbb{1})$



What happens when $\mathcal{C}$=Span?

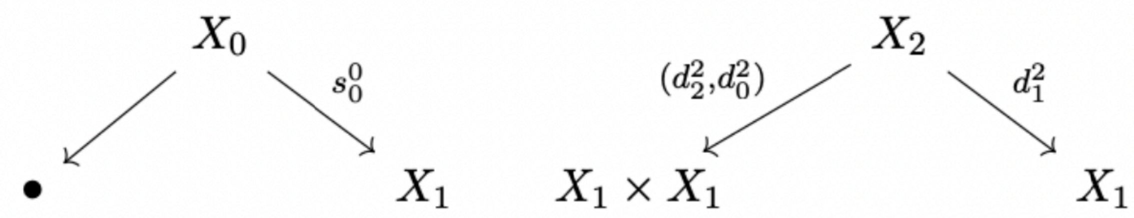
- Monoidal structure: Cartesian Product
- Monoidal unit: $\{\bullet\}$
- $Hom_{\mathcal{C}}(\{\bullet\},\{\bullet\}) = \{$iso-classes of sets$\}=\{$cardinalities $\}$

Theorem (C-, Keller, Mehta '21)

Frobenius objects in Span $\longleftrightarrow$ *simplicial sets* $X_\bullet$ *with conditions*

Conditions on the simplicial sets I

- (Unitality):



Lemma (C-, Keller, Mehta '21)

Let $X_\bullet$ be a 2-truncated simplicial set. The unit axiom holds if and only if for all $\zeta \in X_2$

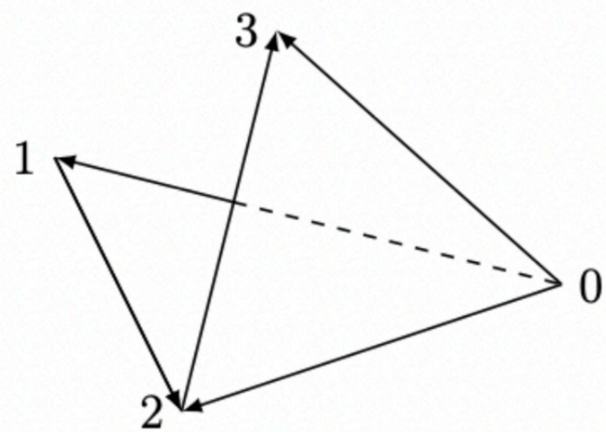
(1) *If $d_2^2\zeta \in im(s_0^0)$, then $\zeta \in im(s_0^1)$*

(2) *If $d_0^2\zeta \in im(s_0^0)$, then $\zeta \in im(s_1^1)$*

Conditions on the simplicial sets II

- (Associativity): We introduced the notion of $(i,j)-$ taco:

$$T_{ij}\mathcal{X} = \{(\zeta, \zeta') \in X_2 \times X_2 | d^2_{j-1}\zeta = d^2_i \zeta'\}.$$



A (13)-taco.

Conditions on the simplicial sets III

Let $S\mathcal{X} = \{(x_{01}, x_{12}, x_{23}, x_{03}) \in (X_1)^4$ such that

$$d_0^1 x_{01} = d_1^1 x_{12}, \qquad d_0^1 x_{12} = d_1^1 x_{23},$$
$$d_0^1 x_{23} = d_0^1 x_{03}, \qquad d_1^1 x_{03} = d_1^1 x_{01}\}.$$

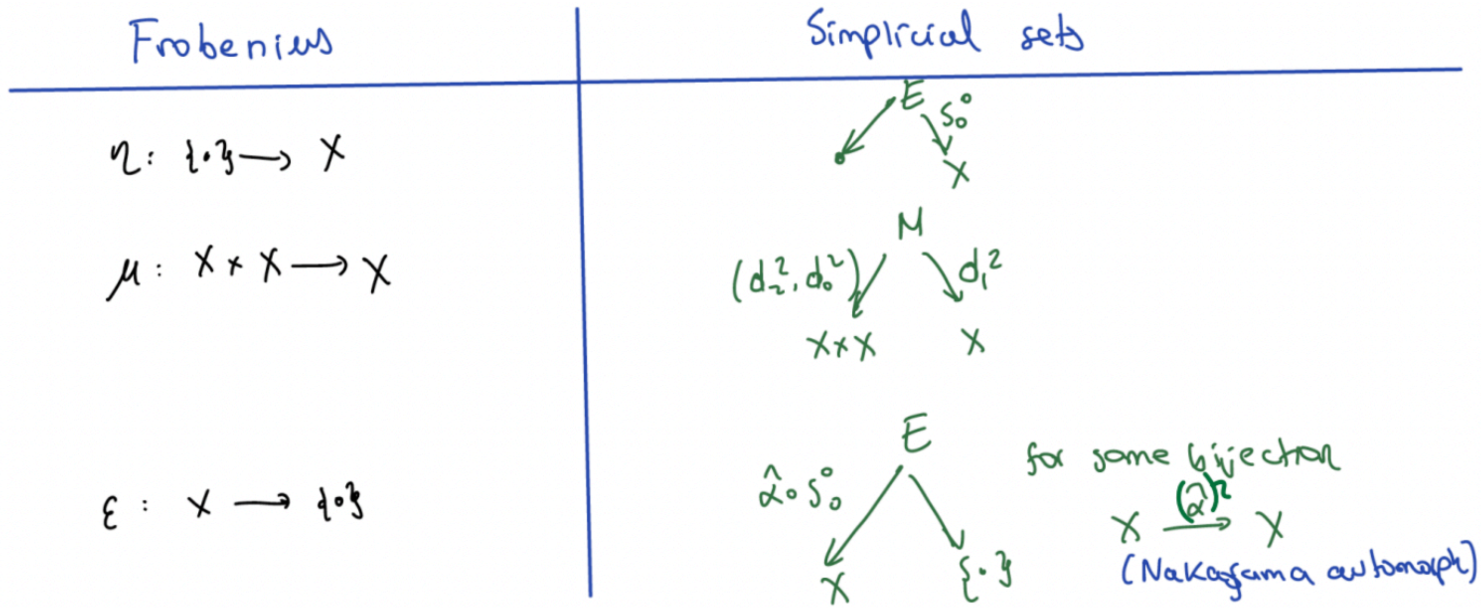
Lemma (C-, Keller, Mehta '21)

Associativity holds if and only if there is a bijection $T_{02}\mathcal{X} \cong T_{13}\mathcal{X}$ that commutes with the boundary maps to $S\mathcal{X}$.

The boundary maps $\partial_{02} : T_{02}\mathcal{X} \to S\mathcal{X}$ and $\partial_{13} : T_{13}\mathcal{X} \to S\mathcal{X}$ are defined by

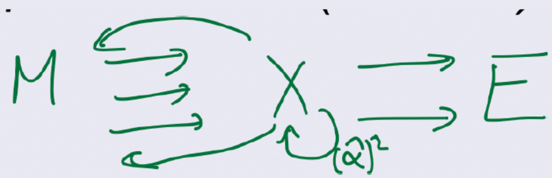
$$\partial_{02}(\zeta, \zeta') = (d_2^2\zeta', d_2^2\zeta, d_0^2\zeta, d_1^2\zeta'),$$
$$\partial_{13}(\zeta, \zeta') = (d_2^2\zeta', d_0^2\zeta', d_0^2\zeta, d_1^2\zeta).$$

Diagrammatics



Theorem (C-, Keller, Mehta '21)

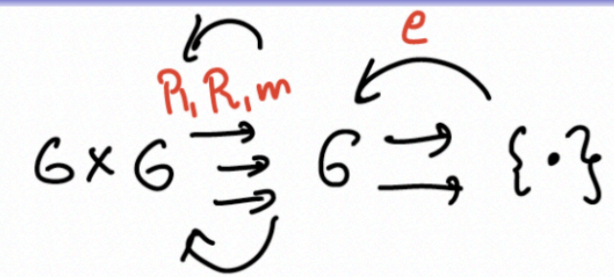
These maps come from a (truncated) simplicial set structure



These conditions are related to the axioms of 2-Segal sets!

Example

Group $G \longrightarrow$ nerve of G:



(Twisted) co-units:

Theorem (C-, Keller, Mehta '22)

If G is finite and abelian, Σ_g is a closed surface with genus g:

$$Z(\Sigma_g) = \begin{cases} |G|^g & \text{if } \omega^g = \omega \\ 0 & \text{otherwise} \end{cases}$$

Recent Work: 2-Segal picture

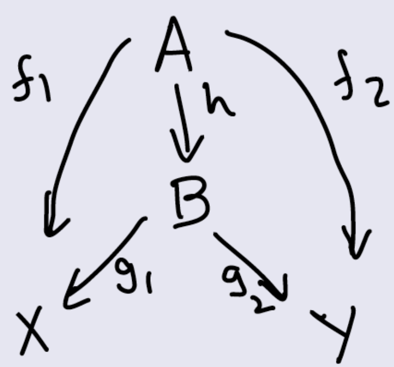
Theorem (Stern '21)

1. *Pseudomonoids in* $\mathbf{Span}_2 \longleftrightarrow$ *2-Segal sets.*
2. *Calabi-Yau objects in* $\mathbf{Span}_2$ *(categorified symmetric Frobenius)* $\longleftrightarrow$ *cyclic 2-Segal sets.*

Question: What about Frobenius pseudomonoids?

Definition (The bicategory $\mathbf{Span}_2$)

- Objects: Sets
- Morphisms: $X \xleftarrow{f_1} A \xrightarrow{f_2} Y$
- 2-morphisms:



Pseudomonids and 2-Segal Sets

Definition (Pseudomonoids)

Let $(\mathcal{C}, \otimes, I)$ be a monoidal bicategory. A pseudomonoid in $\mathcal{C}$ is:

- an object X
- Morphisms: $\eta : I \to X$ (unit)

  $\mu : X \otimes X \longrightarrow X$ (multiplication)
- invertible 2-morphisms: • $a : \mu \circ (\mu \otimes id_X) \Rightarrow \mu \circ (id_X \otimes \mu)$ (associator)

• $l : \mu \circ (\eta \otimes id_X) \Rightarrow id_X$ (left unitor) • $r : \mu \circ (id_X \otimes \eta) \Rightarrow id_X$ (right unitor)

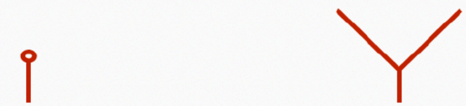
satisfying coherence conditions.

Example

A pseudomonoid in $\mathbf{Cat}_2$ is a monoidal category.

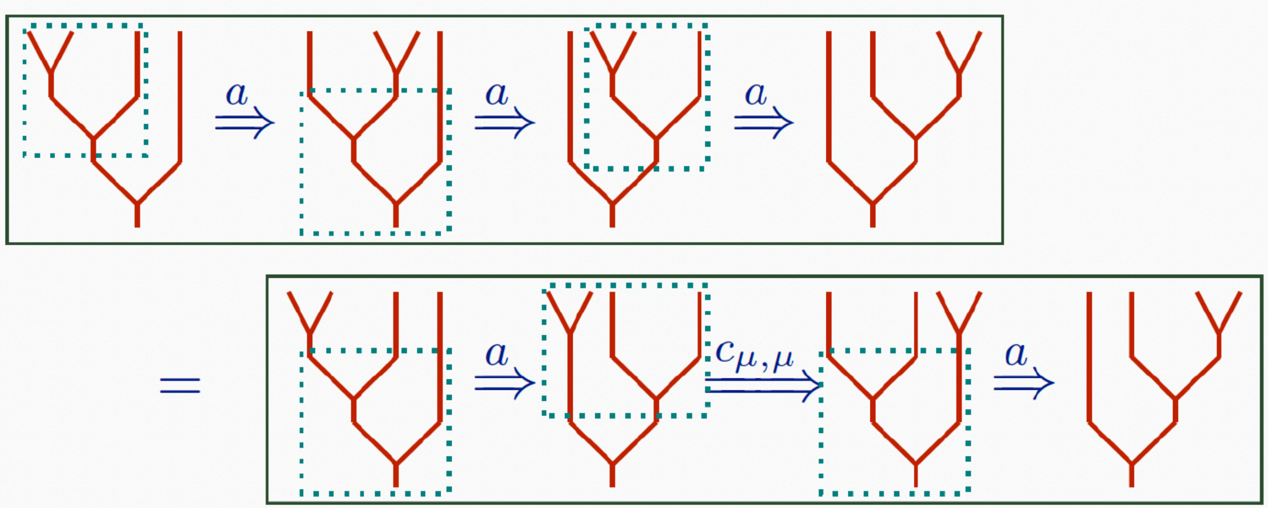
String Diagrams

We'll use string diagrams to denote the unit and multiplication:



Then a, ℓ and r are shown as morphisms of diagrams:

The main coherence condition is the pentagon equation



The lowest 2-Segal condition

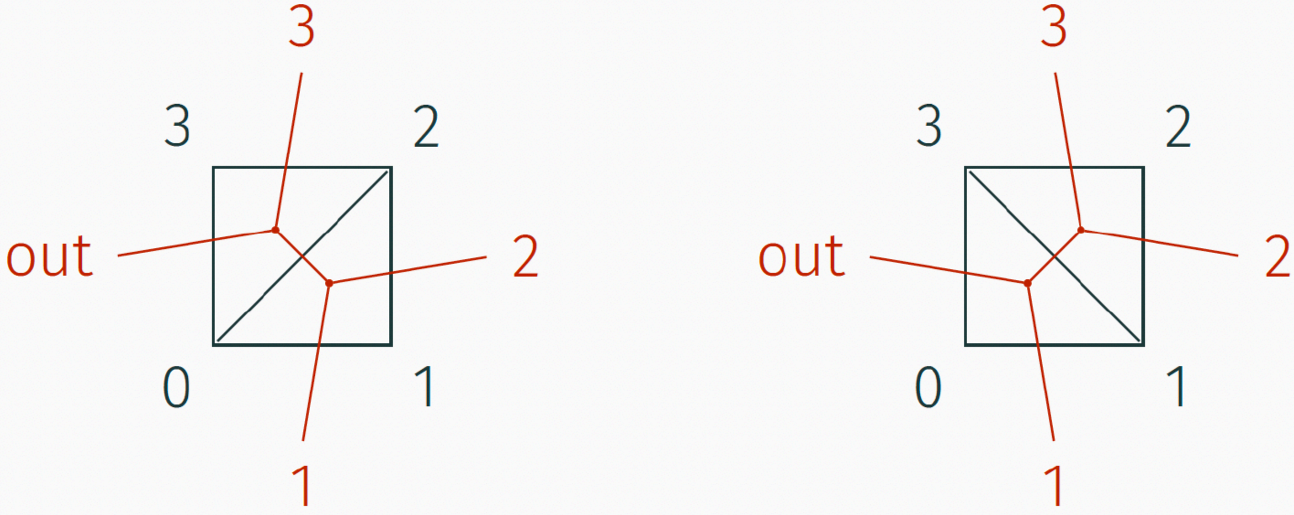
Consider the "taco maps"

$$\tau_{02} : X_3 \xrightarrow{(d_0,d_2)} X_2 \times_{X_1} X_2,$$

$$\tau_{13} : X_3 \xrightarrow{(d_0,d_2)} X_2 \times_{X_1} X_2.$$

The lowest 2-Segal condition says that these two maps should be isomorphisms. The associator is $\tau_{02}\tau_{13}^{-1}$.

These maps correspond to the two triangulations of the square:



Frobenius Pseudomonoids

- A Frobenius pseudomonoid in $\mathcal{C}$ is a pseudomonoid X with a counit morphism $\varepsilon : X \to I$ such that the pairing $\alpha := \varepsilon \circ \mu : X \otimes X \to I$ is nondegenerate.
- In $\mathbf{Span}_2$, a pairing is nondegenerate when it is isomorphic to

$$X \times X \xleftarrow{(\mathrm{id},t)} X \to \bullet$$

for some automorphism $t : X \to X$.
- Compatibility conditions $\implies$ t extends to a paracyclic structure on $X_\bullet$.

Paracyclic Sets

Definition

- A paracyclic set is a simplicial set $X_\bullet$, equipped with automorphisms $t^n : X_n \to X_n$ such that

$$d_i^n t^n = \begin{cases} t^{n-1} d_{i+1}^n, & i < n, \\ d_0^n, & i = n, \end{cases}$$

$$s_i^n t^n = \begin{cases} t^{n+1} s_{i+1}^n, & i < n, \\ (t^{n+1})^2 s_0^n, & i = n. \end{cases}$$

Theorem (C, Mehta, Stern '23)

Frobenius pseudomonoids in $\textbf{Span}_2 \longleftrightarrow$ *paracyclic 2-Segal sets.*

Examples

* Groupoids and Bisections

- Let $G_1 \rightrightarrows G_0$ be a groupoid. Let $\omega \subseteq G_1$ be a bisection.
- The nerve $G_\bullet$ is paracyclic with

$$t(g_1, \ldots, g_n) = (g_2, \ldots, g_n, (g_1 \cdots g_n)^{-1}\omega).$$

- It is cyclic iff ω is central, i.e. $\omega^{-1} g \omega = g$ for all $g \in G_1$.

* Partial addition

- For fixed $N \in \mathbb{N}$, write $[N] = \{0, \ldots, N\}$. This is a partial monoid under addition. (Not a groupoid!)
- The nerve has

$$X_n = \left\{(a_1, \ldots, a_n) \in [N]^n \colon \sum a_i \leq N\right\}.$$

- It is cyclic with

$$t(a_1, \ldots, a_n) = (a_2, \ldots, a_n, N - a_1 - \cdots - a_n).$$

Commutativity and Γ-structures

Definition

Let $\Phi_\star$ denote the category of finite pointed cardinals (i.e. the skeleton of the category $\mathrm{Fin}_\star$ of finite pointed sets). A Γ-set is a functor $\Phi_\star \to \mathrm{Set}$.

Theorem

A Γ-set is equivalent to a simplicial set $X_\bullet$, equipped with an action of S_n on X_n for each n, with extra compatibility conditions.

Theorem (C, Mehta, Stern '23)

Let $X_\bullet$ be a 2-Segal set. There is a one-to-one correspondence between Γ-structures on $X_\bullet$ and equivalence classes of commutative structures on the corresponding pseudomonoid in Span.

Thank you!