# Frobenius Reciprocity and Diffeological Reduction

Based on arXiv:2403.03927v1, joint with J. Watts and F. Ziegler

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# Introduction

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## Definition

The reduced space $X /\!\!/ G$ carries a reduced 2-form if there is a (diffeological) 2-form $\omega_{X /\!\!/ G}$ on it such that $j^*\omega = \pi^*\omega_{X /\!\!/ G}$, where $j$ and $\pi$ are the natural inclusion and projection maps in

$$\begin{array}{ccc} \Phi^{-1}(0) & \xhookrightarrow{j} & X \\ \downarrow{\scriptstyle \pi} & & \\ X /\!\!/ G & & \end{array}$$

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- If we take $X_1 = (\{0\}, 0, 0)$, then $\mathrm{Hom}_G(X_1, X_2) = X_2 /\!\!/ G$.
- If we take $X_1 = G(\mu)$, coadjoint orbit through $\mu \in \mathfrak{g}^*$, then $\mathrm{Hom}_G(X_1, X_2) = \Phi_2^{-1}(\mu)/G_\mu$, i.e. Marsden-Weinstein "shifting trick" (Guillemin-Sternberg, 1982).

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where $H \subset G$ is an aribtrary subgroup, $(Y, \omega_Y, \Psi)$ is a Hamiltonian $H$-space and $L := T^*G \times Y$ is a Hamiltonian $G \times H$-space with action $(g, h)(p, y) = (gph^{-1}, h(y))$, 2-form $\omega_L := d\varpi_{T^*G} + \omega_Y$ and moment map $\phi \times \psi : L \to \mathfrak{g}^* \times \mathfrak{h}^*$:

$$\begin{cases} \phi(p, y) = pq^{-1} \\ \psi(p, y) = \Psi(y) - q^{-1}p_{|\mathfrak{h}} \end{cases}$$

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Note: if we assume $H$ closed, $\mathrm{Ind}_H^G Y$ is a reduced manifold and a Hamiltonian $G$-space with the residual $G$-action and moment map $\Phi_{L/\!\!/H} : \mathrm{Ind}_H^G Y \to \mathfrak{g}^*$ (Khazdan-Kostant-Sternberg, 1978, Weinstein, 1978).

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**Definition**

Let $X$ be a non-empty set and let $\tau_n$ denote the Euclidean topology of $\mathbb{R}^n$. A diffeology $\mathcal{P}$ on $X$ is a subset of $\bigcup_{n\in\mathbb{N}, U\in\tau_n} \mathrm{Maps}(U, X)$, such that the following axioms are satisfied:

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- *Locality*. Let $P : U \to X, U \in \tau_n$; if for any $u \in U$ there exists an open neighbourhood $V \subset U$ of $u$ such that $P|_V \in \mathcal{P}$, then $P \in \mathcal{P}$.

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- *Smooth compatibility*. Let $P : U \to X, U \in \tau_n, V \in \tau_m$ and $\psi \in \mathcal{C}^\infty(V, U)$. If $P \in \mathcal{P}$, then $P \circ \psi \in \mathcal{P}$.

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The couple $(X, \mathcal{P})$ is called *diffeological space* and the maps in $\mathcal{P}$ are called *plots*.

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Example: A manifold $M$ has a natural diffeological space structure if it is endowed with $\mathcal{P} = \bigcup_{n\in\mathbb{N}, U\in\tau_n} \mathcal{C}^\infty(U, M)$.

Other definitions:

- A map $F : (X, \mathcal{P}) \to (Y, \mathcal{Q})$ between diffeological spaces is **smooth** if for any $P \in \mathcal{P}$, $F \circ P \in \mathcal{Q}$.

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- Let $X \subseteq Y$, $Y$ a diffeological space and $i : X \to Y$ the natural inclusion. The **subset diffeology** on $X$ is the coarsest diffeology that makes $i$ smooth; its plots are the maps $P : U \to X$ such that $i \circ P$ is a plot of $Y$.
  Univeral property: a map $F$ to $X$ is smooth if and only if $i \circ F$ is smooth.

- Let $X$ be a diffeological space, $\mathcal{R}$ an equivalence relation on $X$ and $s : X \to X/\mathcal{R}$ the natural projection. The **quotient diffeology** on $X/\mathcal{R}$ is the finest diffeology that makes $s$ smooth; its plots are the maps $P : U \to X/\mathcal{R}$ such that for any $u \in U$ there exist an open neighbourhood $V \subset U$ of $u$ and a plot $Q : V \to X$ such that $P|_V = s \circ Q$.
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  Univeral property: a map $F$ from $X/\mathcal{R}$ is smooth if and only if $F \circ s$ is smooth.
- The **subquotient diffeology** on $X /\!\!/ G = \Phi^{-1}(0)/G$ is obtained by taking the subset diffeology on $\Phi^{-1}(0)$ and then the quotient diffeology, or, equivalently, by taking the quotient diffeology on $X/G$ and then the subset diffeology.

Every map $F : X \to Y$ between diffeological spaces can be decomposed as follows:

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- A **subduction** is a strict surjection.

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A **diffeological** $k$-**form** $\alpha$ on the diffeological space $Y$ is a functional that associates to each plot $P : U \to Y$ a $k$-form on $U$, denoted $P^*\alpha$. Diffeological forms are required to satisfy a compatibility condition: for any $\psi \in \mathcal{C}^\infty(V, U)$, then $(P \circ \psi)^*\alpha = \psi^* P^*\alpha$.

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- Let $F : X \to Y$ be a smooth map between diffeological spaces and $\alpha$ be as in the definition. The **pullback** of $\alpha$ by $F$ is the diffeological form on $X$ such that for any plot $P$ of $X$: $P^*F^*\alpha = (F \circ P)^*\alpha$.

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- The **exterior derivative** $d\alpha$ is the $(k+1)$-form on $Y$ such that for any plot $P$ of $Y$, $P^*d\alpha = dP^*\alpha$.

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*Let $s : X \to Y$ be a subduction and $\alpha$ a $k$-form on $X$. Then the following are equivalent:*

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- *For all $P, Q$, plots of $X$ such that $s \circ P = s \circ Q$, then $P^*\alpha = Q^*\alpha$.*

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- $\alpha = \omega_{|C}$
- If $\beta$ exists, then $\beta = \omega_{X/\!\!/G}$.

**Remark**: Just like symplectic Frobenius reciprocity holds in Hamiltonian $G$-space category and prequantum $G$-space category (Ratiu-Ziegler, 2022), the diffeological Frobenius reciprocity and the results on reduced forms hold in both categories.

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$$t : \mathrm{Hom}_G(X, \mathrm{Ind}_H^G Y) \to \mathrm{Hom}_H(\mathrm{Res}_H^G X, Y).$$

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Moreover, the authors conjectured:

- The map $t$ is a diffeological diffeomorphism.
- The map $t$ preserves the (diffeological) 2-forms that the reduced spaces may carry.

**Theorem**

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- $\phi_M \times \psi_M : M \to \mathfrak{g}^* \times \mathfrak{h}^*$,

$$\begin{cases} \phi_M(p, y) = pq^{-1} - \Phi(x) \\ \psi_M(p, y) = \Psi(y) - q^{-1}p_{|\mathfrak{h}} \end{cases}$$

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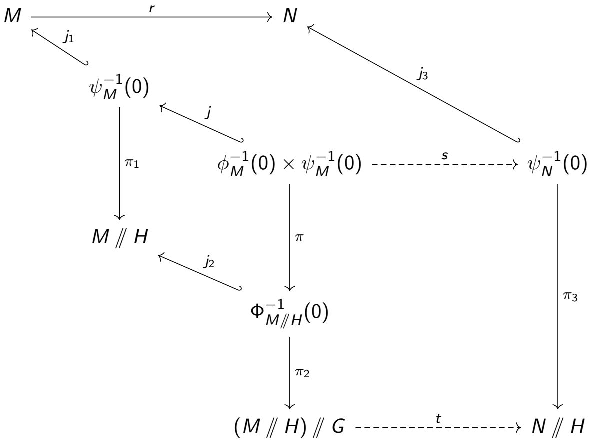
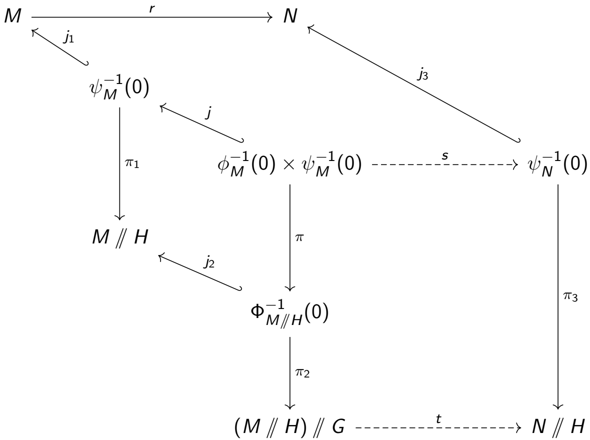
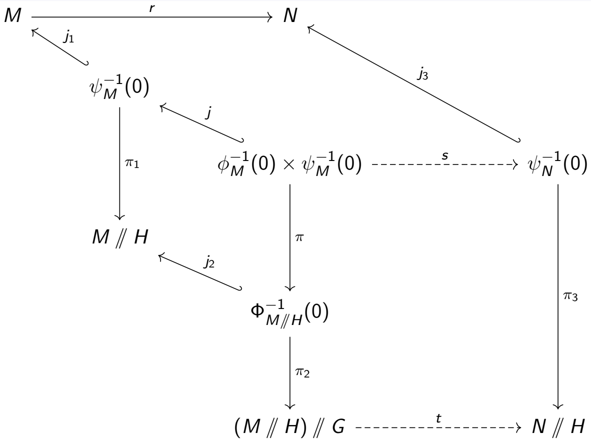

M
N
r
j1
j3
ψ_M^{-1}(0)
j
π1
ϕ_M^{-1}(0) × ψ_M^{-1}(0)
s
ψ_N^{-1}(0)
M // H
π
j2
π3
Φ_{M//H}^{-1}(0)
π2
(M // H) // G
t
N // H




Let $r : M \to N$, $r(x, p, y) := (q^{-1}(x), y)$ and $r' : N \to M$, $r'(x, y) := (x, \Phi(x), y)$. The maps $r$ and $r'$ restrict to maps $s$ and $s'$.


$M$
$r$
$N$
$j_1$
$j_3$
$\psi_M^{-1}(0)$
$j$
$\pi_1$
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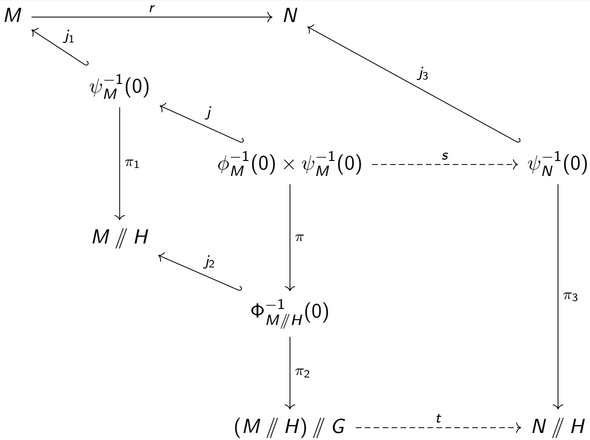


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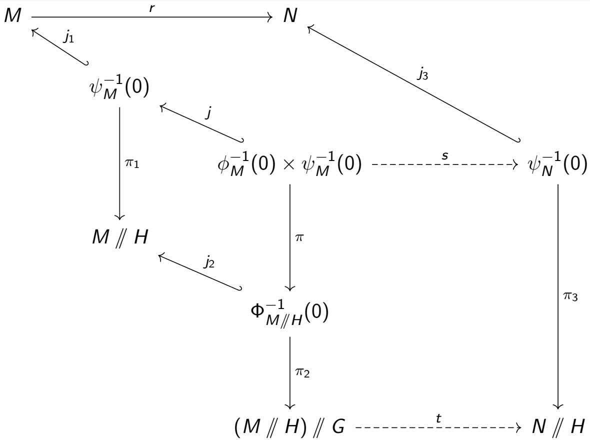


- The map $s$ sends $G \times H$-orbits to $H$-orbits and $s'$ sends $H$-orbits to orbits of the diagonal action of $H$.
- The maps $s$ and $s'$ descend to a bijection $t$ and its inverse $t^{-1}$, respectively (Ratiu-Ziegler, 2022).

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- If only one reduced form exists, the other one can be defined by the equation $\omega_{(M/\!\!/H)/\!\!/G} = t^*\omega_{N/\!\!/H}$.

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# Group actions

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- The action of $G$ on $X$ is **proper** if $\theta$ is proper, i.e. preimages of compact sets via $\theta$ are compact.

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We are going to prove that it suffices to assume the $G$-action strict or locally free or proper.

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- The strictness assumption implies that plots $R : V \to G$, and $S : V \to C$ exist and $(P \times Q_{|V}) = \theta \circ (R \times S)$. Therefore:

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where $H \subset G$ is an arbitrary subgroup, $(Y, \omega_Y, \Psi)$ is a Hamiltonian $H$-space and $L := T^*G \times Y$ is a Hamiltonian $G \times H$-space, it is endowed with 2-form $\omega_L := d\varpi_{T^*G} + \omega_Y$ and moment map $\phi \times \psi : L \to \mathfrak{g}^* \times \mathfrak{h}^*$.

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Example: If $G = \mathbb{T}^2$ and $H = S_\alpha, \alpha \in \mathbb{R} \setminus \mathbb{Q}$, $G/H = T_\alpha$.

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Proof: Local freeness implies:

- $0 \in \mathfrak{g}^*$ is a regular value for $\Phi$ and therefore $C$ is a manifold.
- The $G$-orbits are the leaves of a foliation $\mathcal{F}$ of $C$.

Moreover, $j^*\omega$ is basic, since it is $G$-invariant and $\mathfrak{g}(x) \subset \mathrm{Ker}(j^*\omega)$.

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Moreover, $j^*\omega$ is basic, since it is $G$-invariant and $\mathfrak{g}(x) \subset \mathrm{Ker}(j^*\omega)$. Therefore, it is possible to apply a theorem of *Hector et al.* (2011), which implies that $j^*\omega$ is the pullback of a diffeological 2-form on $C/\mathcal{F} = X /\!\!/ G$.

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*Proof:* Under the properness assumption, $X /\!\!/ G$ is a stratified symplectic space, then, in particular, it is a disjoint union of symplectic manifolds $(C_t/G, \omega_t)$, where $t$ denotes the orbit type (Sjamaar-Lerman, 1991 and Bates-Lerman, 1997).

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The existence of the reduced form $\omega_{X /\!\!/ G}$ can be proved working on the previous equality and applying Souriau's criterion to $j^*\omega$.

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- The theorem follows from Souriau's criterion.

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**Corollary**

*The 2-form $\omega_{X /\!\!/ G}$ on $X /\!\!/ G$ restricts to the Sjamaar-Lerman-Bates 2-form $\omega_t$ on each reduced piece $C_t/G$.*

*Thank you!*