MTH698: Plan B Research Paper

Metrization of Topological Spaces

Kodjo Dieu Donne Youto

supervised by

Dr. Jordan Watts

April 30, 2023

Abstract

This paper provides an overview of the main theorems regarding metrization of topological spaces. While every metric space can be considered as a topological space with open balls as open sets and closed balls as closed sets, it is important to determine the conditions under which a metric can be defined on a topological space. The paper establishes that normality is a necessary but not sufficient condition for metrization. Additionally, the Urysohn metrization theorem is presented as a sufficient but not necessary condition for metrization. Finally, the paper also presents the two main theorems, Smirnov Theorem and the Nagata-Smirnov theorem, which provide both necessary and sufficient conditions for a topological space to be metrizable.

0.1 Preliminaries

The objective of this paper is to present proofs for the three fundamental metrization theorems. Our approach and definitions are based on the work of [1]. To start, we will establish some key concepts that will be frequently referenced in our proofs.

Definition 1. *Let X be a topological space. X is said to be* ***Hausdorff*** *if $\forall\ x, y$ in X such that $x \neq y$, there exist open sets U_x containing x and V_y such that $U_x \cap U_y = \emptyset$.*

Definition 2. *Let X be a topological space. X is said to be* ***regular*** *if for any closed set $A \in X$ and $\forall x$ in $X - A$, there exists disjoint open sets U_x containing x and V_A containing A.*

Definition 3. *Let X be a topological space. X is said to be* ***normal*** *if for any two disjoint closed sets A and $B \in X$, there exist disjoint open sets U_A containing A and V_B containing B.*

Definition 4. *Let X be a topological space. X is said to be* ***second countable*** *if it has a countable basis.*

Theorem 5. *Every second countable regular space is normal.*

Proof. Let A and B be two disjoint closed sets in a second countable regular space X. Given that X is regular, for any arbitrary $x \in A$, there exists an open set U containing x and an open set $W \supset B$ such that $U \cap W = \emptyset$. Consider an open set V_x containing x such that its closure $\overline{V}_x \subset U$. It follows that $\overline{V}_x \cap B = \emptyset$. The collection $(V_x)_{x \in A}$ is an open cover of A not necessarily countable. From second countability, X has a countable basis $\mathcal{B}$ with basis elements that we denote $(\mathcal{O}_n)_{n \in I \subset \mathbb{N}}$. Since V_x can be written as the union of some elements of the basis, there exists some n_0 such that $x \in \mathcal{O}_{n_0} \subset V_x$. The collection of such $\mathcal{O}_{n_0}$ $\forall x \in A$ form a countable open cover of A. Also $\overline{\mathcal{O}}_{n_o} \cap W = \emptyset$.
To simplify the notation in the remaining part of the proof, we denote by $(U_n)_{n \in \mathbb{N}}$ the countable collection of basic open sets covering A as constructed above, and let $(W_n)_{n \in \mathbb{N}}$ a countable collection of basic open sets covering B constructed similarly. For each $n \in \mathbb{N}$, U_n and W_n are disjoint since each U_n is some $\mathcal{O}_{n_o}$ and each $W_n \subset W$. Also $\bigcup_{n \in \mathbb{N}} U_n \supset A$ and $\bigcup_{n \in \mathbb{N}} W_n \supset B$ are open sets but not necessarily disjoint. To

address this, define:

$$U'_n = U_n - \bigcup_{i=1}^{n} \overline{W}_i$$
$$W'_n = W_n - \bigcup_{i=1}^{n} \overline{U}_i$$

Clearly, U'_n and W'_n are open sets as each is the difference of an open set and a finite union of closed sets which is closed. $(U'_n)_{n\in\mathbb{N}}$ covers A since for each $x \in A$, x belongs to some U_n but to none of V_n. Similarly, $(V'_n)_{n\in\mathbb{N}}$ covers B. Define U' to be $\bigcup_{n\in\mathbb{N}} U'_n$ and W' to be $\bigcup_{n\in\mathbb{N}} W'_n$, which contain respectively A and B.
We conclude the proof by showing that U' and W' are disjoint. Assume there exists $x \in U' \cap W'$, then $x \in U'_j \cap W'_k$ for some j, k. Without loss of generality, suppose $j \leq k$. It follows from the definition of U'_j that $x \in U_j$ and from the definition of W'_k that $x \notin \overline{U_j}$,and therefore not in $x \notin U_j$. We have $x \in U_j$ and $x \notin U_j$ which is a contradiction. We have constructed disjoints open sets U' and W'containing respectively closed the arbitrary disjoint closed set A and B. As a result, X is normal. □

Definition 6. *A function $d : X \times X \to \mathbb{R}_+$ is said to be a **metric** (distance function) if the followings holds:*

1. *$d(x, y) = 0 \iff x = y$ for all $x, y \in X$ (nondegeneracy)*
2. *$d(x, y) = d(y, x) \quad \forall x, y \in X$ (symmetry)*
3. *$d(x, y) \leq d(x, z) + d(y, z) \quad \forall x, y, z \in X$ (triangular inequality)*

A metric space (X, d) is a set X with a metric d. Any metric space has an induced topology where the open balls generate the open sets of the topology. However, the converse need not be true as we cannot necessarily define a distance function on any topological space. A topological space is said to be **metrizable** if a metric can be defined on such space.

Example. On the real line with the usual topology, we can define a metric d such that $\forall x, y \in \mathbb{R}, d(x, y) = |x - y|$. This satisfies the definition of a metric and $(\mathbb{R}, d)$ is a metric space. We can similarly define an euclidean distance on $\mathbb{R}^n$.

Theorem 7. *Every metrizable space is normal.*

Proof. Let X be a metrizable space with metric d. Let A and B be two disjoint closed sets in X. $\forall a \in A$ and $b \in B$, there exist for some $r_a > 0$ and $r_b > 0$ disjoint open balls $B(a, r_a)$ and $B(b, r_b)$. The statement remains true for $B(a, \frac{r_a}{2})$ and $B(b, \frac{r_b}{2})$. $(B(a, \frac{r_a}{2}))_{a \in A}$ and $(B(b, \frac{r_b}{2}))_{b \in B}$ are respectively open covers of A and B, in other words, $A \subset U = \bigcup_{a \in A} B(a, \frac{r_a}{2})$ and $B \subset V = \bigcup_{b \in B} B(b, \frac{r_b}{2})$. U and V are open sets as unions of open balls. To prove that they are disjoint, assume that $x \in U \cap V$. There exist some $a \in A$ and $b \in B$ such that $x \in B(a, \frac{r_a}{2})$ and $x \in B(b, \frac{r_b}{2})$. By triangular inequality, $d(a,b) \leq d(a,x) + d(x,b) < \frac{r_a}{2} + \frac{r_b}{2}$. Suppose $r_a \leq r_b$, it follows that $d(a,b) < r_b$ contradicting the fact that $B(a, r_a)$ and $B(b, r_b)$ are disjoints. The same contradiction happens if $r_a \geq r_b$. This shows that U and V are disjoint open sets containing respectively A and B. □

The previous theorem shows that normality is a a **necessary** but not sufficient condition for metrization. The next set of examples from [1] illustrates that normality or regularity is only necessary but not sufficient.

Example.

1. The real line (with the usual topology) is an example that satisfies all the separability axioms and is metrizable.

2. The Sorgenfrey line (real line with topology generated by half-open sets $[a, b)$) satisfies all the separability axioms (T1, Hausdorff, regular, completely regular and normal). However, it is not second countable therefore not metrizable. This shows normality is not sufficient.

3. The Sorgenfrey plane is not normal therefore not metrizable. This illustrate normality as a necessary condition.

0.2 Urysohn Metrization Theorem

This section focuses on the first important metrization theorem: the Urysohn Metrization theorem which provides sufficient but not necessary conditions for metrizability. In order to prove this theorem, lets state without proof the Urysohn Lemma.

Theorem 8 (Urysohn lemma). *Let X be a normal space; let A and B be disjoints closed subsets of A. Let $[a,b] \subset \mathbb{R}$ be a closed interval. Then there exists a continuous map*

$$f : X \longrightarrow [a,b]$$

such that $f(x) = a \ \forall x \in A$ and $f(x) = b \ \forall x \in B$.

Definition 9. *A space X is said to be completely regular if $\forall x_0 \in X$ and any closed $A \subset X$ not containing A, there exists a continuous function $f : X \longrightarrow [0,1]$ such that $f(x_0) = 1$ and $f(x) = 0 \quad \forall x \in A$.*

One important result of the Urysohn lemma is that every normal space is completely regular and completely regular spaces are regular.

Theorem 10 (Urysohn Metrization Theorem). *Every regular second countable space is metrizable.*

It is important to note that this theorem can be stated equivalently as any second countable normal space is metrizable. This is because every regular second countable space is normal.

Proof. Let X be a regular second countable space. We will prove the theorem by embedding X in a metrizable space, and so X is homeomorphic to a subspace Y of the metrizable space, and hence metrizable. One of the two ways is to either take $Y = [0,1]^{\omega}$ with the product topology or $Y = [0,1]^{\omega}$ the topology induced by the uniform metric defined by:

$$\bar{\rho}(x,y) = \sup_{i \in \omega}\{|x_i - y_i|\}$$

where $x = (x_1, x_2, \cdots), y = (y_1, y_2, \cdots)$ and ω some countable index set.

Version 1: In this version of the proof, we show X is homeomorphic to the $Y = [0,1]^{\omega}$ with the product topology.

First, we show there exist a countable collection of continuous functions $f_n : X \longrightarrow [0,1]$ such that $\forall x_0 \in X$ and for any neighborhood U of X, there exists an index n such that f_n vanishes outside U.

Given that X is second countable and regular, X is normal by Theorem 4, and therefore completely regular by Urysohn lemma. This guarantees the existence of a function

such that $f(x_0) = 1$ and f vanishes outside any neighborhood U of x. However, the collection of such functions for each pair (x_0, U), is not countable. We will make use of the fact that it is second countable to achieve this.
Let $\mathscr{B}$ be a countable basis of X, by regularity, we can find B_n and B_m in $\mathscr{B}$ such that $\overline{B}_n \subset B_m$. We construct a function $g_{n,m}$ similar to f, defined by: $g_{n,m} : X \longrightarrow [0,1]$ such that $g_{n,m}(\bar{B}_n) = 1$ and $g_{n,m}(X - B_m) = 0$. Taking any arbitrary x_0 and any neighborhood U of x_0, there exists B_n such that $\bar{B}_n \in U$ and $x_0 \in B_n$, it follows that $g_{n,m}(x_0) = 1$ and $g_{n,m}$ vanishes outside B_n, and therefore vanishes outside U. Moreover such a collection of $g_{n,m}$ is countable since $\mathbb{Z}_+ \times \mathbb{Z}_+$ is countable. We rearrange the index and denote $g_{n,m}$ by f_n.

Next, Given the functions f_n, we consider the product topology on $\mathbb{R}^\omega$ and define a map $F : X \longrightarrow \mathbb{R}^\omega$ by $F(x) = (f_1(x), f_2(x) \dots)$ and show F is an embedding.
F is a continuous map since each f_n is continuous and $\mathbb{R}^\omega$ has the product topology. Considering two arbitrary element x and y in X, there exists an index N such that $f_N(x) > 0$ and $f_N(y) = 0$. It follows that $F(x) \neq F(y)$, and therefore F is injective.
We only need to show F is homeomorphic unto its image $F(X)$ that we denote by Z. Since F is continuous and injective, F defines a bijection from X to $F(X) = Z$. To show F is an homeomorphism from X to Z, it is enough to show for each U open in X, $F(U)$ is open in Z.
Let x_0 be an arbitrary point in X and U an open set containing x_0. We denote by $z_0 = F(x_0)$. There exist an index N such $f_N(x_0) > 0$ and f_N vanishes outside U. Let $V = \pi_N^{-1}((0, \infty))$ where π_N is a projection of $\mathbb{R}^\omega$ to its N-*th* component . V is open in $\mathbb{R}^\omega$ as the inverse image of an open set by a projection π_N. We denote by $W = V \cap Z$. W is open by definition of subspace topology. To show $F(U)$ is open, it is enough to show that $z_0 = F(x_0) \subset W \subset F(U)$, and therefore $F(U)$ would be open in $\mathbb{R}^\omega$.
First $z_0 \in W$ since $\pi_N(z_0) = \pi_N(F(x_0)) = f_N(x_0) > 0$, and so $z_0 \in V$. Therefore, $z_0 \in W = V \cap Z$. Next, we show $W \subset F(U)$. Let $x \in W, z = F(x)$ for some $x \in X$. $\pi_N(z) = \pi_N(F(x)) = f_N(x) > 0$ (by definition of V) and vanishes outside U. If follows that $x \in U$ implying $z = F(x)$ is in $F(U)$. This shows that for any open $U \subset X$, $F(U)$ is open in $Z = F(X)$. X is therefore metrizable since it is homeomorphic to the subset $Y = [0,1]^\omega$ of the metrizable space $\mathbb{R}^\omega$ with the product topology.

Version 2: This version of the proof is pretty similar to the previous one, except this time, X will be embedded in $[0,1]^\omega$ with the metric defined by $\bar{\rho}(x,y) = \sup\{|x_i - y_i|\}$ with $x = (x_1, x_2, \dots)$ and $y = (y_1, y_2, \dots)$.
The construction of the sequence of function f_n is the same. However, this time, we want $f_n(x) < 1/n$. This is obtained by dividing each f_n from the first version by n. As previously, we define the function $F : X \longrightarrow [0,1]^\omega$ by $F(x) = (f_1(x), f_2(x), \cdots)$ and claim F is an embedding on $[0,1]^\omega$ with the metric $\bar{\rho}$.
First F is injective as established in the *version 1* proof. The statement that F is an open map (carries open sets to open sets) remains true as the topology induced by the metric $\bar{\rho}$ on $[0,1]^\omega$ is finer (larger) than the product topology. The only thing we need to prove for the embedding to hold true is the continuity of F as this is not guaranteed by the fact that each f_n is continuous as it was the case with the product topology.
To prove continuity, let x_0 be arbitrary in X. To show F is continuous, we need to show that $\forall \epsilon > 0,$ there exists a neighborhood U of x_0, such that:

$$x \in U \Longrightarrow \bar{\rho}(F(x_0), F(x)) < \epsilon$$

First we choose N large enough, say $N > 2/\epsilon$. It follows $1/N < \epsilon/2$. For each $n < N$, there exists, by continuity of each f_n, a neighborhood U_n of x_0 such that $\forall x \in U_n$, $|f_n(x) - f_n(x_0)| < \epsilon/2$. Let $U = \cap_{n=1}^N U_n$; $x \in U$.
If $n \geq N$ then $|f_n(x) - f_n(x_0)| \leq 2/n \leq 2/N < \epsilon$. We denote such neighborhood of x by V.
On $U \cap V$, $|f_n(x) - f_n(x_0)| < \epsilon$ for all $n \in \mathbb{N}$. It follows that $U \cap V$ is a neighborhood of x_0 such that $x \in U \cap V$ implies $\bar{\rho}(F(x_0), F(x)) = \sup\{|f_n(x) - f_n(x_0)|\} < \epsilon$ which concludes the proof.
□

The second version proved something stronger. We state it here as the following theorem.

Theorem 11 (Embedding Theorem). *Let X be a T_1 space. Suppose there exists a function*

$(f_n)_{n\in\omega}$ where ω is the set of indexes f defined from $f : X \longrightarrow \mathbb{R}$ such that $\forall x_0 \in X$, there exists a neighborhood U of x_0 such that $f(x_0) > 0$ and f vanishes outside U. Then the function $F : X \longrightarrow \mathbb{R}^\omega$ defined by $F(x) = (f_n(x))_{n\in\omega}$ is an embedding of X in $\mathbb{R}^\omega$. Moreover, if f maps X into $[0, 1]$ then F embed X into $[0, 1]^\omega$

0.3 Nagata-Smirnov Metrization Theorem

From first previous section, it was observed that regularity or normality are necessary condition for metrizability. The section two showed that the Urysohn theorem provides a sufficient condition but the assumption of second countability is not really needed to assure measurably. Therefore, a weaker condition that is both necessary and sufficient is required. The Nagata-Smirnov Metrization Theorem replaces second countability with locally countably finite basis, which is a weaker condition. The proof of the Nagata-Smirnov theorem follows a similar pattern to that of the Urysohn metrization theorem. We begin by defining some important notion that will be used in the Nagata-Smirnov theorem.

Definition 12. *A collection $\mathcal{A}$ of subsets of a topological space X is said to be* ***locally finite*** *if $\forall x \in X$, there exists a neighborhood of x that intersects only finitely many elements of $\mathcal{A}$.*

Definition 13. *A collection of subsets $\mathcal{B}$ of a topological space X is said to be* ***countably locally finite*** *if it can be written as a countable union of collections $\mathcal{B}_n$, each of which is locally finite.*

Definition 14. *Let $\mathcal{A}$ be a collection of subsets of a topological space X. A collection $\mathcal{B}$ is said to be a* ***refinement*** *of $\mathcal{A}$ if $\forall B \in \mathcal{B}$, there exists $A \in \mathcal{A}$ such that $B \subset A$. The refinement is open if the elements of $\mathcal{B}$ are open set and closed if the elements of $\mathcal{B}$ are closed.*

Lemma 15. *Let X be a metrizable space and $\mathcal{A}$ an open cover of X. There is an open covering $\mathcal{B}$ of X refining A that is locally countably finite.*

Proof. The proof provides us with an algorithm of constructing a refinement from any cover of a metrizable set.
Let $\mathscr{A}$ be the collection of open sets $(A_i)_{i\in I}$ that covers X. Note that the index set I is not necessarily countable. We can however order the index set creating therefore an

order in the A_is. We define:

$$T_n(A_i) = S_n(A_i) - \bigcup_{j<i} A_j$$

where $S_n(A_i) = \{x \mid B(x, 1/n) \subset A_i\}$ is the subset of A_i obtained by "shrinking" A_i a distance $1/n$. The sets $T_n(A_i)$ are separated by a distant at least $1/n$ (i.e taking $i \neq j$, $\forall x \in T_n(A_i)$ and $\forall y \in T_n(A_j), d(x, y) > 0$).
Next we define $E_n(A_i)$ by expanding slightly $T_n(A_i)$. Specifically, let $E_n(A_i)$ be the $1/3n$-neighborhood of T_n i.e. $E_n(A_i)$ be the union of open balls $B(x, 1/3n)$ for $x \in T_n(A_i)$. The sets $E_n(A_i)$ are separated by a distance of at least $1/3n$.

Let's define

$$\mathcal{E}_n = \{E_n(A_i) \mid A_i \in \mathscr{A}\}.$$

$\mathcal{E}_n$ is a refinement of $\mathscr{A}$ since each $E_n(A_i) \subset \mathscr{A} \in A_i$. Also $\mathcal{E}_n$ is locally finite since any arbitrary $x \in X$, there exist a neighborhood that intersects finitely many elements of $\mathcal{E}_n$. In our case, the $1/6n$-neighborhood of x $B(x, 1/6n)$ intersect at the most one element in $\mathcal{E}_n$ since each element $E_n(A_i)$ are separated from each other by a distance of at least $1/3n$. It only remains to show it covers X and is countable. Unfortunately, $\mathcal{E}_n$ does not cover X.
We then define:

$$\mathcal{E} = \bigcup_{n \in \mathbb{N}} \mathcal{E}_n$$

Such collection is countably locally finite, we also claim it covers X. To prove this, we will show that each $x \in X$ belongs to some element in $\mathcal{E}$.
For any arbitrary $x \in X$, there is an element $A_i \in \mathscr{A}$ that contains x. We can reorder the elements such that A_i is the first element containing x. Since A_i is open, there exists an open ball $B(x, 1/n) \subset A_i$ for some n. By definition of $S_n(A_i)$, $B(x, 1/n) \in S_n(A_i)$. It follows that $x \in S_n(A_i)$. Given A_i is the first element containing x, removing the other A_i will not remove x. Since $E_n(A_i)$ is obtained by expanding $T_n(A_i)$, it follows $x \subset E_n(A_i) \in \mathcal{E}_n \subset \mathcal{E}$. This concludes $\mathcal{E}$ covers X. □

Definition 16. *Let A be a subset of a topological space X. A is said to be a G_δ* ***set*** *if there exists a countable collection of opens sets $\mathcal{O}_n$ of X such that $A = \cap_{n \in \mathbb{N}} \mathcal{O}_n$*

Lemma 17. *Every regular space with a basis that is countably locally finite is normal and every closed set in such space is a G_δ-set.*

Proof. Let X be such a space, and let $\mathscr{B}$ be a countably locally finite basis. First, we show that for any open set $V \subset X$ there exists a countable collection U_n of open sets such that

$$V = \bigcup_{n\in\mathbb{N}} U_n = \bigcup_{n\in\mathbb{N}} \overline{U}_n.$$

Given that basis $\mathscr{B}$ is countably locally finite, there exists a locally finite collection $\mathscr{B}_n$ such that $\mathscr{B} = \bigcup_{n\in\mathbb{N}} \mathscr{B}_n$.

Let $\mathcal{C}_n$ be the collection of $B \in \mathscr{B}_n$ such that $\bar{B} \in V$. $\mathcal{C}_n$ is locally finite as a sub-collection of $\mathscr{B}_n$.

Let

$$U_n = \bigcup_{B\in\mathcal{C}_n} B$$

From the locally finiteness, it follows that

$$\overline{U}_n = \bigcup_{B\in\mathcal{C}_n} \overline{B}.$$

$\overline{B} \subset V$ implies $\bigcup U_n \subset \bigcup \overline{U}_n \subset V$. Conversely, $\forall x \in V$, by regularity x belongs to some B such that $\overline{B} \in V$. Hence $x \in U_n \subset \bigcup U_n$ which concludes the equality.

Next we show every closed set in X is a G_δ-set.

Let C be a closed set. There exists an open set V in X that is the complement of C in X i.e. $C = X - V$. From the first step, there exists a collection of open sets U_n such that $V = \bigcup_{n\in\mathbb{N}} \overline{U}_n$. Hence

$$\begin{aligned} C &= X - \bigcup_{n\in\mathbb{N}} \overline{U}_n \\ C &= \bigcap_{n\in\mathbb{N}} (X - \overline{U}_n) \end{aligned}$$

is a G_δ-set as a countable intersection of open sets.

The final step consists in showing X is normal. Consider two closed sets A and B in X. The sets $X - A$ and $X - B$ are open sets and from step one there exists a collection of open sets $\{U_n\}_{n\in\mathbb{N}}$ such that

$$\bigcup_{n\in\mathbb{N}} U_n = \bigcup_{n\in\mathbb{N}} \overline{U}_n = X - B$$

Then $\{U_n\}_{n\in\mathbb{N}}$ covers A and each $\overline{U}_n$ is disjoint from B. We can define $\{V_n\}_{n\in\mathbb{N}}$ which also covers B and each $\overline{V}_n$ is disjoint from A. The remaining is exactly the same as in the case of a regular space with a countable basis (*theorem 5*).

We define

$$U'_n = U_n - \bigcup_{i=1}^{n} \overline{V}_i \quad \text{and} \quad V'_n = V_n - \bigcup_{i=1}^{n} \overline{U}_i$$
$$U' = \bigcup_{n\in\mathbb{N}} U'_n \quad \text{and} \quad V' = \bigcup_{n\in\mathbb{N}} V'_n.$$

The sets U' and V' are disjoint open sets that contain respectively the closed sets A and B. □

Lemma 18. *Let X be a normal space and A, a closed G_δ set in X. There exists a continuous function $f : X \longrightarrow [0,1]$ such that $f(x) = 0$ for $x \in A$ and $f(x) > 0$ $\forall x \notin A$.*

Proof. Let A be a closed G_δ-set. It follows that there exists a countable collection of open sets $\mathcal{O}_n, n \in \mathbb{N}$ such that $A = \cap_{n\in\mathbb{N}}\mathcal{O}_n$. For each n, we construct a function $f_n : X \longrightarrow [0,1]$ such that f_n vanishes on A and $f_n \equiv 1$ on $X - \mathcal{O}_n$. Let $f = \sum_{n\in\mathbb{N}} \frac{f_n}{2^n}$.
Clearly f vanishes on A and $f \equiv 1$ on $X - \cup_{n\in\mathbb{N}}\mathcal{O}_n$.
To show f is continuous, recall, $f_n \leq 1 \quad \forall\ n$. It follows that $f \leq \sum_{n\in\mathbb{N}} \frac{1}{2^n}$ which is absolutely convergent. f is therefore continuous. □

Theorem 19 (**Nagata-Smirnov Metrization Theorem**). *A space is metrizable if and only if it is regular and has a basis that is countably locally finite.*

Proof. We will prove this in two steps. In step 1 we will assume X is metrizable and show X is regular and has a basis that is countably locally finite. Step 2 will prove the converse.

Step 1: *Let X be metrizable, with a metric d. We will show X is regular and has countable locally finite basis*

To prove the regularity of X, we recall *theorem 7* that states that every metrizable space is normal. Also, metrizable spaces are Hausdorff since $\forall x, y \in X$ such that

$d(x,y) = r > 0$, there exists open disjoints balls $B(x, r/3)$ and $B(y, r/3)$. In Hausdorff spaces, normality implies regularity. Therefore X is regular.
To prove X has a countable locally finite basis, let $\mathcal{A}_n$ be a covering of X by open balls of radius $1/n$. By *lemma 15*, there exists a locally finite refinement $\mathcal{B}_n$ of open balls of diameter at the most $2/n$ (since each ball of $\mathcal{B}_n$ is in a ball of $\mathcal{A}_n$ of radius at the most $1/n$) covering X. Let $\mathcal{B} := \bigcup_{n \in \mathbb{N}} \mathcal{B}_n$. $\mathcal{B}$ is countably locally finite since each $\mathcal{B}_n$ is locally finite. We only need to show it is a basis for X.
To prove $\mathcal{B}$ is a basis, we show for any arbitrary ball $B(x, \epsilon)$, there exists an element B of $\mathcal{B}$ containing x such that the ball $B \subset B(x, \epsilon)$. We choose n large enough such that $1/n < \epsilon/2$. Given $\mathcal{B}_n$ covers X, there is a B in the cover $\mathcal{B}_n$ that contains x and have a diameter at the most $2/n < \epsilon$. It follows that $B \subset B(x, \epsilon)$ which implies that $\mathcal{B}$ is a basis of X that is countably locally finite.

Step 2*: To prove the converse, show that any regular space with countably locally finite basis is normal and the remaining follows from the second version of the proof of Urysohn metrization theorem.*
Assume X is regular with a countable locally finite basis $\mathcal{B}$. It follows from *lemma 17* that X is normal and every closed set is a G_δ set in X. We show X is metrizable by embedding X in a metric space $(\mathbb{R}^J, \bar{\rho})$ for some countable index set J.
First, we prove the existence of a sequence of continuous functions $f_n : X \longrightarrow [0, 1/n]$, such that $\forall x_0 \in X$, and for each open neighborhood U of x_0, $f(x_0) > 0$ on U but vanishes outside U. In other words the functions separate points from closed sets.
Let $\mathcal{B} = \cup_{n \in \mathbb{N}} \mathcal{B}_n$ where each collection $\mathcal{B}_n$ is locally finite. For each n and for each element B in $\mathcal{B}_n$, we choose a function $f_{n,B} : X \longrightarrow [0, 1/n]$ such that $f_n(x) > 0 \quad \forall x \in B$ and vanishes outside B. Let $x_0 \in X$ and U be a neighborhood of x_0. There exists a $B \in \mathcal{B}_n$ for some $n \in \mathbb{N}$, such that $x_0 \in B \subset U$ such that $f_{n,B}(x_0) > 0$ and vanishes outside B therefore vanishes outside U.
Let J be the set of all pair (n, B) such that $B \in \mathcal{B}_n$ and define $F : X \longrightarrow [0,1]^J$ such that $F(x) = (f_{n,B}(x))_{(n,B) \in J}$. From *theorem 11(Embedding Theorem)*, F is an embedding of X into $[0,1]^J$ with respect to the product topology. As proven in the *version 2* of the Urysohn metrization theorem, F is continuous, injective and F is unto the image space $Z = F(X)$. All these results except the continuity are preserved when we move to the

finer (larger) topology $(\mathbb{R}^J, \bar{\rho})$ with $\bar{\rho}$ being the uniform metric.
The only remaining step is to prove F is continuous which does not follow from the continuity of each $f_{n,B}$ since we are no longer in the product topology. Recall that the uniform metric is defined by $\bar{\rho}(x,y) = \sup_{n\in\mathbb{N}}\{|x_n - y_n|\}$ where $x = (x_1, x_2, \cdots)$ and $y = (y_1, y_2, \cdots)$. To prove F is continuous, we show $\forall x_0 \in X$ and $\forall \epsilon > 0$, there exist a neighborhood W of x_0, such that

$$x \in W \Longrightarrow \bar{\rho}(F(x_0), F(x)) < \epsilon$$

For a fixed n, we choose a neighborhood U_n of x_0 that intersects all but finitely many elements of the collection $\mathcal{B}_n$. As B ranges over $\mathcal{B}_n$ all but finitely many $f_{n,B}$ are identically 0. Because each function $f_{n,B}$ is continuous, we can choose a neighborhood V of x_0 such that $f_{n,B}$ for $B \in \mathcal{B}_n$ varies by at most $\epsilon/2$ on V. Next we choose such neighborhood V of x_0 and choose $N \in \mathbb{N}$ such that $1/N \leq \epsilon/2$ and define $W = \cap_{n=1}^{N} V_n$. W is the desired neighborhood. The proof follows *verbatim* from the continuity of F in the second version of the proof of *Urysohn metrization theorem (theorem 8)* .
□

0.4 Smirnov Metrization Theorem

As an equivalent statement of the Nagata-Smirnov theorem, the Smirnov metrization theorem provides a condition that extends the property of local metrizability to the metrizability of the whole topological space. Paracompactness, a weaker form of compactness and local metrizability are necessary and sufficient conditions for metrizability. We define some key notions needed for the proof.

Definition 20. *A space X is said to be* ***compact*** *if every open cover has a finite open subcover.*

Definition 21. *A space X is said to be* ***paracompact*** *if every open cover has a* ***locally*** *finite open refinement.*

From the definitions, it is clear that the difference between compactness and paracompactness is the fact that paracompactness is a local property. Also, paracompactness require a refinement, which can be taken from any subsets of the elements (sets) of the cover whereas subcover only refers to picking some elements of the cover. It

follows that every subcover is a refinement while refinements are not necessarily subcovers. It is important to note that some books add an extra condition to the definition of paracompactness, the space must be Hausdorff. Our definition of paracompact here does not make this a requirement
As established in *section 1* that normality is a necessary necessary condition, it is important to investigate if there is some connection between paracompacteness and normality. Here comes our next theorem.

Theorem 22. *Every Paracompact Hausdorff space is normal.*

Proof. We begin by proving regularity. Let x be an arbitrary point in a paracompact Hausdorff space X and $F \subset X$ be a closed set. $X - F$ is open since F is closed. Given that X is Hausdorff, $\forall a \in F$, there exists an open set $U_a \in X$ whose closure is disjoint with x. The collection of open sets $(U_a)_{a \in F}$ together with the open set $X - F$ is an open cover X. From paracompacteness, there is a locally finite open refinement $\mathcal{C}$ covering X. Let $\mathcal{D}$ be the collection of elements of $\mathcal{C}$ that intersect F; $\mathcal{D}$ is a collection of open sets that covers F.
Let D be an element of $\mathcal{D}$. D belongs to $\overline{U}_a$ for some $a \in F$. It follows that $\overline{D}$ is disjoint from x since $\overline{U}_a$ is disjoint from x. Let $V = \cup_{D \in \mathcal{D}} D$. V is an open set containing F and the closure of V denoted by $\overline{V}$ is disjoint from x since the closure of each $D \in \mathcal{D}$ is disjoint from x which concludes X is regular.

To prove normality, we repeat exactly the same process replacing Hausdorff by regularity and repeating the point x by another closed set. □

Theorem 23. *Every metrizable space is paracompact and Hausdorff.*

Proof. First, every metrizable space is Hausdorff. Given a metric d and any arbitrary points a, b such that $d(a, b) = r > 0$, there exist disjoint open balls $B(a, r/4)$ and $B(b, r/4)$.
By *lemma 15*, given the space is metrizable, every cover has a refinement that is locally countably finite. The locally finite refinement is equivalent to paracompactness. □

Theorem 24 (Smirnov Metrization Theorem). *A topological space is metrizable if and only if it is locally metrizable, paracompact, and Hausdorff.*

Proof. Assume X is metrizable. It follows X is locally metrizable. From *theorem 23* X is paracompact and Hausdorff.

To prove the converse, assume X is metrizable and paracompact and Hausdorff. *Theorem 22* guarantees the normality of X . We only need to show X has a basis that is countably locally finite and conclude by the the *Nagata-Smirnov Metrization theorem* that X is metrizable.
Assuming X is a locally metrizable i.e $\forall x \in X$, there exists an open neighborhood U_x of x that is metrizable. There exists on U_x a metric d_x, allowing us to define define open balls $\{B(x,\epsilon) := \{y \in X \mid d_x(x,y) < \epsilon\}$.
Let define

$$\mathcal{C}_{n,x} = \{B(y,1/n) \subset U_x \mid y \in U_x\} \text{ and } \mathcal{A}_n = \bigcup_{x \in X} \mathcal{C}_{n,x}.$$

$\mathcal{A}_n$ covers X. From paracompactness of X, we can find a refinement $\mathcal{D}_n$ of $\mathcal{A}_n$ that is locally finite. Letting $\mathcal{D} = \cup_{n \in \mathbb{N}} \mathcal{D}_n$, we see that $\mathcal{D}$ is countably locally finite, being a countable union of locally finite collections. Finally, we need to show that $\mathcal{D}$ is a basis for X. This is to show that for any arbitrary $x \in X$ and any open neighborhood $\mathcal{O}_x$ of x, there exists $D \in \mathcal{D}$ such that $x \in D \subset \mathcal{O}_x$.
$x \in \mathcal{O}_x$ and the space X being metrizable i.e a open neighborhood U_x of x that is metrizable. $\mathcal{O}_x \bigcap U_x$ is open and metrizable therefore it contains an open ball $B(x,1/n) \in \mathcal{A}_n$ for some very large $n \in \mathbb{N}$. Given $\mathcal{D}_n$ is a refinement of $\mathcal{A}_n$, there exists an open set $D \in \mathcal{D}_n$ that contains x. Hence $x \in D \subset B(x_0,1/n) \subset \mathcal{O}_x \bigcap U_x \subset \mathcal{O}_x$ where $D \in \mathcal{D}_n \subset \mathcal{D}$ which concludes $\mathcal{D}$ is a basis for X. □

Conclusion

Following the the proofs in [1], we have proven the three most important metrization theorems of topological spaces. The Urysohn Metrization theorem provides a sufficient condition while the Nagata-Smirnov theorem and Smirnov theorem provide necessary and sufficient conditions on metrizability.

Bibliography

[1] James R Munkres. Topology. *Prentice-Hall*, 2000.