The Structure of Symplectic Quotients by Hamiltonian Circle Actions

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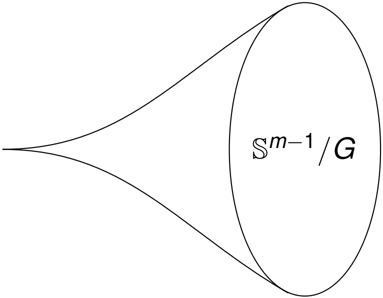
Linear Actions of Compact Groups

Consider an effective linear action of a compact group G on $\mathbb{R}^m$.

(WOLOG: assume $G \subseteq \mathrm{O}(m)$.)

Note that $\mathbb{S}^{m-1} \subseteq \mathbb{R}^m$ is invariant.

$\mathbb{R}^m/G$ looks like:



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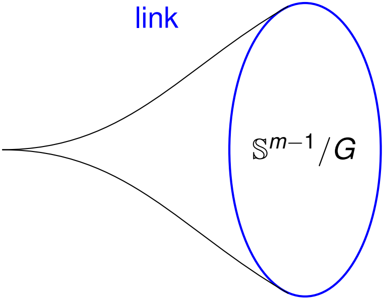

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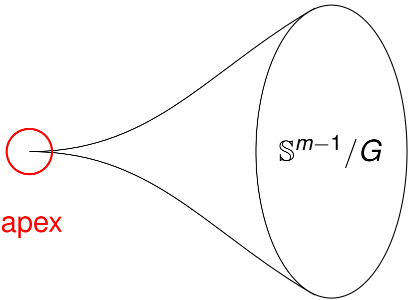
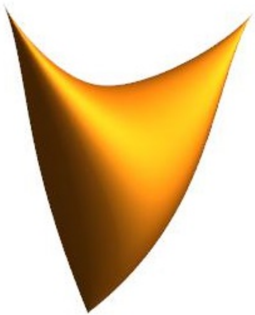


Linear Actions of Compact Groups

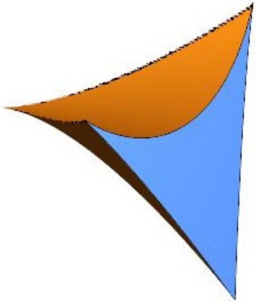
Facts:

1. $\mathbb{R}^m/G$ is a closed semi-algebraic variety.
2. It obtains a differentiable structure by restricting smooth functions on the ambient space.
3. These "differentiable spaces" are examples of **Sikorski differential spaces**.
4. Orbit spaces of proper Lie groupoids locally are diffeomorphic to these types of "cones".
5. **Theorem (W. '15):** There is an essentially injective functor from effective orbifold groupoids to Sikorski differential spaces.

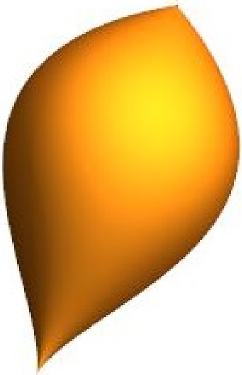
Example: $\mathbb{S}^2/A_4$



Example: $\mathbb{R}^3/T$ (T = tetrahedral group)



Example: $\mathbb{S}^3/\mathbb{S}^1$ with weights -2 and 3



Linear $\mathbb{S}^1$-Actions on $\mathbb{C}^m$

Let $\mathbb{S}^1$ act on $\mathbb{C}^m$ via:

$$e^{i\theta} \cdot (z_1, \ldots, z_m) = (e^{i\alpha_1\theta} z_1, \ldots, e^{i\alpha_m\theta} z_m).$$

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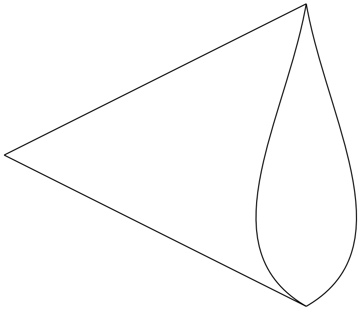

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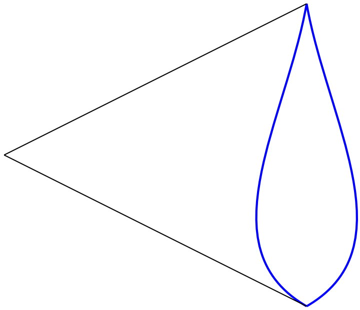


Weighted Projective Space: $\mathbb{CP}(\alpha_1, \ldots, \alpha_m) := \mathbb{S}^{2m-1}/\mathbb{S}^1$

Linear $\mathbb{S}^1$-Actions on $\mathbb{C}^m$

Theorem (Craig, Downey, Goad, Mahoney, W. '16):
There is an essentially injective functor from effective linear circle actions (as groupoids) to differential spaces.

Corollary:
There is an essentially injective functor from effective circle actions on manifolds (as groupoids) to differential spaces in which the codimension-1 orbit-type strata are decorated with either a 0 or a 1.

Symplectic Manifolds

Recall that a symplectic manifold is an (even-dimensional) manifold M

along with a **symplectic form** ω:

a 2-form satisfying

1. $d\omega = 0$,
2. $\forall x \in M, \ \forall u \in T_x M$,

$$\omega(u, v) = 0 \ \forall v \in T_x M \ \Rightarrow u = 0.$$

Symplectic Circle Actions

Let (M, ω) a symplectic manifold.

A **symplectic circle action** on (M, ω) is a Lie group action of $\mathbb{S}^1$ on M such that $g^*\omega = \omega$ for all $g \in \mathbb{S}^1$.

This is equivalent to $\mathcal{L}_{\xi_M}\omega = 0$ where ξ_M is the infinitesimal generator of the action.

By Cartan's magic formula,

$$\mathcal{L}_{\xi_M}\omega = d(\xi_M \lrcorner\, \omega) + \xi_M \lrcorner\, (d\omega) = d(\xi_M \lrcorner\, \omega).$$

Conclude: an $\mathbb{S}^1$-action is symplectic if and only if $\xi_M \lrcorner\, \omega$ is closed.

Hamiltonian Circle Actions

Let $\mathbb{S}^1$ act symplectically on (M, ω).

The action is **Hamiltonian** if $\xi_M \lrcorner \omega$ is exact;

that is, there exists a smooth $\mathbb{S}^1$-invariant real-valued function $\Phi \colon M \to \mathbb{R}$ such that

$$\xi_M \lrcorner \omega = -d\Phi.$$

We call Φ the **momentum map**.

Symplectic Quotient

The level set $\Phi^{-1}(0)$ is invariant under the $\mathbb{S}^1$-action,

we call the quotient space $M/\!/_0\,\mathbb{S}^1 := \Phi^{-1}(0)/\mathbb{S}^1$ the **symplectic quotient.**

Facts:

1. If 0 is a regular value of Φ and $\mathbb{S}^1$ acts freely on $\Phi^{-1}(0)$, then $M/\!/_0\,\mathbb{S}^1$ is a symplectic manifold.
2. If 0 is a regular value, then $\mathbb{S}^1$ acts locally freely on $\Phi^{-1}(0)$ and $M/\!/_0\,\mathbb{S}^1$ is a symplectic orbifold.
3. In general, $M/\!/_0\,\mathbb{S}^1$ is a **symplectic stratified space.**

Question: When is $M/\!/_0\,\mathbb{S}^1$ diffeomorphic to the underlying semi-algebraic variety of an orbifold?

The Commutative Diagram

$$
\begin{array}{ccc}
\Phi^{-1}(0) & \xrightarrow{\;i\;} & M \\
{\scriptstyle \pi'}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\
M/\!\!/_0\,\mathbb{S}^1 & \xrightarrow[\;j\;]{} & M/\mathbb{S}^1
\end{array}
$$

Linear Symplectic Circle Action

Given a linear circle action,

$$e^{i\theta} \cdot (z_1, \ldots, z_m) = (e^{i\alpha_1\theta} z_1, \ldots, e^{i\alpha_m\theta} z_m),$$

(WOLOG: assume all α_js are non-zero)

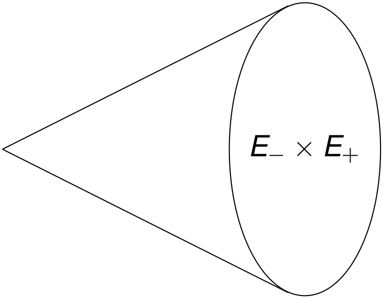
the action preserves the standard symplectic form $\omega = \frac{i}{2} \sum_{j=1}^{m} dz_j \wedge d\overline{z_j}$.

In fact, this action is Hamiltonian with homogeneous quadratic momentum map

$$\Phi(z_1, \ldots, z_m) = \frac{1}{2} \sum_{j=1}^{m} \alpha_j |z_j|^2.$$

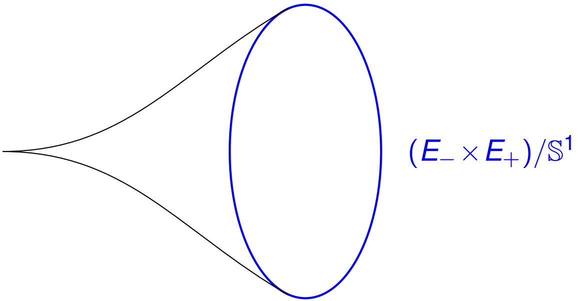
The Level Set

The level set $\Phi^{-1}(0)$ is a cone over a product of ellipsoids E_- and E_+:



The Symplectic Quotient

The symplectic quotient $\mathbb{C}^m /\!/_0 \, \mathbb{S}^1$ is “cone-like” with link an orbifold $(E_- \times E_+)/\mathbb{S}^1$.



Questions

Question: When is $\mathbb{C}^m /\!/_0 \, \mathbb{S}^1$ diffeomorphic to the underlying semi-algebraic variety of an orbifold?

Necessary Condition: $(E_- \times E_+)/\mathbb{S}^1$ is diffeomorphic to $\mathbb{S}^k/\Gamma$ for a finite group Γ.

Work by Herbig-Iyengar-Pflaum, Farsi-Herbig-Seaton, Herbig-Schwarz-Seaton ...

Partial-Answer [HSS]: $\mathbb{C}^m /\!/_0 \, \mathbb{S}^1$ is "regular"-diffeomorphic to the underlying semi-algebraic variety of an orbifold if and only if

$$\dim_{\mathbb{R}}(\mathbb{C}^m /\!/_0 \, \mathbb{S}^1) < 4.$$

Questions

Question: When is $\mathbb{C}^m /\!\!/_0 \mathbb{S}^1$ diffeomorphic to the underlying semi-algebraic variety of an orbit space of a proper Lie group action on a manifold?

Necessary Condition: $(E_- \times E_+)/\mathbb{S}^1$ is diffeomorphic to $\mathbb{S}^k/H$ for a compact group H.

Issue: It is currently unknown when the above condition is satisfied.

Simplifying Assumption: $H \ltimes \mathbb{S}^k$ *is* an orbifold groupoid.

Classifying Spaces

By the theorem on orbifolds, since $\mathcal{G} := \mathbb{S}^1 \ltimes (E_- \times E_+)$ and $\mathcal{H} := H \ltimes \mathbb{S}^k$ are both (effective) orbifold groupoids yielding diffeomorphic quotients, they are Morita equivalent.

If follows that their classifying spaces $B\mathcal{G}$ and $B\mathcal{H}$ are homotopy equivalent.

Since $B\mathcal{G}$ is homotopic to $E\mathbb{S}^1 \times_{\mathbb{S}^1} (E_- \times E_+)$ and $B\mathcal{H}$ is homotopic to $EH \times_H \mathbb{S}^k$,

we have fibrations:

$$\mathbb{S}^1 \to E\mathbb{S}^1 \times (E_- \times E_+) \to B\mathcal{G},$$
$$H \to EH \times \mathbb{S}^k \to B\mathcal{H}.$$

Fibration Long-Exact Sequences

We now can use the long-exact sequences of homotopy groups determined by these fibrations to gather information on H, and the dimensions of E_- and E_+, and we find that H is finite, and either E_- or E_+ has dimension 1.

Conclude: if $\mathbb{C}^m /\!/_0 \mathbb{S}^1$ is diffeomorphic to an orbit space of a proper Lie group action on a manifold (with the simplifying assumption), then it is diffeomorphic to the underlying semi-algebraic variety of an orbifold. In this case, there is at most one negative weight, or at most one positive weight.

Generalising to Manifolds

Generalising to an effective Hamiltonian circle action on a symplectic manifold M,

Theorem (W. '16): If $M/\!/_0\,\mathbb{S}^1$ is diffeomorphic to an orbit space of a Lie group action on a manifold with the simplifying assumption (using local models), then $M/\!/_0\,\mathbb{S}^1$ is diffeomorphic to an orbifold. In this case, at each $\mathbb{S}^1$-fixed point in $\Phi^{-1}(0)$, there is at most one negative weight, or at most one positive weight.

Thank you!

References and Acknowledgements

- All images produced with *Mathematica* 11.
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