# When is a Symplectic Quotient a Diffeological Orbifold?

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# Lie Groupoids

## Definition

A **groupoid** is a small category in which every arrow is invertible.

## Notation

We often denote a groupoid $\mathcal{G} = (\mathcal{G}_1 \rightrightarrows \mathcal{G}_0)$, where $\mathcal{G}_1$ is the set of arrows and $\mathcal{G}_0$ is the set of objects. Also, we have

- the **source map** $\mathrm{s}\colon \mathcal{G}_1 \to \mathcal{G}_0$ sending an arrow ${}_x\overset{g}{\curvearrowright}{}_y$ to $x$,
- the **target map** $\mathrm{t}\colon \mathcal{G}_1 \to \mathcal{G}_0$ sending an arrow ${}_x\overset{g}{\curvearrowright}{}_y$ to $y$,
- the **unit map** $\mathrm{u}\colon \mathcal{G}_0 \to \mathcal{G}_1$ sending an object $x$ to its identity arrow ${}_x\overset{u_x}{\curvearrowright}{}_x$,
- the **multiplication map** $\mathrm{m}\colon \mathcal{G}_1 {}_s\times_t \mathcal{G}_1 \to \mathcal{G}_1$ given by composition, and
- the **inversion map** $\mathrm{inv}\colon \mathcal{G}_1 \to \mathcal{G}_1$ sending ${}_x\overset{g}{\curvearrowright}{}_y$ to ${}_x\overset{g^{-1}}{\curvearrowleft}{}_y$.

# Lie Groupoids

## Definition

A **Lie groupoid** $\mathcal{G}$ is a groupoid in which $\mathcal{G}_1$ and $\mathcal{G}_0$ are smooth manifolds; $\mathrm{s}$, $\mathrm{t}$, $\mathrm{u}$, $\mathrm{m}$, and $\mathrm{inv}$ are all smooth maps; and $\mathrm{s}$ and $\mathrm{t}$ are submersive.

## Definition

A **Lie groupoid homomorphism** $\varphi\colon \mathcal{G} \to \mathcal{H}$ is a **smooth functor**; that is, a pair of maps $\varphi_1\colon \mathcal{G}_1 \to \mathcal{H}_1$ and $\varphi_0\colon \mathcal{G}_0 \to \mathcal{H}_0$ respecting the source and target maps:

$$\begin{array}{ccc} \mathcal{G}_1 & \xrightarrow{\varphi_1} & \mathcal{H}_1 \\ \downarrow\downarrow & & \downarrow\downarrow \\ \mathcal{G}_0 & \xrightarrow[\varphi_0]{} & \mathcal{H}_0 \end{array}$$

# Examples

**Example**

Any Lie group $G$ is a Lie groupoid $G \rightrightarrows *$.

**Example**

Any Lie group action $G \circlearrowright M$ induces a Lie groupoid $G \ltimes M$, called its **action groupoid**,

$$G \times M \rightrightarrows M,$$

with source $(g, x) \mapsto x$ and target $(g, x) \mapsto g \cdot x$.

A $G$-equivariant map $f\colon M \to N$ between $G$-manifolds $M$ and $N$ induces a smooth functor $G \ltimes M \to G \ltimes N$.

# Examples

## Example

Any vector field $X$ on a smooth manifold $M$ has a **local flow**, which is a Lie groupoid $U \rightrightarrows M$ where $U$ is an open neighbourhood of $\{0\} \times M$ in $\mathbb{R} \times M$; the source map is $(t, x) \mapsto x$ and the target map is $(t, x) \mapsto \exp(tX)(x)$, where

$$\frac{d}{dt}\Big|_{t=0} \exp(tX)(x) = X|_x.$$

# Examples

## Example

Any manifold $M$ can be viewed as a **trivial groupoid** $M \rightrightarrows M$, where source and target maps are the identity maps. Any Lie groupoid $\mathcal{G}$ admits a smooth embedding of its units $(\mathcal{G}_0 \rightrightarrows \mathcal{G}_0) \hookrightarrow \mathcal{G}$.

## Example

Any manifold has its **pair groupoid** $M \times M \rightrightarrows M$, where the source and target maps are the first and second projections maps, resp. Any Lie groupoid $\mathcal{G}$ admits the smooth functor

$$\begin{array}{ccc} \mathcal{G}_1 & \xrightarrow{(s,t)} & \mathcal{G}_0 \times \mathcal{G}_0 \\ \downarrow\downarrow & & \downarrow\downarrow \\ \mathcal{G}_0 & = & \mathcal{G}_0. \end{array}$$

# Orbit Spaces

**Definition**

Given a Lie groupoid $\mathcal{G}$, the **orbit** of $x \in \mathcal{G}_0$ is the set $\mathrm{t}(\mathrm{s}^{-1}(x))$. The **stabiliser** of $x$ is the set $\mathrm{s}^{-1}(x) \cap \mathrm{t}^{-1}(x)$. The **orbit space** of $\mathcal{G}$ is the set of orbits $\mathcal{G}_0/\mathcal{G}_1$ equipped with the quotient diffeology.

- Any smooth functor between Lie groupoids $\varphi\colon \mathcal{G} \to \mathcal{H}$ descends to a smooth map $\widehat{\varphi}\colon \mathcal{G}_0/\mathcal{G}_1 \to \mathcal{H}_0/\mathcal{H}_1$.
- This induces a functor $Q$ from the category of Lie groupoids $\mathbf{LieGpoid}$ to the category of diffeological spaces $\mathbf{Diffeol}$.

# Orbifolds

- After the work of Moerdijk, Pronk, and others, effective orbifolds are now often treated as Lie groupoids that are proper, effective, and étale (or Morita equivalent to one of these). [P96, MP97]
- The work of Iglesias-Zemmour – Karshon – Zadka [IZKZ] implies that $Q$ restricted to effective orbifolds as Lie groupoids is "essentially injective" onto effective **diffeological orbifolds**: diffeological spaces locally diffeomorphic to linear quotients of finite groups.
- Essential injectivity means that if $Q(\mathcal{G}) \cong Q(\mathcal{H})$, then $\mathcal{G}$ and $\mathcal{H}$ are "Morita equivalent". To make this precise, we need to extend $\mathbf{LieGpoid}$.

# Morita Equivalence

**Definition**

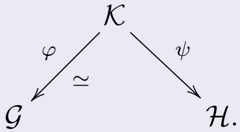
Recall that Lie groupoids are categories. So there should be a notion of "equivalence of categories" for them. Define a **weak equivalence** $\varphi\colon \mathcal{G} \xrightarrow{\simeq} \mathcal{H}$ to be a smooth functor that is "smoothly fully faithful" and "smoothly essentially surjective". (See the work of Moerdijk, Pronk, Lerman, or del Hoyo for definitions [P96, MP97, L10, dH].)

- We would like weak equivalences to be invertible; however, they typically are not.

# Morita Equivalence

## Definition

Define a **generalised morphism from $\mathcal{G}$ to $\mathcal{H}$** to be the span




A **Morita equivalence** is a generalised morphism in which $\psi$ is also a weak equivalence.

- A Morita equivalence can be inverted by switching $\varphi$ and $\psi$.
- We now have a bicategory $\mathbf{LieGpoid}[W^{-1}]$ (there is no need for us to discuss the $2$-arrows).

# $Q$

- The functor $Q$ extends to a pseudofunctor from $\mathbf{LieGpoid}[W^{-1}]$ to $\mathbf{Diffeol}$ (the latter viewed as a $2$-category with trivial $2$-arrows) [W22a].
- Restricting $Q$ to the full sub-bicategory of effective orbifolds, the result is essentially injective (See [W17].)
- In fact, $Q$ can be extended even further to a pseudofunctor $\mathbf{DGpoid}[DW^{-1}]$ to $\mathbf{Diffeol}$, where $\mathbf{DGpoid}[DW^{-1}]$ is the bicategory of diffeological groupoids [vdS21, W22a].

# Sikorski Spaces

## Definition

Fix a set $X$. A **Sikorski structure** $\mathcal{F}$ on $X$ is a family of functions $X \to \mathbb{R}$ such that

1. for any $g \in C^\infty(\mathbb{R}^n)$ and for any $f_1, \ldots, f_n \in \mathcal{F}$,
$$g(f_1, \ldots, f_n) \in \mathcal{F};$$
and
2. if $f\colon X \to \mathbb{R}$ satisfies for every $x \in X$ that there exist an open neighbourhood of $x$ (with respect to the initial topology on $X$ induced by $\mathcal{F}$) and $\widetilde{f} \in \mathcal{F}$ such that $f|_U = \widetilde{f}|_U$, then $f \in \mathcal{F}$.

Call $(X, \mathcal{F})$ a **Sikorski space** (also called a **differential space** in the literature). A map between Sikorski spaces $\varphi\colon (X, \mathcal{F}_X) \to (Y, \mathcal{F}_Y)$ is **Sikorski smooth** if $\varphi^* f \in \mathcal{F}_X$ for every $f \in \mathcal{F}_Y$.

## Example

Let $(X, \mathcal{D})$ be a diffeological space. The diffeologically smooth real-valued functions $C^\infty(X)$ is a Sikorski structure, making $(X, C^\infty(X))$ into a Sikorski space. In fact, there is a functor

$$\Phi\colon \mathbf{Diffeol} \to \mathbf{Sik}$$

from diffeological spaces to Sikorski spaces that sends a diffeologically smooth map to itself.

- It turns out that restricting $\Phi \circ Q$ to effective orbifolds as Lie groupoids is again essentially injective: call the objects in the image of this functor "Sikorski orbifolds".
- Many interesting questions remain about $\Phi \circ Q$, applied to Lie group actions and other Lie groupoids.

# Symplectic Quotients

## Definition

- Let $(M, \omega)$ be a symplectic manifold admitting a Hamiltonian action of a compact Lie group $G$: this means that $g^*\omega = \omega$ for all $g \in G$ and that there is an equivariant momentum map $\mu\colon M \to \mathfrak{g}^*$.
- If $0 \in \mathfrak{g}^*$ is a regular value of $\mu$, then $Z := \mu^{-1}(0)$ is an embedded $G$-invariant submanifold; if additionally the action $G \circlearrowright Z$ is free, then $M/\!/_0\, G := Z/G$ is a symplectic manifold, called the **reduced space** or **symplectic quotient** at $0$.
- Without the freeness assumption of $G \circlearrowright Z$, this action is automatically locally free and the symplectic quotient $Z/G$ is a symplectic orbifold. (That is, $G \ltimes Z$ is an orbifold.)
- If $0 \in \mathfrak{g}^*$ is a critical value, then $Z$ is no longer necessarily a submanifold of $M$, but it is $G$-invariant, and the symplectic quotient $Z/G$ is a symplectic stratified space [SL].

# Symplectic Quotients

- Note that $G \ltimes Z$ will always be a diffeological groupoid, and so the quotient space $Q(G \ltimes Z)$ with its quotient diffeology is another way of thinking about the symplectic quotient besides as a stratified space. Yet another way is thinking of $Z/G$ as a Sikorski space with this the subquotient Sikorski structure (induced by $M$).
- Sometimes the subquotient Sikorski structure on $Z/G$ is a Sikorski orbifold when $Z$ is a critical level set, which is somewhat unexpected. This has been studied in detail by Seaton, Farsi, Herbig, Schwarz, et al. [FHS13, HSS15].

# Example (Cushman-Sjamaar)

## Example

- Consider $\mathbb{S}^1 \circlearrowleft \mathbb{C}^2$ given by $e^{i\theta} \cdot (z_1, z_2) := (e^{i\theta} z_1, e^{-i\theta} z_2)$. This is a Hamiltonian action with respect to the standard symplectic form with momentum map $\mu(z_1, z_2) = |z_1|^2 - |z_2|^2$.
- $Z = \{|z_1| = |z_2|\}$, which is the quadratic cone of a torus.
- The diagonal map $\Delta\colon \mathbb{C} \to Z\colon z \mapsto (z, z)$ is an induction, and the image intersects every $\mathbb{S}^1$-orbit in $Z$ at exactly two points ($(z, z)$ and $(-z, -z)$), except for the fixed point $(0, 0)$.
- It follows that $\mathcal{F}_{Z/\mathbb{S}^1} = \mathcal{F}_{\mathbb{C}/\mathbb{Z}_2}$; this is exactly the cone

$$s^2 + t^2 = u^2, \ u \geq 0.$$

# Questions

- Is $\Phi \circ Q(\mathbb{S}^1 \ltimes Z) = \mathcal{F}_{Z/\mathbb{S}^1}$?
- Is $Q(\mathbb{S}^1 \ltimes Z)$ a diffeological orbifold?
- Under what conditions are symplectic quotients, reduced at critical level sets, diffeological orbifolds?
- Note: $\mathbb{S}^1 \ltimes Z$ is not Morita equivalent as a diffeological groupoid to an orbifold groupoid.

Thank you!

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