Dimension in Diffeology

Bishal Panthee

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1 Definition

1.1 Parametrisation in sets:

Let X be a non empty set. A parametrisation of X is a map $p : A \to X$ defined on a Euclidean domain.
We denote the set of all parametrisations in X by 'Param(X)'.

1.2 Diffeology.

Given a nonempty set X. Any subset $\mathcal{P}$ of 'Param(X)' is said to be the diffeology on set X if it satisfies the following three axioms:

(1) (Covering) The constant parametrisations belongs to set $\mathcal{P}$.
(2) (Locality) Let $p : A \to X$ be a parametrisation in X.Then p belongs to $\mathcal{P}$ if for every point a of A, there is an open neighbourhood B of a such that $p|B$ belongs to $\mathcal{P}$.
(3) (Smooth Compatibility) For all $p \in \mathcal{P}$, for every real domain B and for every smooth map $f : B \to A$, the composite $p \circ f : B \to X$ belongs to $\mathcal{P}$.

The pair $(X, \mathcal{P})$ is a diffeological space where X is the underlying set with its diffeology $\mathcal{P}$.

1.2.1 Example 1.

The set of all smooth parametrisation in a domain A of $\mathbb{R}^n$ is a diffeological space.

1.3 Plots of the Diffeological Space.

The elements of the diffeology $\mathcal{P}$ of the diffeological space X are called the plot of the diffeological space X.

1.4 Diffeological Smooth Map.

Given X_1 and X_2 are two diffeological spaces. We say that the map $g : X_1 \to X_2$ is diffeologically smooth map if for every plot p of X_1, the composition $g \circ p$ is a plot of X_2 .

1.4.1 Example 1.

Every infinitely differentiable maps from $\mathbb{R}^m$ to $\mathbb{R}^n$ are diffeological smooth maps.

1.5 Subset Diffeology.

Suppose the pair $(X, \mathcal{D})$ is a diffeological space and Y is a subset of X.
Then, the subset diffeology on set Y is the set of all plots in $\mathcal{D}$ with the image in Y.

1.5.1 Example 1.

Let $A = [0,1] \times \{0,1\} \cup \{0,1\} \times [0,1]$ be a square and $A \subset \mathbb{R} \times \mathbb{R}$.

The set of the parametrisations of the square which, regarded as a parametrisations of $\mathbb{R} \times \mathbb{R}$ are smooth , is a diffelogical space.
Such diffeology is a subset diffeology with respect to the diffeolgical space $\mathbb{R} \times \mathbb{R}$.

1.6 Quotient Diffeology.

Let X be a diffeological space and let $\phi : X \to Y (= X/\sim)$ be the quotient map where $\sim$ is an equivalence relation on X.

The set of all plots $p : U \to Y$ is said to be the the quotient diffeology on set Y if for every $u \in U$, there is an open neighbourhood V of u and a plot $p' : V \to X$ such that $\phi \circ p' = p|V$.

1.6.1 Example 1.

Let $S^1 = \{z \in C : z\bar{z} = 1\} \subset \mathbb{C}$ be a circle. The parametrisations $p : U \to S^1$ satisfying: for all u in U, there exists an open neighbourhood V of u and a smooth parametrisation $p' : V \to \mathbb{R}$ such that $p|V : r \mapsto exp(2\pi i p'(r))$ forms a quotient diffeology of $\mathbb{R}$ and $S^1 \simeq \mathbb{R}/\mathbb{Z}$.

1.6.2 Example 2.

Let $\alpha \in \mathbb{R} - \mathbb{Q}$. Let T_α be the quotient set $\mathbb{R}/(\mathbb{Z} + \alpha\mathbb{Z})$. Let $\phi_\alpha : \mathbb{R} \to T_\alpha$ be the quotient map. Let D be the set of parametrisations $p : U \to T_\alpha$ such that for all $u \in U$, there exists an open neighbourhood V of u and a smooth parametrisation $p' : V \to \mathbb{R}$ such that $p|V = \pi_\alpha \circ p'$.
Then, D forms a quotient diffeology of T_α

1.7 Dimension of a diffelogical space.

Let X be a space equipped with the diffeology $\mathcal{D}$. The dimension of X, denoted by dim(X), is defined as the infimum of the dimension of the generating families $\mathcal{F}$ of $\mathcal{D}$.

That is:
$dim(X) = inf\{dim(\mathcal{F})|\mathcal{F} \subset \mathcal{D} \text{ and } \mathcal{D} = \langle\mathcal{F}\rangle\}$

where $dim(\mathcal{F}) = sup\{dim(\boldsymbol{F})|\boldsymbol{F} \in \mathcal{F}\}$

Also, note that if $U \in Domains(\mathbb{R}^n)$, then $dim(U) = n$

But, if the diffeology $\mathcal{D}$ has no generating family with finite dimension, then $dim(X) = \infty$

Also, the dimension is a diffeological invarint.(Art 1.79 of [IZ])

1.8 Pushforward of diffeologies.

Given $(X, \mathcal{D})$ is a diffeological space and Y is any set. The pushforward of the diffeology $\mathcal{D}$ of X by any map $f : X \to Y$ is defined as the finest diffeology of Y such that the map f is smooth.
We denote it by $f_*(\mathcal{D})$.

1.9 Subduction.

Consider a map $f : X \to Y$ between two diffeological spaces X and Y. We say that the map f is a subduction if it holds the following conditions:

a. f is surjective.

b. The diffeology $\mathcal{D}'$ of Y is the pushforward of the diffeology $\mathcal{D}$ of X.That is $f_*(\mathcal{D}) = \mathcal{D}'$

2 Dimension of $\mathbb{R}^n/O(n,\mathbb{R})$ and half lines.

2.1 Dimension of $\mathbb{R}^n/O(n,\mathbb{R})$

Assume that $O(n)$ is the group of rotations and reflections of $\mathbb{R}^n$ and $\mathbb{R}^n/O(n,\mathbb{R})$ is the quotient diffeological space obtained via the equivalence relation induced by $O(n)$.

Set, for each natural number $n, \Delta_n = \mathbb{R}^n/O(n,\mathbb{R})$

We shall first establish the following two statements and we then deduce the dimension of $\mathbb{R}^n/O(n,\mathbb{R}) = n$.

1. Δ_n is equivalent to the set $[0,\infty)$ equipped with the pushforward of the standard diffeology of $\mathbb{R}^n$ by the function $v_n : \mathbb{R}^n \to [0,\infty)$ with $v_n(x) = ||x||^2$

2. The plot v_n can not be lifted locally at the point 0 along a p -plot with $p < n$.

Claim 1:

The quotient space Δ_n is equivalent to the set $[0,\infty)$ equipped with the pushforward of the standard diffeology of $\mathbb{R}^n$ by the function $v_n : \mathbb{R}^n \to [0,\infty)$ with $v_n(x) = ||x||^2$

Proof:

For $x, x' \in \mathbb{R}^n$, $x \sim x'$ if there exists an element $\mathcal{A}$ of $O(n,\mathbb{R})$ such that $x' = \mathcal{A}x$.

Let $[0,\infty)$ be the set and the map $v_n : \mathbb{R}^n \to [0,\infty)$ be a surjection such that $v_n(x) = ||x||^2$

Moreover, $||x|| = ||x'||$ iff $x' = Ax$ for some A in $O(n,\mathbb{R})$. Thus there is a bijection between the orbits of $O(n,\mathbb{R})$.
Suppose that $\pi_n : \mathbb{R}_n \to \Delta_n$ is the projection map from $\mathbb{R}_n$ onto its quotient. Now, there is a natural bijection map $f : \Delta_n \to [0,\infty)$ such that $f \circ \pi_n = v_n$.

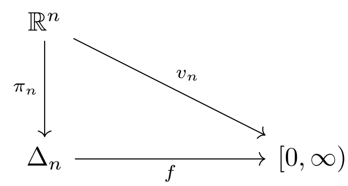
Since in each quotient diffeology, the projection class is a subduction $f \circ \pi_n = v_n$ is subduction. Applying the definition of smooth maps from quotients, we get,

$f \circ \pi_n = v_n$ is subduction if and only if f is subduction.

Hence, by the uniqueness of quotients, f is differomishm.(Art 1.52 of [IZ])

Thus the map $f : class(x) \to v_n(x)$ is a diffeomorphism from Δ_n to a set $[0,\infty)$ where $[0,\infty)$ is equipped with the push forward diffeology of $\mathbb{R}^n$ by v_n.

Diagram:



Claim 2: The plot v_n can not be lifted locally at the point 0 along a p -plot with $p < n$.

Proof: Suppose that the space $([0,\infty),\mathcal{D}_n)$ is the representation of Δ_n where $\mathcal{D}_n$ is the pushforward of the standard diffeology of $\mathbb{R}^n$ by v_n.
Also, the elements of $\mathcal{D}_n$ consists of the parametrisations which locally can be lifted along v_n by smooth parametrisations of $\mathbb{R}_n$.

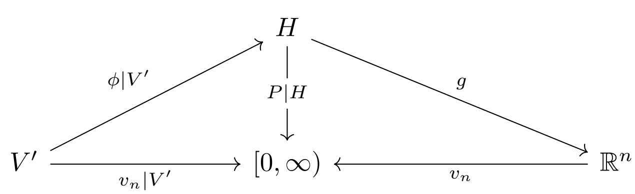
Assume that 0_k represents the zero of $\mathbb{R}^k$. Since $dim(v_n) = n$, the dimension of $\Delta_n \leq n$.

Suppose, if possible, the plot v_n, an element of $\mathcal{D}_n$, can be lifted at the point 0_n along a p-plot $P : U \to \Delta_n$,with $dim(P) = p < n$.

So, there is a smooth parametrisation $\phi : V \to U$ such that $P \circ \phi = v_n|V$
Without loss of generality, suppose that $P(0_p) = 0$ and $\phi(0_n) = 0_p$.

Again, since $P \in \mathcal{D}_n$, it can also be lifted locally at the poiint 0_p along v_n. Then, there is a smooth parametrisation $g : H \to \mathbb{R}^n$ such that $v_n \circ g = P|H$ where H is an open subset of U containing 0_p.

Commutative diagram:



Set $V' = \phi^{-1}(H)$,$F = g \circ \phi|V'$

Then, $v_n|V' = v_n \circ F$ with $F \in C^\infty(V', \mathbb{R}^n), 0_n \in V'$ and $F(0_n) = 0_n$

That is: $\||x||^2 = \||F(x)||^2$

Taking first derivative,

$\forall x \in V'$ and $\forall \Delta x \in \mathbb{R}^n$, we have $2 \cdot x \cdot \Delta x = 2 \cdot F(x) \cdot D(F)(x) \cdot \Delta x$

But , at the point 0_n, $F(0_n) = 0_n$,
the second derivative gives $[D(F)(0_n)]^t \cdot [D(F)(0_n)] = 1_n$

Denote $C = [D(F)(0_n)]$ and C^t is the transpose matrix of C. Also, we have $F = g \circ \phi|V'$.
We can further write $D(F)(0_n) = D(g)(0_p) \circ D(\phi)(0_n)$

$D(F)(0_n) = A \circ B$ where $A = D(g)(0_p)$ and $B = D(\phi)(0_n)$ and A and B are both tangent linear maps at the points 0_p and 0_n respectively.

So clearly A belongs to the space of linear maps from $\mathbb{R}^p$ to $\mathbb{R}^n$ and B belongs to the space of linear maps from $\mathbb{R}^n$ to $\mathbb{R}^p$.

Now, $D(F)(0_n) = D(g)(0_p) \circ D(\phi)(0_n)$
$D(F)(0_n) = AB$
$C = AB$ and
$1_n = C^tC = B^tA^tAB$

Since B belongs to the space of linear maps from $\mathbb{R}^n$ to $\mathbb{R}^p$, the rank of the B must be less or equal to p.

But by our hypothesis, $dim(P) = p < n$
and it further implies that the rank of 1_n is also less than n.

This is a contradiction since the rank of 1_n is n. Hence , the plot v_n can not be lifted locally at the point 0 along a p -plot with $p < n$.

Now,
For the dimension of $\mathbb{R}^n/O(n, \mathbb{R})$.

Since v_n is the generator of the diffeology of $\Delta_n = O(n, \mathbb{R}^n)$ which is represented by the space $([0, \infty), D_n)$, the set $\mathcal{F} = \{v_n\}$ is a generating family for Δ_n

So, $dim(\Delta_n) = inf\{dim(\mathcal{F}) | \mathcal{F} \subset \Delta_n \text{ and } \Delta_n = \langle\mathcal{F}\rangle\} \leq n$.

Further assume that $dim(\Delta_n) = p$ where $p < n$. v_n can be lifted locally at the point 0_n along an element P' of some generating family $\mathcal{F}'$ for Δ_n since v_n is a plot of Δ_n. This implies $dim(\mathcal{F}') = p$

So, we have $dim(P') \leq p < n$, which is not possible by our claim 2: the plot v_n can not be lifted locally at the point 0_n along a p -plot with $p < n$.

This means we must have $dim(\Delta_n) = n$

Also,

Since the dimension is diffeological invariant, for all $n \neq m$ $\Delta_n = \mathbb{R}^n/O(n,\mathbb{R})$ and $\Delta_m = \mathbb{R}^m/O(m,\mathbb{R})$ are not diffeomorphic. (Art 1.79 of [IZ])

2.2 Dimension of the half line.

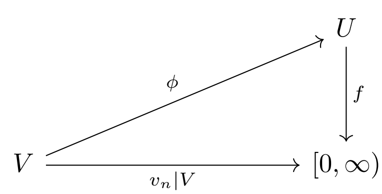
Suppose $\Delta_\infty = [0,\infty) \subset \mathbb{R}$ is equipped with the subset diffeology D_∞. We shall show that $dim(\Delta_\infty) = \infty$

For this, let $dim(\Delta_\infty) = N$ where N is a finite number. Define a map $v_n : \mathbb{R}^n \to \Delta_\infty$ by $v_n(x) = ||x||^2$ and v_n are plots of Δ_∞

Also, v_n are smooth parametrisations of $\mathbb{R}$ and $v_n(x) = ||x||^2$ lies in D_∞.

So v_n can be lifted locally at the point 0_n along some p-plot of Δ_∞ with $p \leq N$ where $P \in \mathcal{D}_\infty$ with $dim(P) = p$.
Again, for any $n > N$, there is a smooth parameterisation $f : U \to \mathbb{R}$ such that the function values lies in $[0,\infty)$. This means f is a p- plot of Δ_∞ and there exists a smooth parametrisation $\phi : V \to U$ such that $f \circ \phi = v_n|V$.

Diagram:



Without loss of generality, suppose $0_p \in U$ and $\phi(0_n) = 0_p$.
Then $f(0_p) = 0$.

Also, we have $f \circ \phi = v_n|V$.

Taking the first derivative of v_n at a point x on V' and $V' = \phi^{-1}(V)$,

we get $D(f)\phi(x) \circ D(\phi)(x) = x$

since $f \in C^\infty(U,\mathbb{R})$, non-negative and $f(0) = 0$, we get $D(f)(0_p) = 0$

Again, taking second derivative at the point 0_n,

we get $1_n = [D(\phi)(0)]^t[D^2(f)(0)][D(\phi)(0)]$

The matrix $D(\phi)(0)$ represents the tangent map of f at 0_p But, since $n > N$ was choosen and $p \leq N$, we have $p < n$.

So, the tangent map $D(\phi)(0)$ of f at 0_p has a non zero kernel and then it implies that matrix $[D(\phi)(0)]^t[D^2(f)(0)][D(\phi)(0)]$ is degenerate.

This is not possible since 1_n is not degenerate. Hence the dimension of $\Delta_\infty = \infty$.

References:

[IZ] Patrick Iglesias-Zemmour, *Diffeology, Mathematical Surveys and Monographs, Vol. 185*, American Mathematical Society, 2010,