The Differential Structure of an Orbifold

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Introduction

Let $G_1 \rightrightarrows G_0$ be a Lie groupoid.

Consider the “quotient”, or “orbit space”, of the groupoid.

This probably is not a manifold. So what is it?

Introduction

Answer: it could be a stack, sheaf of sets on the site of smooth manifolds, a diffeological space, a (Sikorski) differential space, a topological space... (There are many other answers as well.)

All of the examples of possible structures just mentioned give different answers to what our quotient space is; however, they are all related.

The stack induces the sheaf of sets, the sheaf induces the diffeology, the diffeology induces the differential structure, and the differential structure induces the topology.

Each time you move along this chain, you lose information.

Introduction

Question: Given an effective orbifold (viewed as an effective proper étale Lie groupoid) how far along the "chain of quotients" can we go while still maintaining enough invariants with which we can rebuild the Lie groupoid?

The *quotient topology* obviously is a bad candidate, and cannot tell the difference between the group actions of $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ ($n > 0$) on the plane $\mathbb{R}^2$ by rotations.
That is, all topological quotients $\mathbb{R}^2/\mathbb{Z}_n$ are homeomorphic.

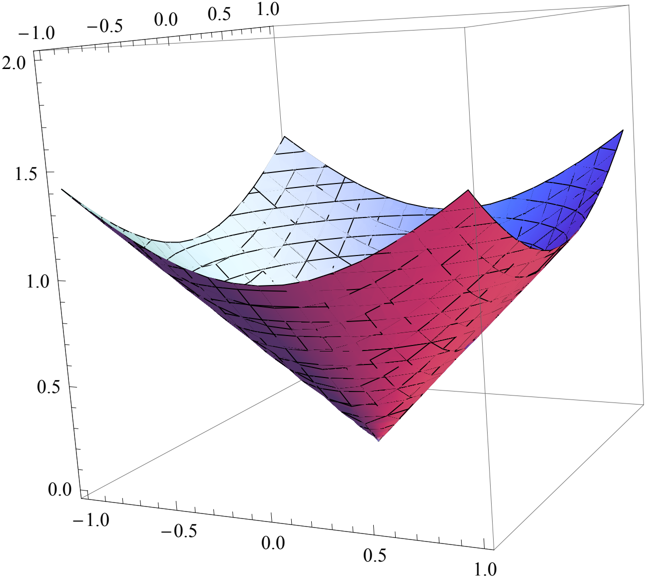
Differential Structure

Let $G_1 \rightrightarrows G_0$ be an effective proper étale Lie groupoid.

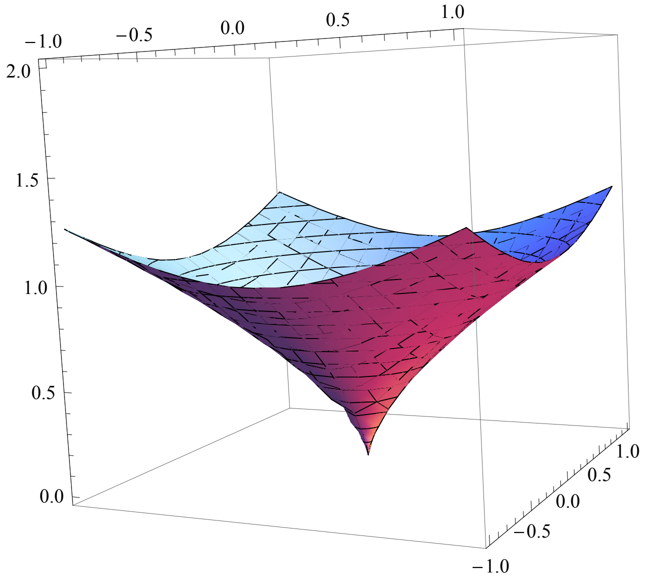
The "differential structure" on the quotient G_0/G_1 is the set of real-valued functions f on G_0/G_1 which pull back to smooth invariant functions on G_0.

By a result of Gerald Schwarz, the differential structure is locally generated via invariant polynomials on the local (linear) charts. Furthermore, locally, G_0/G_1 can be embedded into Euclidean space.

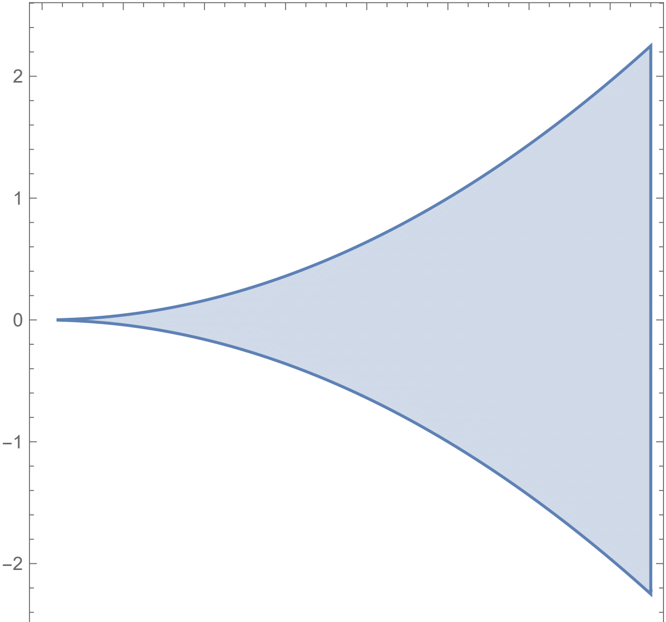
$\mathbb{R}^2/\mathbb{Z}_2$



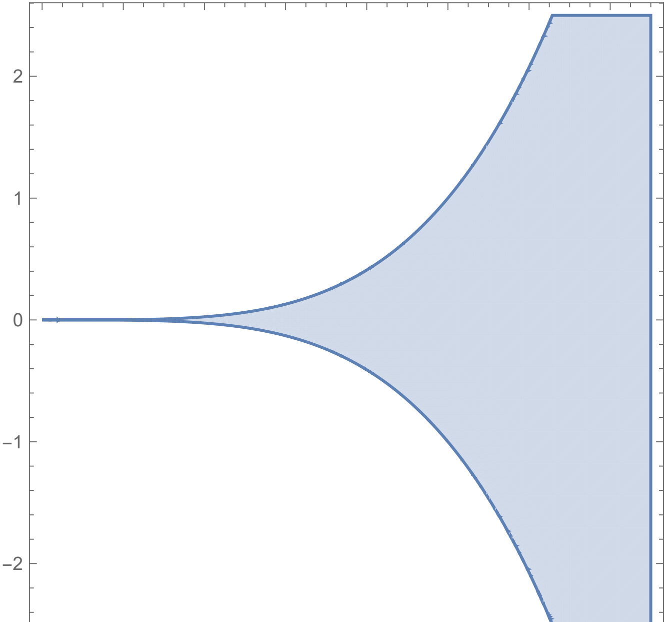
$\mathbb{R}^2/\mathbb{Z}_3$



$\mathbb{R}^2/D_4$



$\mathbb{R}^2/D_8$



The Main Result

Theorem (W. 2015): Let $G_1 \rightrightarrows G_0$ be an effective proper étale Lie groupoid.

The differential structure on G_0/G_1 contains enough information in order to completely reconstruct the groupoid $G_1 \rightrightarrows G_0$ (up to Morita equivalence).

Reconstructing the Charts

To reconstruct the groupoid, it is enough to reconstruct the charts of the orbifold.

To reconstruct a chart about a point, it is enough to know what the link at that point is (as an orbifold).

It follows that one can reconstruct a chart using induction on the dimension of the orbifold, provided one knows three things:

1) the topology on the orbifold,
2) the natural stratification on the orbifold,
3) and the isotropy groups at points in low-codimension strata.

The Topology

It is an exercise in point-set topology to show that the initial topology of the differential structure is the standard topology on the orbifold, and this topology is an invariant of the differential structure.

(Recall the initial topology is the weakest topology making all of the functions of the differential structure continuous.)

The Stratification

There is a notion of vector field of a differential structure. Using theory developed by Śniatycki, one can obtain:

Theorem: (W.)
The natural stratification of G_0/G_1 is given exactly by the admissible sets of the family of all vector fields of the differential structure on G_0/G_1.

(Actually, this theorem holds for any proper Lie groupoid $G_1 \rightrightarrows G_0$.)

The Stratification

Isomorphisms of differential structures take vector fields to vector fields.

So, the stratification is an invariant of the differential structure.

Isotropy Groups at Codimension-2 Strata

A (consequence of a) theorem of Haefliger and Ngoc Du states that the isotropy group at any point can be obtained using the topology, stratification, and orders of isotropy groups at each codimension-2 stratum.

So we now only need to find the order of the isotropy groups at codimension-2 strata, and then we will be done.

This, in turn, means we need to know how to find the order of a finite group Γ given an orbit space $\mathbb{R}^2/\Gamma$, where Γ acts on $\mathbb{R}^2$ orthogonally.

Dihedral and Cyclic Groups

A standard result, attributed to Leonardo di Vinci, is that the only such (effective) group actions are when Γ is a dihedral group D_k, or a cyclic group $\mathbb{Z}_k$.

The order of the dihedral group quotient is $2(\mu+1)$, and that of the cyclic group is $\mu+1$, where μ is the Milnor number of $f : \mathbb{R}^n \to \mathbb{R}$ at $0 \in \mathbb{R}^n$, where

$$f^{-1}(0) = \mathbb{R}^2/\Gamma \subset \mathbb{R}^n$$

and $n = 2$ (in the case $\Gamma = D_k$) or $n = 3$ (in the case $\Gamma = \mathbb{Z}_k$).

The function f is an invariant of the differential structure up to diffeomorphism, and the Milnor number is invariant under diffeomorphism.

This completes the proof.

In Terms of Functors

The theorem on the differential structure of an orbifold can be rephrased in the following form:

Corollary to Main Theorem: There is a functor from the (weak 2-)category of effective proper étale Lie groupoids to (Sikorski) differential spaces that is essentially injective on objects.

Thank you!

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