Infinite-Dimensional Calculus & Symplectic Topology

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March 2025

Joint Work

- This is joint work with Yael Karshon (in progress).

Motivation

- Symplectic topologists are interested in infinite-dimensional groups such as $\mathrm{Diff}(M)$, $\mathrm{Symp}(M, \omega)$, and $\mathrm{Ham}(M, \omega)$, where M is a compact manifold, and (M, ω) a compact symplectic manifold.
- They often employ either methods from functional analysis, as well as purely topological constructions, to study these groups.

Motivation

- For instance, the Flux Conjecture (proved by Ono in [O06]) is that the flux group Γ_ω in the following exact commutative diagram [McD04] is discrete:

$$
\begin{array}{ccccc}
\pi_1(\mathrm{Ham}(M,\omega)) & \longrightarrow & \pi_1(\mathrm{Symp}_0(M,\omega)) & \xrightarrow{\mathrm{Flux}} & \Gamma_\omega \\
\downarrow & & \downarrow & & \downarrow \\
\widetilde{\mathrm{Ham}}(M,\omega) & \longrightarrow & \widetilde{\mathrm{Symp}}_0(M,\omega) & \xrightarrow{\mathrm{Flux}} & H^1(M,\mathbb{R}) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Ham}(M,\omega) & \longrightarrow & \mathrm{Symp}_0(M,\omega) & \xrightarrow{\mathrm{Flux}} & H^1(M,\mathbb{R})/\Gamma_\omega
\end{array}
$$

- Here $\mathrm{Flux}(\varphi_t) := \int_0^1 [\omega(\dot{\varphi}_t, \cdot)]\, dt \in H^1(M,\mathbb{R})$.

Motivation

Question

What can theories generalising the differential geometry/topology of smooth manifolds, such as **diffeology** or **Sikorski spaces**, do for us?

- Diffeology provides an “internal” perspective on a space, whereas Sikorski structures provide an “external” perspective. Considering both structures together yields a lot of information about the space.

Definition ([Ha82,Mi84])

- Let E and F be locally convex spaces, and let $U \subseteq E$ be open.
- A function $f\colon U \to F$ is C^1 if it is continuous, and for every $u \in U$ and $h \in E$, the limit

$$Df(u;h) := \lim_{t \to 0} \frac{f(u+th) - f(u)}{t}$$

  exists and is continuous as a map $U \times E \to F$.
- Continuing recursively, one define C^k functions, and then **infinitely-differentiable** functions as $\bigcap_k C^k(U,F)$.

Diffeology

Definition ([IZ13])

- Let X be a set.
- A **parametrisation** $p\colon U_p \to X$ is a map from an open subset U_p of some Euclidean space.
- A **diffeology** $\mathcal{D}_X$ on X is a family of parametrisations satisfying
  1. all constant parametrisations are in $\mathcal{D}_X$,
  2. if p is a parametrisation and $\{U_\alpha\}$ an open cover of U_p such that for each α
     $$p|_{U_\alpha} \in \mathcal{D}_X$$
     then $p \in \mathcal{D}_X$,
  3. if $p \in \mathcal{D}_X$ and $f\colon V \to U_p$ is smooth with V an Euclidean open subset then $p \circ f \in \mathcal{D}_X$.
- Call $(X, \mathcal{D}_X)$ a **diffeological space** and each $p \in \mathcal{D}_X$ a **plot**.

Diffeology

Definition

A map $F\colon (X, \mathcal{D}_X) \to (Y, \mathcal{D}_Y)$ is **diffeologically smooth** if $F \circ p \in \mathcal{D}_Y$ for every $p \in \mathcal{D}_X$.

- We obtain a "complete, co-complete quasi-topos" [BH11], denoted $\mathbf{Diffeol}$. In particular, we obtain a category admitting all subsets, quotients, products, coproducts, and function spaces.

Example

A (smooth) manifold comes with a standard diffeological structure: all smooth parametrisations into it. In fact, the category of smooth manifolds forms a full subcategory of $\mathbf{Diffeol}$.

Irrational Tori

Definition

Let X be a diffeological space with an equivalence relation $\sim$ and quotient map $\pi\colon X \to X/{\sim}$. A parametrisation $p\colon U_p \to X/{\sim}$ is a plot in the **quotient diffeology** if for every $u \in U_p$ there exists an open neighbourhood V of u and a plot $q\colon V \to X$ such that $p|_V = \pi \circ q$.

Example

Fix an irrational number α. Consider the action of the group $\mathbb{Z}^2$ on $\mathbb{R}$ by

$$(m,n)\cdot x = x + m + \alpha n.$$

The quotient group $T_\alpha := \mathbb{R}/\mathbb{Z}^2$ has trivial topology, but its diffeology is rich. This space is an example of an **irrational torus**.

Diffeology for Locally Convex Spaces

Example

Let E be a locally convex space. The collection of all infinitely-differentiable parametrisations of E forms a diffeology, denoted $\mathcal{D}_E$.

Example ([L92])

Fréchet spaces form a full subcategory of $\mathbf{Diffeol}$.

D-Topology

Definition

Let $(X, \mathcal{D}_X)$ be a diffeological space. The **D-topology** $\tau_{\mathcal{D}_X}$ on X is the strongest topology making all plots continuous.

Question

Given a locally convex space E with topology τ_E, when is $\tau_E = \tau_{\mathcal{D}_E}$?

- It is always true that $\tau_E \subseteq \tau_{\mathcal{D}_E}$.

Sikorski Spaces

Definition ([Ś13])

Let X be a set. A **Sikorski (differential) structure** on X is a family of real-valued functions $\mathcal{F}$ on X satisfying

1. if $g \in C^\infty(\mathbb{R}^n)$ and $f_1, \dots, f_n \in \mathcal{F}$, then $g(f_1, \dots, f_n) \in \mathcal{F}$; and
2. with respect to the initial topology $\tau_\mathcal{F}$ on X generated by $\mathcal{F}$, if $f\colon X \to \mathbb{R}$ admits a function $f_x \in \mathcal{F}$ for each $x \in X$ satisfying
$$f|_{U_x} = f_x|_{U_x}$$
on an open neighbourhood U_x of x, then $f \in \mathcal{F}$.

$(X, \mathcal{F})$ is called a **Sikorski (differential) space.**

Definition

A map $\varphi\colon (X, \mathcal{F}_X) \to (Y, \mathcal{F}_Y)$ is **Sikorski smooth** if $\varphi^* f \in \mathcal{F}_X$ for every $f \in \mathcal{F}_Y$.

- Sikorski spaces form a category $\mathbf{Sik}$ admitting subspaces, products, coproducts, and quotients.
- Function spaces are a little more difficult to deal with.

Example

Manifolds come with a standard Sikorski structure: all smooth real-valued functions. In fact, manifolds form a full subcategory of $\mathbf{Sik}$.

Singular Varieties

Definition

Given a Sikorski space $(X, \mathcal{F}_X)$ and a subset $Y \subseteq X$, the **subspace Sikorski structure** $\mathcal{F}_Y$ on Y is given by all real-valued functions $f\colon Y \to \mathbb{R}$ such that for any $y \in Y$ there is an open neighbourhood $U \subseteq X$ of y and a function $\widetilde{f} \in \mathcal{F}_X$ such that

$$f|_{U \cap Y} = \widetilde{f}|_{U \cap Y}.$$

Example

Any level set of a smooth function, such as a real algebraic/analytic variety, comes equipped with a subspace Sikorski structure.

Locally Convex Space Sikorski Structure

Example

Let E be a locally convex space. The collection of all infinitely-differentiable real-valued functions forms a Sikorski structure, denoted $\mathcal{F}_E$.

Question

Given a locally convex space E with topology τ_E, when is $\tau_E = \tau_{\mathcal{F}_E}$, where $\tau_{\mathcal{F}_E}$ is the initial topology generated by $\mathcal{F}_E$?

- It is always true that $\tau_{\mathcal{F}_E} \subseteq \tau_E$.

Reflexivity

- Given a diffeological space $(X, \mathcal{D}_X)$, the set of diffeologically smooth real-valued functions, denoted $\Phi\mathcal{D}_X$, is a Sikorski structure on the underlying set of X.
- In fact, these spaces $(X, \Phi\mathcal{D}_X)$ form a subcategory of $\mathbf{Sik}$ isomorphic to the category of Frölicher spaces.
- In the other direction, given a Sikorski space $(X, \mathcal{F}_X)$, the set of all Sikorski smooth parametrisations into X, denoted $\Pi\mathcal{F}_X$, is a diffeology.
- Again, the diffeological spaces $(X, \Pi\mathcal{F}_X)$ form a subcategory of $\mathbf{Diffeol}$ isomorphic to the category of Frölicher spaces.

Reflexivity

- Φ and Π in the previous slide are in fact functors that send maps to themselves:

$$\mathbf{Diffeol} \underset{\Pi}{\overset{\Phi}{\rightleftarrows}} \mathbf{Sik}$$

Definition ([BKW23])

- If $\Pi \circ \Phi(X, \mathcal{D}_X) = (X, \mathcal{D}_X)$, then $\mathcal{D}_X$ is called **reflexive.**
- If $\Phi \circ \Pi(X, \mathcal{F}_X) = (X, \mathcal{F}_X)$, then $\mathcal{F}_X$ is called **reflexive.**

Reflexivity

Examples

- Manifolds have reflexive diffeological and Sikorski structures.
- Manifolds with corners have reflexive Sikorski structures. They have reflexive diffeological structures if these are locally induced by subset diffeologies on the orthants used to make the charts.
- Quotient spaces of proper Lie group actions (or proper Lie groupoids) have reflexive Sikorski structures.
- The union of the three axes of $\mathbb{R}^3$ has a reflexive Sikorski structure, but the union of three distinct lines through the origin in $\mathbb{R}^2$ does not.

Question

Given a locally convex space E, when is $\mathcal{D}_E$ and/or $\mathcal{F}_E$ reflexive?

Sequential Spaces

Definition

A topological space X is **sequential** if for every subset $S \subseteq X$,

$$\overline{S} = \{x \in X \mid \exists (x_n) \text{ in } S \text{ s.t. } x_n \to x\}.$$

Example

Any first-countable space is sequential.

Example

The D-topology of a diffeological space is sequential. (This follows from the fact that diffeological spaces are colimits of their plot domains.)

Definition

Given a locally convex space E, a sequence (x_n) in E **converges fast** to x_∞ if for every $k \in \mathbb{N}$, the set

$$\{n^k(x_n - x_\infty)\}_{n\in\mathbb{N}}$$

is bounded.

Theorem

Given a sequential locally convex space E, if every convergent sequence in E admits a fast-converging subsequence, then $\tau_E = \tau_{\mathcal{D}_E}$.

Corollary

If E is a metrisable locally convex space, then $\tau_E = \tau_{\mathcal{D}_E}$.

Definition

A locally convex space E is **smoothly regular** if for any $x \in X$ and open neighbourhood $U \ni x$, there is a function $f \in \mathcal{F}_E$ such that $f(x) = 1$ and $\mathrm{supp}(f) \subseteq U$.

Theorem

Given a locally convex space E, $\tau_E = \tau_{\mathcal{F}_E}$ if and only if τ_E is smoothly regular.

Corollary

If τ_E is generated by semi-norms that are smooth on the complements of their zero-sets, then τ_E is smoothly regular.

Reflexivity

- Given a locally convex space E, let E^* denote the continuous real linear functionals (which are infinitely-differentiable).

Theorem

1. *If E is a metrisable locally convex space, then $\mathcal{F}_E = \Phi\mathcal{D}_E$.*
2. *If E is a sequentially complete locally convex space, then $\mathcal{D}_E = \Pi E^*$ (and hence $\mathcal{D}_E = \Pi\mathcal{F}_E$).*
3. *If E is a Fréchet space, then both $\mathcal{D}_E$ and $\mathcal{F}_E$ are reflexive.*

Theorem

Let M be a Fréchet manifold locally modelled on a smoothly regular Fréchet space. The natural diffeology $\mathcal{D}_M$ and Sikorski structure $\mathcal{F}_M$ on M are reflexive, and the topology on M is unambiguous.

Functional Diffeology

Definition

Given diffeological spaces X and Y, the set of diffeologically smooth functions between them $\mathbf{Diffeol}(X, Y)$ admits the **functional diffeology**, in which a parametrisation p is a plot if

$$p^\sharp : U_p \times X \to Y : (u, x) \mapsto p(u)(x)$$

is smooth.

- The functional diffeology satisfies the Exponential Law:

$$\mathbf{Diffeol}(X, \mathbf{Diffeol}(Y, Z)) \cong \mathbf{Diffeol}(X \times Y, Z).$$

Reflexivity of Function Spaces

Question

Given diffeological spaces X and Y, under what conditions is the functional diffeology on $\mathbf{Diffeol}(X, Y)$ reflexive?

Theorem (Karshon-W.)

If X and Y are diffeological spaces with Y reflexive, then the functional diffeology of $\mathbf{Diffeol}(X, Y)$ is reflexive.

Spaces of Maps

Proposition ([CSW14])

Let M be a compact manifold. The functional diffeology of $C^\infty(M,\mathbb{R}^n)$ coincides with $\mathcal{D}_{C^\infty(M,\mathbb{R}^n)}$, where $C^\infty(M,\mathbb{R}^n)$ is given the smoothly regular Fréchet space structure of the C^∞-topology.

Corollary

Let M and N be manifolds with M compact. The functional diffeology of $C^\infty(M,N)$ coincides with $\mathcal{D}_{C^\infty(M,N)}$, where $C^\infty(M,N)$ is given the Fréchet manifold structure.

Corollary ([Hi94,CSW14])

Let M be a compact manifold. Then $\mathrm{Diff}(M)$ is an open subset of $C^\infty(M,M)$ with respect to the D-topology, and hence inherits all of the nice properties of $C^\infty(M,M)$.

Symplectomorphisms

Proposition (Karshon-W.)

Given a compact symplectic manifold (M, ω)*, the group of symplectomorphisms* $\mathrm{Symp}(M, \omega)$ *is a closed subgroup of* $\mathrm{Diff}(M)$ *in an unambiguous way. In particular, the smooth identity component* $\mathrm{Symp}_0(M, \omega)$ *has an unambiguous definition.*

Corollary

Thus $\mathrm{Symp}_0(M, \omega)$ *inherits a Fréchet manifold structure in an unambiguous way.*

Hamiltonian Diffeomorphisms

Definition

Let (M, ω) be a compact symplectic manifold. Given a function $H\colon [0,1] \times M \to M$, a **hamiltonian isotopy generated by H** is a smooth path $\Psi\colon [0,1] \to \mathrm{Diff}(M)\colon t \mapsto \Psi_t$ such that

1. $\Psi_0 = \mathrm{id}_M$ and,
2. the time-dependent vector field ξ_t given by

$$\xi_t \lrcorner \omega = dH(t, \cdot) \qquad \text{satisfies} \qquad \frac{d}{dt}\Psi_t = \xi_t \circ \Psi_t.$$

The **group of hamiltonian diffeomorphisms of (M, ω)** is

$$\mathrm{Ham}(M, \omega) := \{\psi \mid \exists \text{ a hamiltonian isotopy } \Psi \text{ s.t. } \psi = \Psi_1\}.$$

- We are still working on $\mathrm{Ham}(M, \omega)$, but expect everything to go through.

Covering Spaces

Definition

Given a connected topological space X, denote by $\widetilde{X}_\tau$ the space

$$\widetilde{X}_\tau := C^0([0,1],X)\Big/ C^0\text{-homotopy.}$$

- In the case that X is connected, locally path-connected, and semi-locally simply-connected, then $\widetilde{X}_\tau$ is a topological universal cover of X.

Definition

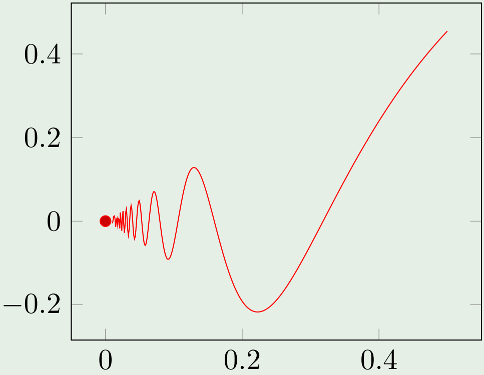
Given a connected diffeological space X, denote by $\mathrm{stPaths}(X)$ the smooth maps $[0,1] \to X$ constant near 0 and 1, and by $\widetilde{X}_D$ the **diffeological universal cover**

$$\widetilde{X}_D := \mathrm{stPaths}(X)\Big/ \text{diffeologically smooth homotopy.}$$

Covering Spaces

Example

Let X be the graph of $f(x) = \begin{cases} x\sin(1/x) & x > 0, \\ 0 & x = 0. \end{cases}$ Then X is simply-connected, but not even smoothly path-connected.



Covering Spaces

Example

If X is an irrational torus, then $\widetilde{X}_D \cong \mathbb{R}$ but $\widetilde{X}_\tau \cong X$.

Question

Given a diffeological space X, when is the underlying topological space of $\widetilde{X}_D$ the same as $\widetilde{X}_\tau$?

- There is always a continuous map $\sigma\colon \widetilde{X}_D \to \widetilde{X}_\tau$.

Covering Spaces

Theorem (Karshon-W.)

- *Let X be a smoothly regular Fréchet manifold. Then $\sigma\colon \widetilde{X}_D \to \widetilde{X}_\tau$ is a homeomorphism.*
- *Using the fact that topological covering maps are local homeomorphisms, there is a natural diffeology one can put on $\widetilde{X}_\tau$ that turns σ into a diffeomorphism in the case that $\widetilde{X}_\tau$ is a covering space.*

Examples

Given manifolds M and N with M compact, $X = C^\infty(M, N)$ satisfies $\widetilde{X}_D \cong \widetilde{X}_\tau$. If M is a compact manifold, then $X = \mathrm{Diff}(M)$, then $\widetilde{X}_D \cong \widetilde{X}_\tau$. If (M, ω) is a compact symplectic manifold, then for $X = \mathrm{Symp}(M, \omega)$ or $= \mathrm{Ham}(M, \omega)$, then again $\widetilde{X}_D \cong \widetilde{X}_\tau$.

Thank you!

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