We begin our exploration of *Baire's Theorem* with the following definitions:

**Definition:** If a set $A \subseteq \mathbb{R}$ can be written as the countable (finite or countably infinite) union of closed sets, then we say $A$ is an $F_\sigma$ set. If a set $B \subseteq \mathbb{R}$ can be written as the countable intersection of open sets, then we say that $B$ is a $G_\delta$ set.

From even a cursory glance at these definitions, a natural question which arises is if there are any relationships between $F_\sigma$ and $G_\delta$ sets. As we will see in the following proof, such a relationship does exist, and should be familiar to readers who are aware of the differences between open and closed sets.

**Statement:** A set $A$ is a $G_\delta$ set if and only if its complement is an $F_\sigma$ set.

**PROOF:** ($\Rightarrow$) Assume that $A$ is a $G_\delta$ set. Thus, $A = \bigcap_{n\in\mathbb{N}} S_n$, where every $S_n$ is an open set. By definition, $A^C = \mathbb{R} - A = \mathbb{R} - \bigcap_{n\in\mathbb{N}} S_n$. Further, De Morgan's Laws tell us that $A - (B \cap C) = (A - B) \cup (A - C)$, so applying this to the countable union here implies that $A^C = \bigcup_{n\in\mathbb{N}}(\mathbb{R} - S_n)$. It follows that $(\mathbb{R} - S_n) = S_n^C$, and since each $S_n$ is an open set, each $S_n^C$ is a closed set. Hence, $A^C$ can be written as the countable union of closed sets, so by definition $A^C$ is an $F_\sigma$ set.

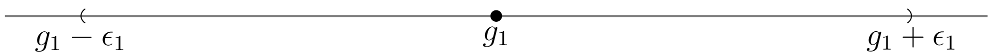
($\Leftarrow$) Assume that $A^C$ is an $F_\sigma$ set. Thus, $A^C = \bigcup_{n\in\mathbb{N}} S_n$, where every $S_n$ is a closed set. Further, $A = \mathbb{R} - A^C = \mathbb{R} - \bigcup_{n\in\mathbb{N}} S_n$. Once again by De Morgan's Laws, we have that $A = \bigcap_{n\in\mathbb{N}}(\mathbb{R} - S_n)$. Since each $S_n$ is a closed set, $(\mathbb{R} - S_n) = S_n^C$ is an open set. Hence, $A$ can be written as the countable intersection of open sets, so $A$ is a $G_\delta$ set. ■

A similar argument can be produced to prove that a set $A$ is an $F_\sigma$ set if and only if $A^C$ is a $G_\delta$ set.

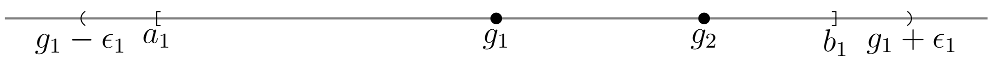
Next, we transition to an exploration of a less-general form of *Baire's Theorem*, keeping in mind that a set $G \subseteq \mathbb{R}$ is *dense* in $\mathbb{R}$ if, given any two real numbers $a < b$, it is possible to find a point $x \in G$ with $a < x < b$.

**Theorem:** If $\{G_1, G_2, G_3, ...\}$ is a countable collection of dense, open sets, then the intersection $\bigcap_{n=1}^{\infty} G_n$ is not empty.

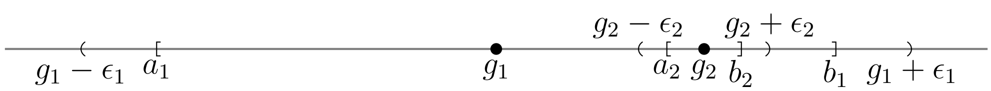
**PROOF:** First, we pick a $g_1 \in G_1$, and we can note that since $G_1$ is open, we have that $\exists\ \epsilon_1 > 0$ such that $V_{\epsilon_1}(g_1) \subseteq G_1$.




Further, by density, we have that $\exists\ g_2 \in G_2$ such that $g_1 < g_2 < g_1 + \epsilon_1$. Also, we add in a closed interval $[a_1, b_1]$ which is a subset of $(g_1 - \epsilon_1, g_1 + \epsilon_1)$ and still contains both $g_1$ and $g_2$.




Again, by openness, we have that $\exists\ \epsilon_2 > 0$ such that $V_{\epsilon_2}(g_2) \subseteq G_2$. We require that $\epsilon_2 < (b_1 - g_2)/2$. Again, we will build a closed interval inside of this new $\epsilon-$ interval.




Note that everything inside of this new closed interval $[a_2, b_2]$ is an element of $G_2$ and is also an element of $G_1$. Further, $[a_2, b_2]$ is contained in $[a_1, b_1]$. In general, we could keep constructing these intervals and we would see that each of these subsets is a subset of each of the previous subsets. Further, since each of these closed intervals are nested, nonempty, compact sets, the *Nested Interval Property* tells us that that intersection $\bigcap_{n=1}^{\infty}[a_n, b_n] \neq \emptyset$. Thus, there is at least one element which is in all of these $G_n$ sets, so we can conclude that $\bigcap_{n=1}^{\infty} G_n \neq \emptyset$. ■

Next, we will turn our attention to the topic of nowhere-dense sets, which we define as follows:

**Definition:** A set $E$ is *nowhere-dense* if $\overline{E}$ contains no nonempty open intervals. Additionally, we will use the definition that a set $E$ is nowhere-dense in $\mathbb{R}$ if and only if the complement of $\overline{E}$ is dense in $\mathbb{R}$.

One common example of a nowhere-dense set in $\mathbb{R}$ is $\mathbb{Z}$. Since $\mathbb{Z}$ has no limit points, $\overline{\mathbb{Z}} = \mathbb{Z}$, and clearly $\mathbb{Z}$ contains no nonempty open intervals (due to the inherent gaps between elements). Another familiar set which is nowhere-dense in $\mathbb{R}$ is $A = \{1/n : n \in \mathbb{N}\}$. $\overline{A} = A \cup \{0\}$, and again, this set contains no nonempty open intervals. This example shows that a nowhere-dense set is not necessarily a closed set.

Finally, with these definitions, we are able to state *Baire's Theorem* as follows:

**Baire's Theorem:** The set of real numbers $\mathbb{R}$ cannot be written as the countable union of nowhere-dense sets.

**PROOF:** We prove by contradiction. Assume that $\mathbb{R} = \bigcup_{n\in\mathbb{N}} E_n$, where each $E_n$ is a nowhere-dense set in $\mathbb{R}$. Further, since $\overline{E_n} = E_n \cup E_n'$, and $\forall\, x \in E_n', x \in \mathbb{R}$, we can write that $\mathbb{R} = \bigcup_{n\in\mathbb{N}} \overline{E_n}$. Thus, $\mathbb{R}^c = \mathbb{R} - \bigcup_{n\in\mathbb{N}} \overline{E_n} = \bigcap_{n\in\mathbb{N}}(\mathbb{R} - \overline{E_n}) = \bigcap_{n\in\mathbb{N}} \overline{E_n}^c$. By previous definition, we have that $\overline{E_n}^c$ is dense, and further it follows that $\overline{E_n}^c$ is open. Thus, we can write this intersection as $\bigcap_{n\in\mathbb{N}} G_n$, where $G_n$ is a dense, open set. Note that $\mathbb{R}^c = \emptyset$, so $\emptyset = \bigcap_{n\in\mathbb{N}} G_n$. However, we have by our previous theorem that $\bigcap_{n\in\mathbb{N}} G_n \neq \emptyset (\Rightarrow\Leftarrow)$. Thus, we can conclude that our assumption was false, and $\mathbb{R} \neq \bigcup_{n\in\mathbb{N}} E_n$, where $E_n$ is a nowhere-dense set. ■

Thus, *Baire's Theorem* offers us another way to look at $\mathbb{R}$, this time in terms of nowhere-dense sets. Nowhere-dense sets are considered to be "thin" sets due to the inherent gaps between the elements, and so any set which is a countable union of nowhere-dense sets is called a "meager" set or a set of "first category." In contrast to these sets are sets of the "second category," which can not be written as the countable union of these "thin" sets. Therefore, as *Baire's Theorem* shows, $\mathbb{R}$ is of second category.

*Definitions and theorems were taken from Understanding Analysis, Second Edition, by Stephen Abbott. Proofs were supplied by the author of this paper.*