The Quotient Manifold Theorem

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April 21, 2020

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1 Smooth Manifolds

1.1 Topological Manifolds

Definition 1.1. Suppose M is a topological space. We say that M is a **topological manifold of dimension n** if it satisfies the following properties:

- M is a ***Hausdorff space (or*** T_2): for every two distinct points $p, q \in M$, there are open subsets $U, V \subseteq M$ which are disjoint and $p \in U$ and $q \in V$.
- M is ***second-countable***: we can find a countable basis for the topology on M.
- M is ***locally Euclidean of dimension n***: there is a neighborhood about each point of M that is homeomorphic to an open subset of $\mathbb{R}^n$

Remark. A non-empty n-dimensional topological manifold can only be homeomorphic to an m-dimensional manifold if $m = n$.

The first two properties listed in the above definition are typically easier to check for newly formed spaces because both subspaces and finite products of manifolds inherit them. Showing that a space is locally Euclidean with dimension n requires a bit more terminology, however.

1.2 Coordinate Charts

Definition 1.2. Let M be a topological n-manifold. Suppose there exists an open $U \subseteq M$ and a homeomorphism $\varphi : U \longrightarrow \widehat{U}$ such that $\varphi(U) = \widehat{U} \subseteq \mathbb{R}^n$ is also open. Then the ordered pair (U, φ) is called a **coordinate chart**.

We can think of these coordinate charts as relating unknown spaces such as M to the more familiar Euclidean space. When we can make this type of coordinate chart about each point $p \in M$ and M is both T_2 and second-countable, it is a topological manifold.

Remark. For a given coordinate chart (U, φ), if $\varphi(U)$ is an open ball in $\mathbb{R}^n$, we call U a *coordinate ball.* Similarly, if $\varphi(U) \subseteq \mathbb{R}^n$ is an open cube, we call U a *coordinate cube.*

1.3 Smooth Manifolds

If we would like to see how calculus applies to topological manifolds, then we must make some additional restrictions to our definition by adding more structure to topological manifolds. Suppose F is a function from $\mathbb{R}^n$ to $\mathbb{R}^m$ such that each component function of F has continuous partial derivatives of all orders. Then F is said to be *smooth* (or C^∞, or *infinitely differentiable*). Also suppose F has a smooth inverse. Then F is said to be a *diffeomorphism.*

Definition 1.3. Let M be a topological n-manifold.

1. If (U, φ), (V, ψ) are two coordinate charts where $U \cap V \neq \varnothing$, then the composition map $\psi \circ \varphi^{-1} : \varphi(U \cap V) \longrightarrow \psi(U \cap V)$ is called the **transition map from φ to ψ**.

2. We call two coordinate charts (U, φ), (V, ψ) **smoothly compatible** if either $U \cap V = \varnothing$ or the transition map $\psi \circ \varphi^{-1}$ is a diffeomorphism. We can think of this as if the two coordinate charts agree on their overlap.

3. We call any collection of coordinate charts $\mathcal{A}$, whose domains cover M, an **atlas**. The analogy to an actual atlas is a great one. A geographical atlas consists of a set of maps which comprise the whole planet.

4. If any two coordinate charts from an atlas are smoothly compatible, then we say it is a **smooth atlas**. Additionally, any coordinate chart from a smooth atlas is called a **smooth coordinate chart**.

5. We call a smooth atlas on M a **smooth structure on M** if it is not properly contained in a larger smooth atlas. We say that such a smooth atlas is *maximal*.

Remark. The transition map is a composition of homeomorphisms, so it is also a homeomorphism. The smoothness of the transition map is considered in the ordinary sense since it is from an open subset of $\mathbb{R}^n$ to another open subset of $\mathbb{R}^n$.

Definition 1.4. A **smooth manifold** is the ordered pair $(M, \mathcal{A})$ where M is a topological manifold and $\mathcal{A}$ is a smooth structure on M.

Remark. In literature, one may see smooth structures identified as *differentiable structures*, or *C^∞ structures.*

If M is a topological manifold, every smooth atlas $\mathcal{A}$ for M is contained in a unique maximal smooth atlas, called the *smooth structure determined by $\mathcal{A}$*. Two smooth atlases for M determine the same smooth structure if and only if their union is a smooth atlas.

We can say that a Hausdorff set has a smooth structure if we can find a collection of subsets–which are a countable cover–that are homeomorphic to an open subset of $\mathbb{R}^n$ and such homeomorphisms agree on overlaps between subsets.

1.4 Smooth Functions and Smooth Maps

Definition 1.5. Let M and N be smooth m- and n-manifolds, respectively, and suppose k is a non-negative integer.

- We say a function $f : M \longrightarrow \mathbb{R}^k$ is a **smooth function** if for every $p \in M$, there exists a smooth coordinate chart (U, φ) for M whose domain contains p and such that the composition function $f \circ \varphi^{-1}$ is smooth on the open subset $\widehat{U} = \varphi(U) \subseteq \mathbb{R}^n$. This composite function $f \circ \varphi^{-1} : \mathbb{R}^n \longrightarrow \mathbb{R}^k$ can be considered smooth in the usual sense.

- We say a map $F : M \longrightarrow N$ is a **smooth map** if for every $p \in M$ there exists smooth coordinate charts (U, φ), (V, ψ) such that for $p \in U$, $F(U) \subseteq V$, and the composition map $\psi \circ F \circ \varphi^{-1}$ is smooth from $\varphi(U)$ to $\psi(V)$. Similarly, $\psi \circ F \circ \varphi^{-1} : \mathbb{R}^n \longrightarrow \mathbb{R}^m$.

Earlier we were reminded of the usual definition for diffeomorphisms regarding smooth functions. Let us be more general now.

Definition 1.6. Suppose that M, and N are smooth manifolds. A bijective, smooth map $F : M \longrightarrow N$ that also has a smooth inverse is called a **diffeomorphism**.

2 Tangent Vectors

2.1 Tangent Spaces

Before we can talk about calculus over arbitrary smooth manifolds, we must define what we mean by taking the *derivation* at a point. Let us start in a more familiar setting.

Suppose $a \in \mathbb{R}^n$. A function $w : C^\infty(\mathbb{R}^n) \longrightarrow \mathbb{R}$ is called an $\mathbb{R}$-*derivation at* a if it is $\mathbb{R}$-linear and if it satisfies

$$w(fg) = f(a)wg + g(a)wf$$

for all $f, g \in C^\infty(\mathbb{R}^n)$.

Definition 2.1.

- Suppose M is a smooth manifold and $p \in M$. A linear map $v : C^\infty(M) \longrightarrow \mathbb{R}$ is called a **derivation at p** if it satisfies

  $$v(fg) = f(p)vg + g(p)vf$$

  for all $f, g \in C^\infty(M)$.

- The **tangent space to M at p**, denoted by T_pM, is the set of all derivations of $C^\infty(M)$ at p. If a vector is tangent to p then it is an element of T_pM.

2.2 The Differential of a Smooth Map

Definition 2.2. Suppose M, N are smooth and $F : M \longrightarrow N$ is a smooth map. For each $p \in M$ we define the map

$$dF_p : T_pM \longrightarrow T_{F(p)}N,$$

which we call the **differential of F at p** by the following rule:

$$dF_p(v)(f) = v(f \circ F)$$

where $v \in T_pM$ and $f \in C^\infty(N)$.

Remark. $dF_p(v)$ is a *derivation at* $F(p)$. Also, if F is a diffeomorphism, then $dF_p : T_pM \longrightarrow T_{F(p)}N$ is an isomorphism, and $(dF_p)^{-1} = d(f^{-1})_{F(p)}$.

Proposition 2.3. *If M is an n-dimensional smooth manifold, then for each $p \in M$, the tangent space T_pM is an n-dimensional vector space.*

We have the tangent space for each point of M just above, but sometimes it is useful to consider all of the tangent spaces simultaneously.

Definition 2.4. Let M be a smooth manifold. Then we call the disjoint union of tangent spaces at every point in M:

$$TM = \bigsqcup_{p \in M} T_pM$$

the **tangent bundle of $\boldsymbol{M}$**, denoted $\boldsymbol{TM}$.

Proposition 2.5. *For a smooth manifold M, there is a natural topology and smooth structure for the tangent bundle TM making it a $2n-$dimensional smooth manifold.*

3 Submersions, Immersions, and Embeddings

3.1 Submersions and Immersions

Definition 3.1. Suppose M, N are smooth manifolds and $F : M \longrightarrow N$ is a smooth map.

1. We call the **rank of F at p** to be the rank of the linear map $dF_p : T_pM \longrightarrow T_{F(p)}N$, for each $p \in M$.

2. We say that **F has constant rank** if F has the same rank at each point in M.

3. We say that **F has full rank at p** if the rank of F at p is equal to the $\min\{\dim M, \dim N\}$.

4. We say **F has full rank** if F has full rank for all $p \in M$.

5. We say that F is a **smooth submersion** if $\dim N < \dim M$ and F has full rank.

6. We say that F is a **smooth immersion** if $\dim N > \dim M$ and F has full rank

Remark. For a smooth map between smooth manifolds such that the map is surjective, then there exists a restriction of the domain such that the restricted map is a smooth submersion.

Theorem 3.2. *Suppose $F : M \longrightarrow N$ is a smooth map between smooth manifolds. Let $\dim M = m$ and $\dim N = n$. If F is a smooth submersion, then there exists smooth coordinate charts (U, φ) for M at p and (V, ψ) for N centered at $F(p)$ such that $F(U) \subseteq V$ and $\tilde{F}$ is the coordinate representation of F then*

$$\tilde{F}(x^1, \ldots, x^n, x^{n+1}, \ldots, x^m) = (x^1, \ldots, x^n).$$

Similarly, if F is a smooth immersion with the same coordinate charts as described above, then $\tilde{F}$ becomes

$$\tilde{F}(x^1, \ldots, x^m) = (x^1, \ldots, x^m, 0, \ldots, 0).$$

Definition 3.3. Suppose $\pi : M \longrightarrow N$ is a continuous map.

- We call a continuous right-inverse for π, i.e., $\sigma : N \longrightarrow M$ such that $\pi \circ \sigma = \mathrm{Id}_N$, a **section of π**.

- Sometimes there do not exist full right-inverses. We denote a ***local* section of π** to be a continuous map $\sigma : U \longrightarrow M$ defined on an open $U \subseteq N$ such that $\pi \circ \sigma = \mathrm{Id}_U$.

Remark. A smooth map between smooth manifolds is a smooth submersion if and only if every point in the domain is in the image of a smooth local section.

Theorem 3.4. *Suppose M and N_1 are smooth manifolds and $\pi : M \longrightarrow N_1$ is a surjective smooth submersion. For a smooth manifold N_2, a map $F : N_1 \longrightarrow N_2$ is smooth if and only if $F \circ \pi$ is smooth.*

Since π is a submersion, there will be local sections σ such that $F \circ \pi \circ \sigma : U_1 \subseteq N_1 \longrightarrow N_2$ which is essentially the same as $F|_{U_1}$. From here it is clearly apparent that the smoothness of the composition depends on F.

3.2 Embeddings

Definition 3.5. Suppose M, N are smooth manifolds.

1. A **smooth embedding of M into N** is a smooth immersion $F : M \longrightarrow N$ that is also a topological embedding, i.e., a homeomorphism onto its image $F(M) \subseteq N$ in the subspace topology.

2. An **embedded submanifold of M** is a subset $S \subseteq M$ that is a manifold in the subspace topology, endowed with a smooth structure with respect to which the inclusion map $S \hookrightarrow M$ is a smooth embedding.

3. We call an embedded submanifold $S \subseteq M$ **properly embedded** if the inclusion $S \hookrightarrow M$ is a proper map, i.e., the preimage of a compact set is a compact set.

Proposition 3.6. *Suppose M and N are smooth manifolds. For each $p \in N$, the subset $M \times \{p\}$, called the **slice** of the product manifold, is an embedded submanifold of $M \times N$ diffeomorphic to M.*

Remark. To ensure that an embedded submanifold is properly embedded, it need only to be closed.

Smooth maps restricted to immersed or embedded submanifolds are also smooth. A submanifold is immersed if each of the coordinate charts in its corresponding smooth structure are immersions into the larger smooth structure.

3.3 Slice Charts

Definition 3.7.

- For $k \in \{0, \dots, n\}$, a **k-dimensional slice of $U \subseteq \mathbb{R}^n$** (or a **k-slice**) is any subset of the form
$$S = \{(x^1, \dots, x^k, x^{k+1}, \dots, x^n) \in U : x^{k+1} = c^{k+1}, \dots, x^n = c^n\}$$
where $c^{k+1}, \dots, c^n$ are constants.

- Let (U, φ) be a smooth coordinate chart on an n-dimensional smooth manifold M. We say that **S is a k-slice of U** if $S \subseteq U$ is such that $\varphi(S)$ is a k-slice of $\varphi(U)$.

- Given $S \subseteq M$ and a non-negative integer k, we say that S satisfies the **local k-slice condition** if each point of S is contained in the domain of a smooth coordinate chart (U, φ) for M such that $S \cap U$ is a single k-slice in U.
  - Any such coordinate chart is called a **slice chart for S in M**, and the corresponding coordinates are called **slice coordinates**.

4 Lie Groups

4.1 Lie Group Homomorphisms

Definition 4.1.

- A **Lie group** G is a smooth manifold that also holds the group properties such that the multiplication map $m : G \times G \longrightarrow G$ and the inversion map $i : G \longrightarrow G$, each given by:

$$m(g, h) = gh, \qquad i(g) = g^{-1}$$

  are both smooth.

- Suppose G, H are Lie groups. A **Lie group homomorphism from G to H** is a smooth map $F : G \longrightarrow H$ that is also a group homomorphism.
  - F is called a **Lie group isomorphism** if it is also a diffeomorphism.

Remark. Every Lie group homomorphism has constant rank.

4.2 Group Actions

Definition 4.2.

- Suppose G is a group and M is a set. A **left action of G on M** is a map $G \times M \longrightarrow M$, by the rule $(g, p) \longmapsto g \cdot p$ that satisfies

$$g_1 \cdot (g_2 \cdot p) = (g_1 g_2) \cdot p \quad \text{for all } g_1, g_1 \in G \text{ and } p \in M;$$

$$e \cdot p = p \qquad \text{for all } p \in M.$$

  - If M is a smooth manifold, G is a Lie group, and the defining map is smooth, then the left action is a **smooth action**.

- Suppose $\theta : G \times M \longrightarrow M$ is a left action of a group G on a set M.
  - For each $p \in M$, the **orbit of p**, denoted $G \cdot p$, is the set of all images under the action by every element in G:
$$G \cdot p = \{g \cdot p : g \in G\}.$$
  - We call the action **free** if only the identity in G fixes (i.e., $g \cdot p = p$) an element of M.

- A continuous left action of a Lie group G on a manifold M is said to be a **proper action** if the map $G \times M \longrightarrow M \times M$ defined by $(g, p) \mapsto (g \cdot p, p)$ is a proper map.

Remark. For a smooth left action θ of a Lie group G on a smooth manifold M and for each $p \in M$, the orbit map $\theta^{(p)} : G \longrightarrow M$ is smooth and has constant rank. We can see then that the orbit is an immersed submanifold of M.

5 Lie Algebras and Vector Fields

5.1 Vector Fields

Remember that TM, or the tangent bundle of M, is the disjoint union of tangent spaces at every point in M. We need this for the following definition.

Definition 5.1. Let M be a smooth manifold. We define a **smooth vector field on M** as a smooth section of the map $\pi : TM \longrightarrow M$. What we are saying is that a vector field is a smooth map $X : M \longrightarrow TM$, by the rule $p \mapsto X_p$, such that $\pi \circ X = \mathrm{Id}_M$. This is the same as $X_p \in T_pM$ for each $p \in M$ where X_p denotes a tangent vector at p.

Remark. If M is a smooth manifold, let $\mathfrak{X}(M)$ denote the set of all smooth vector fields on M and is a vector space under point-wise addition and scalar multiplication. If $f \in C^\infty(M)$ and $X \in \mathfrak{X}(M)$ we define $fX : M \longrightarrow TM$ by the rule $(fX)_p = f(p)X_p$.

Remark. If $X \in \mathfrak{X}(M)$ and f is a smooth real-valued function defined on an open subset $U \subseteq M$, we can obtain a new function by $Xf : U \longrightarrow \mathbb{R}$, defined by the rule $(Xf)(p) = X_pf$.

Definition 5.2. Let $F : M \longrightarrow N$ be smooth and let X be a vector field on M. If there exists a vector field Y on N such that for each $p \in M$, $dF_p(X_p) = Y_{F(p)}$, then we say that the vector fields X and Y are ***F*-related**.

Remark. If $F : M \longrightarrow N$ is a smooth map between manifolds and $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$, then X and Y are F-related if and only if for every smooth real-valued function f defined on an open subset of N,

$$X(f \circ F) = (Yf) \circ F.$$

5.2 Lie Brackets

Definition 5.3. Let $X, Y \in \mathfrak{X}(M)$ for a smooth manifold M. Given a smooth function $f : M \longrightarrow \mathbb{R}$, we call [X,Y] the **Lie bracket**, defined by the rule

$$[X, Y]f = XYf - YXf.$$

Remark. The Lie bracket of any pair of smooth vector fields is a smooth vector field and the value of the Lie bracket at $p \in M$ is the derivation at p given by

$$[X, Y]_pf = X_p(Yf) - Y_p(Xf).$$

Proposition 5.4. *The Lie bracket satisfies the following identities for all* $X, Y, Z \in \mathfrak{X}(M)$*:*

1. *Bilinearity: For* $a, b \in \mathbb{R}$*,*
$$[aX + bY, Z] = a[X, Z] + b[Y, Z],$$
$$[Z, aX + bY] = a[Z, X] + b[Z, Y].$$

2. *Anti-symmetry:*
$$[X, Y] = -[Y, X].$$

3. *Jacobi Identity:*
$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0.$$

5.3 Lie Algebras

Definition 5.5. Let G be a Lie group. G acts smoothly and transitively on itself by left translation, i.e., $L_g(h) = gh$. A vector field X on G is said to be **left-invariant** if it is L_g-related to itself for every $g \in G$
$$d(L_g)_{g'}(X_{g'}) = X_{gg'}, \quad \text{for all } g, g' \in G.$$

Remark. If G is a Lie group, and X and Y are smooth left-invariant vector fields on G, then $[X, Y]$ is also left-invariant.

Definition 5.6.

- A **Lie algebra** (over $\mathbb{R}$) is a real vector space $\mathfrak{g}$ endowed with a map called the **Lie bracket** from $\mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$, denoted by the rule $(X, Y) \mapsto [X, Y]$ such that it satisfies the *Bilinearity, Anti-symmetry,* and *Jacobi Identity* properties.

- A linear subspace $\mathfrak{h} \subseteq \mathfrak{g}$ is called a **Lie subalgebra of $\mathfrak{g}$** if it is closed under Lie brackets.

- If $\mathfrak{g}$ and $\mathfrak{h}$ are Lie algebras, a linear map $K : \mathfrak{g} \longrightarrow \mathfrak{h}$ is called a **Lie algebra homomorphism** if it preserves its Lie brackets, i.e., $K[X, Y] = [KX, KY]$ for $X, Y \in \mathfrak{g}$.

- An invertible Lie algebra homomorphism is called a **Lie algebra isomorphism**.

Remark. A Lie algebra is a vector space where each of its Lie brackets are equal to zero. When all of the Lie brackets for a Lie algebra are zero it is said to be *abelian.*

Definition 5.7. The Lie algebra of all smooth left-invariant vector fields on a Lie group G is called the **Lie algebra of G**, and is denoted $\mathrm{Lie}(G)$.

Remark. Every Lie subgroup has a Lie algebra canonically isomorphic to a Lie subalgebra which is related via the inclusion map.

6 Integral Curves and Flows

6.1 Integral Curves

Definition 6.1. For a vector field V on a smooth manifold M and a subset $J \subseteq \mathbb{R}$, a differentiable curve $\gamma : J \longrightarrow M$ is called an **integral curve of V** such that the velocity at each point of γ is equal to the value of V at that point.

6.2 Flows

Definition 6.2.

- We define a continuous map $\theta : \mathbb{R} \times M \longrightarrow M$, or a continuous left $\mathbb{R}$-action on a smooth manifold M, to be a **global flow**. Such a map should satisfy both

$$\theta(t, \theta(s, p)) = \theta(t + s, p) \quad \text{and} \quad \theta(0, p) = p$$

  for all $t, s \in \mathbb{R}$ and $p \in M$.

  We adopt the following notation on global flows:

  - For each $t \in \mathbb{R}$, define $\theta_t : M \longrightarrow M$ by

$$\theta_t(p) = \theta(t, p).$$

  - For each $p \in M$, define $\theta^{(p)} : \mathbb{R} \longrightarrow M$ by

$$\theta^{(p)}(t) = \theta(t, p).$$

- If $\theta : \mathbb{R} \times M \longrightarrow M$ is a smooth global flow, for each $p \in M$ we define a tangent vector $V_p \in T_pM$ by

$$V_p = \theta^{(p)\prime}(0).$$

  The assignment of $p \mapsto V_p$ is a vector field on M, which we call the **infinitesimal generator of θ**.

Proposition 6.3. *Let $\theta : \mathbb{R} \times M \longrightarrow M$ be a smooth global flow on a smooth manifold M. The infinitesimal generator V of θ is a smooth vector field on M, and each curve $\theta^{(p)}$ is an integral curve of V.*

7 Distributions and Foliations

7.1 Distributions and Integral Manifolds

Definition 7.1.

- Let M be a smooth manifold. We pay special attention to what are called **distributions on $\boldsymbol{M}$ of rank $\boldsymbol{k}$**. We will define a distribution D as follows: for each $p \in M$, there is a linear subspace $D_p \subseteq T_pM$ of dimension k, and then we define $D = \bigcup_{p \in M} D_p$. We say D is a *smooth* distribution if and only if each point of M has a neighborhood U on which there are smooth vector fields $X_1, \dots, X_k : U \longrightarrow TM$ such that $X_1|_q, \dots, X_k|_q$ form a basis for D_q at each $q \in U$.

- Let $D \subseteq TM$ be a smooth distribution of rank k. For non-trivial immersed submanifolds $N \subseteq M$, we call a N an **integral manifold of $\boldsymbol{D}$** if for each $p \in N$, the tangent space of N at p, T_pN, is a linear subspace of rank k and is equal to D_p, where D_p is defined above.

- Let D be a smooth distribution on a smooth manifold M. We say that D is **involutive** if given any pair of smooth local sections of D, i.e., smooth vector fields X, Y defined on an open subset U of M such that $X_p, Y_p \in D_p$ for each $p \in M$, then $[X_p, Y_p]$ is also a local section of D.

- A smooth distribution D on M is said to be **integrable** if each point of M is contained in some integral manifold of D.

- Let $D \subseteq TM$ be a rank-k smooth distribution. We say a smooth coordinate cube (U, φ) on M is **flat for $\boldsymbol{D}$** if for each point in U, D is spanned by the first k coordinate vector fields $\partial/\partial x^1, \dots, \partial/\partial x^k$. In such charts, each k-slice having constants for the $k+1, \dots, m = \dim M$ coordinates is an integral manifold of D.

- Let $D \subseteq TM$ be a smooth distribution. We say it is **completely integrable** if there exists a flat chart for D in a neighborhood of each point in M.

Remark. Through our definitions it may be apparent that when a distribution is completely integrable it implies it is integrable, which then implies that it is involutive. *Frobenius' Theorem* states that every involutive distribution is completely integrable. This means that all of the *implied* statements just mentioned are actually equivalence statements.

7.2 Foliations

Definition 7.2. Let M be a smooth n-manifold, and let $\mathcal{F}$ be any collection of k-dimensional submanifolds of M.

- We say that a smooth coordinate cube (U, φ) is **flat for** $\mathcal{F}$ if each submanifold in $\mathcal{F}$ intersects U in either the empty set or a countable union of k-dimensional slices.
- We call $\mathcal{F}$ a **foliation of dimension *k* on *M*** if it is a collection of disjoint, connected, nonempty, immersed k-dimensional submanifolds of M such that their union is M and there exists a flat chart for $\mathcal{F}$ in a neighborhood of each point $p \in M$.

Proposition 7.3. *The collection of all maximal connected integral manifolds of an involutive distribution on a smooth manifold M forms a foliation on M.*

7.3 Lie Subalgebras

Definition 7.4. A distribution D on a Lie group G is said to be **left-invariant** if it is invariant under every left translation:

$$d(L_g)(D) = D \quad \text{for each } g \in G.$$

Lemma 7.5. *Let G be a Lie group. If $\mathfrak{h}$ is a Lie subalgebra of Lie(G), then the subset $D = \bigcup_{g \in G} D_g \subseteq TG$, where*

$$D_g = \{X_g : X \in \mathfrak{h}\} \subseteq T_g G,$$

is a smooth left-invariant involutive distribution on G.

8 The Exponential Map

8.1 One-Parameter Subgroups

Definition 8.1. Let G be a Lie group. We call the Lie group homomorphism $\gamma : \mathbb{R} \longrightarrow G$, with $\mathbb{R}$ being considered as a Lie group under addition, a **one-parameter subgroup of *G***. In this instance $\gamma(\mathbb{R})$ is a Lie subgroup of G when endowed with a suitable smooth manifold structure.

Remark. If we let G be a Lie group, then the one-parameter subgroups of G are precisely the maximal integral curves of left-invariant vector fields starting at the identity.

Definition 8.2. Let γ be the maximal integral curve of some left invariant vector field $X \in \mathrm{Lie}(G)$ starting at the identity. Then γ is called the **one-parameter subgroup generated by *X***.

Definition 8.3. Let G be a Lie group with Lie algebra $\mathfrak{g}$. We define the map $\exp : \mathfrak{g} \longrightarrow G$, called the **exponential map of *G***. For any $X \in \mathfrak{g}$,

$$\exp X = \gamma(1),$$

where γ is the one-parameter subgroup generated by X. This is also denoted as the integral curve starting at the identity.

Example 8.4. For any $A \in \mathfrak{gl}(n, \mathbb{R})$, let

$$e^A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k = I_n + A + \frac{1}{2} A^2 + \cdots .$$

This series converges to an invertible matrix $e^A \in \mathrm{GL}(n, \mathbb{R})$, and the one-parameter subgroup of $\mathrm{GL}(n, \mathbb{R})$ generated by $A \in \mathfrak{gl}(n, \mathbb{R})$ is $\gamma(t) = e^{tA}$. When $t = 1$, e^A is the integral curve starting at the identity.

Proposition 8.5. *Let G be a Lie group and let $\mathfrak{g}$ be its Lie algebra.*

1. *The exponential map is a smooth map from $\mathfrak{g}$ to G.*

2. *For any $X \in \mathfrak{g}$ and $s, t \in \mathbb{R}$, $\exp(s + t)X = \exp sX \, \exp tX$.*

3. *For any $X \in \mathfrak{g}$ and $n \in \mathbb{Z}$, $(\exp X)^n = \exp(nX)$.*

4. *The differential $(d \exp)_0 : T_0\mathfrak{g} \longrightarrow T_eG$ is the identity map, with the canonical identifications of both $T_0\mathfrak{g}$ and T_eG with $\mathfrak{g}$.*

5. *There is a restriction of the exponential map such that it is a diffeomorphism from some neighborhood of 0 in $\mathfrak{g}$ to a neighborhood of e in G.*

6. *If H is another Lie group, $\mathfrak{h}$ is its Lie algebra, and $\Phi : G \longrightarrow H$ is a Lie group homomorphism, then $\mathfrak{h}$ is Φ-related to $\mathfrak{g}$ and $\exp \circ \Phi_\star = \Phi \circ \exp$.*

Remark. For a Lie group G and a Lie subgroup $H \subseteq G$, with $\text{Lie}(H)$ considered as a subalgebra of $\text{Lie}(G)$ in the usual way, the exponential map of H is the restriction of the exponential map of G to $\text{Lie}(H)$. Additionally,

$$\text{Lie}(H) = \{X \in \text{Lie}(G) \mid \exp tX \in H \text{ for all } t \in \mathbb{R}\}.$$

Theorem 8.6. *If G is a Lie group and H is any subgroup of G, the following are equivalent:*

1. *H is closed in G.*
2. *H is an embedded submanifold of G.*
3. *H is an embedded Lie subgroup of G.*

Remark. The preceding theorem gets us that each of the orbits of a Lie group are embedded submanifolds, since they are closed.

8.2 Infinitesimal Generators of Group Actions

We mentioned the following definition earlier, but are now generalizing it to all Lie groups acting on smooth manifolds, rather than just $\mathbb{R}$.

Definition 8.7. Let G be a Lie group on a smooth manifold M. Suppose $\theta : G \times M \longrightarrow M$ is a smooth left action of G on M. We call the **infinitesimal generator of** θ the map $\widehat{\theta} : \text{Lie}(G) \longrightarrow \mathfrak{X}(M)$ defined by the rule $\widehat{\theta}(X) = \widehat{X}$, where

$$\widehat{X}_p = \frac{d}{dt}\Big|_{t=0}((\exp tX) \cdot p) = d(\theta^{(p)})_e(X_e),$$

and $\theta^{(p)} : G \longrightarrow M$ is the orbit map $\theta^{(p)}(g) = g \cdot p$.

9 Quotient Manifolds

9.1 Orbits

Definition 9.1. Let $\theta : G \times M \longrightarrow M$ be a left action. Recall that the *orbit* of a point $p \in M$ is the set of images of p under all elements of the group denoted as $\theta^{(p)}(g) = G \cdot p = \{g \cdot p : g \in G\}$. We can define an equivalence relation on M by setting $p \sim q$ if there exists $g \in G$ such that $g \cdot p = q$. The equivalence classes are exactly the orbits of G in M. We denote the set of equivalence classes, or the orbits, by M/G. When this set of orbits is equipped with the quotient topology, we call it the **orbit space** of the action.

Lemma 9.2. *For any continuous action of a topological group G on a topological space M, the quotient map $\pi : M \longrightarrow M/G$ is an open map.*

Proof. For some $g \in G$, $U \subseteq M$, we denote $g \cdot U = \{g \cdot x : x \in U\} \subseteq M$. The image $\pi(U)$ is a set of orbits in the orbit space. Then $\pi^{-1}(\pi(U))$ is the union of all $g \cdot U$ for all $g \in G$. If U is open, then so is $\pi^{-1}(\pi(U))$. Then $\pi(U) \subseteq M/G$ is open. Therefore, $\pi : M \longrightarrow M/G$ is an open map. ∎

Proposition 9.3. *If a Lie group acts continuously and properly on a manifold, then the orbit space is Hausdorff.*

Proof. Let G be a Lie group that acts continuously and properly on a manifold M via the proper map $\Theta : G \times M \longrightarrow M \times M$, given by $\Theta((g, p)) = (g \cdot p, p)$. Let $\pi : M \longrightarrow M/G$ be the quotient map and let us define the set $\mathcal{O} = \{(g \cdot p, p) : (g, p) \in G \times M\}$ or simply put, the image of our proper action.

From Lemma 9.2, π is an open map and we have that M/G is Hausdorff if and only if the orbit relation given by $\mathcal{O}$ is closed in $M \times M$. Since Θ is proper and continuous, it is a closed map. In particular, $\mathcal{O} \subseteq M \times M$ is closed. ∎

Proposition 9.4. *Let M be a manifold, and let G be a Lie group acting continuously and properly on M. If (p_i) is a sequence in M and (g_i) is a sequence in G such that both (p_i) and $(g_i \cdot p_i)$ converge, then a subsequence of (g_i) converges.*

Remark. If we have a smooth proper free action of a Lie group on a smooth manifold, then each of the orbit maps are closed and proper. They are also smooth embeddings into the smooth manifold and the orbit is a properly embedded submanifold.

Definition 9.5. Suppose $\dim G = k$ and $\dim M = m$. We say that a smooth coordinate chart (U, φ) is **adapted to the G-action** or that it is an **adapted coordinate chart** if it is a coordinate cube with coordinate functions $(x, y) = (x^1, \ldots, x^k, y^{k+1}, \ldots, y^m)$, such that each G-orbit intersects U either in the empty set or in a single slice of the form $(y^{k+1}, \ldots, y^m) = (c^{k+1}, \ldots, c^m)$ where $c^{k+1}, \ldots, c^m$ are constants.

10 The Quotient Manifold Theorem

Theorem 10.1. *Suppose G is a Lie group acting smoothly, freely, and properly on a smooth manifold M. Then the orbit space M/G is a topological manifold of dimension equal to* $\dim M - \dim G$*, and has a unique smooth structure with the property that the quotient map $\pi : M \longrightarrow M/G$ is a smooth submersion.*

Let us break apart and examine the different statements of the theorem. Let us suppose that G is a Lie group acting smoothly, freely, and properly on a smooth manifold M. Let $\dim G = k$ and $\dim M = m$. First, we start with a claim that, if proven true, will lighten our load a bit.
We make the following claim: *For each $p \in M$, there exists an adapted chart centered at p.*

Proof. Since $\theta : G \times M \longrightarrow M$ is smooth, free and proper as an action, the G-orbits, $\theta^{(p)}$, are properly embedded submanifolds of M diffeomorphic to G.

Define a subset $D \subseteq TM$ by

$$D = \bigcup_{p \in M} D_p, \quad \text{where } D_p = T_p(G \cdot p).$$

To see that D is a smooth distribution, we look at each X in the Lie algebra of G, denoted $\mathfrak{g}$. Let $\widehat{X}$ be the vector field on M defined in Definition 8.7. If $(X_1, \ldots, X_k)$ is a basis for $\mathfrak{g}$, then $(\widehat{X_1}, \ldots, \widehat{X_k})$ is a global frame for D, so D is smooth. Since each point is contained within a G-orbit, which is an integral manifold of D, D is involutive.

Let $p \in M$ and (U, φ) be a smooth coordinate chart for M centered at p such that it is flat for D with coordinate functions $(x, y) = (x^1, \ldots, x^k, y^{k+1}, \ldots, y^m)$. Then each G-orbit intersects U in either the empty set or in a countable union of k-slices.

Suppose that there does not exist a coordinate cube $U_0 \subseteq U$ centered at p that will intersect each G-orbit in at most a single slice. For each positive integer i, let U_i be a coordinate cube contained in U consisting of points whose coordinates are all less than $1/i$ in absolute value. If $m - k = n$, let Y be the n-dimensional submanifold of M consisting of points in U whose coordinate representations are of the form $(0, \ldots, 0, c^{k+1}, \ldots, c^m)$, and for each i let $Y_i = U_i \cap Y$. Since each k-slice of U_i intersects Y_i in exactly one point, our assumption implies that for each i there exist distinct points $p_i, g_i \cdot p_i \in Y_i$ that are in the same G-orbit. By the choice of $\{Y_i\}$, both sequences (p_i) and $(g_i \cdot p_i)$ converge to p. Since G acts properly on M, Proposition 9.4 says that we may convert to a subsequence and assume that $g_{i_j} \to g \in G$. But then, by continuity,

$$g \cdot p = \lim_{i \to \infty} g_i \cdot p_i = p.$$

And since G acts freely on M, $g = e$.

Denote $\theta^Y : G \times Y \longrightarrow M$ to be the restriction of the action to $G \times Y$. As noted prior, dim $Y = m - k$, so dim $G \times Y = k + (m - k) = m =$ dim M. Let us restrict this map twice more into two other maps. The first restriction $\{e\} \times Y$ is isomorphic to Y and so is essentially an inclusion map into M. The second restriction $G \times \{p\}$ is essentially the orbit map and is isomorphic to G. Both of these restrictions are embeddings (by their identifications). Then, since $T_pM = T_p(G \cdot p) \oplus T_pY$, $d(\theta^Y)_{(e,p)}$ is an isomorphism. Then there exists a neighborhood W in $G \times Y$ about the point (e, p) such that $\theta^Y\big|_W$ is a diffeomorphism (which is injective). But for large enough i, we can see that both (g_i, p_i) and $(e, g_i \cdot p_i)$ are in W. This means that $\theta^Y(g_i, p_i) = g_i \cdot p_i = \theta^Y(e, g_i \cdot p_i)$ contradicts the injectivity of the diffeomorphism, which in turn contradicts what we were supposing in the first place: that there does not exist a cubical subset $U_0 \subseteq U$ centered at p that will intersect each G-orbit in at most a single slice.

■

Let us get back to the initial problem. Let $m - k = n$.

We need to show: M/G, *with the quotient topology, is a topological* n*-manifold.*

Proof. First, we get that M/G is *Hausdorff* by Proposition 9.3. Second, we look to show that M/G is second-countable. Let $\pi : M \longrightarrow M/G$ be the quotient map, which is an open mapping by Lemma 9.2. Since M is a smooth manifold, it is second-countable. Let $\{B_i\}$ be a countable basis on M. Since π is continuous and an open map (by Lemma 9.2), then $\{\pi(B_i)\}$ covers M/G since the quotient map is also surjective. We just need to show that the intersection of any two open sets in M/G is indeed the union of basis elements. Let $B_1, B_2 \subseteq M$ open. Then $\pi(B_1) \cap \pi(B_2) \subseteq M/G$ is also open. But then

$$\pi^{-1}(\pi(B_1) \cap \pi(B_2)) = \pi^{-1}(\pi(B_1)) \cap \pi^{-1}(\pi(B_2))$$

which is open in M. This is the intersection of two open sets in M which means that

$$\pi^{-1}(\pi(B_1)) \cap \pi^{-1}(\pi(B_2)) = \bigcup_{i=1}^{k} B_i$$

$$\pi(B_1) \cap \pi(B_2) = \bigcup_{i=1}^{k} \pi(B_i)$$

Thus, M/G is second-countable.

Third, we look to show that M/G is locally Euclidean and has dimension $m - k = n$. To do this, we must construct an atlas of homeomorphisms from M/G to open subsets of $\mathbb{R}^n$. For $p \in M$, denote $q = \pi(p) \in M/G$. Let (U, φ) be an adapted coordinate chart centered at p. By definition then, $\varphi(U)$ is an open cube in $\mathbb{R}^m = \mathbb{R}^k \times \mathbb{R}^n$. For open cubes U_k, U_n in $\mathbb{R}^k$ and $\mathbb{R}^n$, respectively, denote $\varphi(U) = U_k \times U_n$. Let $V = \pi(U) \subseteq M/G$. Since U is open in M, V is also open. Also let $Y \subseteq U$ be the submanifold of U as defined earlier in the previous proof for adapted charts. By this definition, the restriction of π to Y is a bijection to V. We remember π is an open map. Since it is also continuous, it is a homeomorphism. We can see that a restriction of a continuous map is also continuous. Take an open set $W \subseteq V$. Then $\pi^{-1}|_Y(W) = \pi^{-1}(W) \cap Y \subseteq Y$ is open (relatively). We can also see that π will be open. Take $Z \subseteq Y$ open. Notice $Z = Y \cap U_o$, for some open U_o in the topology on Y. Noting that π is a homeomorphism:

$$\pi|_Y(Y \cap U_o) = \pi(Y \cap U_o) = \pi(Y) \cap \pi(U_o),$$

where $\pi(U_o)$ is open, then $\pi(Y) \cap \pi(U_o) \subseteq \pi(Y)$ is open (relatively). We then have that the restriction $\pi\big|_Y : Y \longrightarrow V$ is then also a homeomorphism.

Now, let $\sigma : V \longrightarrow Y$ be the local section of π such that $\sigma = (\pi\big|_Y)^{-1}$. Let us define a new composite function with the explicit purpose of creating a homeomorphism from $V \subseteq M/G$ to $U_n \subseteq \mathbb{R}^n$. Let $\eta : V \longrightarrow U_n$ be the following composition:

$$\eta = \pi_n \circ \varphi \circ \sigma,$$

defined by $\eta([(v^1, \cdots, v^k, v^{k+1}, \cdots, v^{k+n})]) = (v^{k+1}, \cdots, v^{k+n})$ and where we let $\pi_n : U_k \times U_n \longrightarrow U_n$ be the projection onto the second factor. Since $\pi\big|_Y$ is a homeomorphism, so is σ. Also, φ is a homeomorphism by definition. Not only that, but note that we are really only looking at a *restriction* of φ since the image of σ is $Y \subseteq U \subseteq M$. The image of V under $\varphi \circ \sigma$ is the submanifold $\{0, \ldots, 0\} \times U_n$. Not unlike ignoring the first $k-$dimensional zero point, the projection π_n is a homeomorphism onto U_n.

The composition of homeomorphisms is again a homeomorphism. Two sets are homeomorphic only if they have the same dimension. Since η is then a homeomorphism, we can see that M/G is locally Euclidean and then its has dimension equal to n. ■

Remark. We can think of Y as the points in the 'vertical thread' through p. This thread intersects each G-orbit in only a single constant slice. We can think of V as condensing each G-orbit of U to a single point (equivalence class) in M/G.

Remark. We can think of η as sending "points" (or equivalence classes) in $V \subseteq M/G$ back to U where it can be any point in the G-orbit, then to an open cube in $\mathbb{R}^k \times \mathbb{R}^n$ and then to an open cube in $\mathbb{R}^n$ by ignoring the additional coordinates we added to move between equivalence classes to the G-orbits.

We still have three non-trivial statements left to confirm: (1) *that there exists a smooth structure on M/G, (2) π is a smooth submersion, and* (3) *that the smooth structure on M/G is unique.*

Proof. (1) Given an atlas $\{(U_\alpha, \varphi_\alpha)\}$ of adapted coordinate charts for M, we will look at the atlas $\{(V_\alpha, \eta_\alpha)\}$ corresponding to the quotient map such that $\pi(U_\alpha) = V_\alpha$. First we will focus specifically on whether two coordinate charts for M centered at the same point are smoothly compatible with respect to their charts in M/G. After that we will show that the compatability of any two charts for M/G will boil down to this same conclusion.

Since we have already shown they exist: let (U, φ) and $(\widehat{U}, \widehat{\varphi})$ be two adapted coordinate charts centered at the same point $p \in M$. Also, let (V, η) and $(\widehat{V}, \widehat{\eta})$ be the corresponding coordinate charts for M/G. If we look at the coordinate functions, $(x, y) = (x_1, \ldots, x_k, y_{k+1}, \ldots, y_m)$ and $(\widehat{x}, \widehat{y}) = (\widehat{x}_1, \ldots, \widehat{x}_k, \widehat{y}_{k+1}, \ldots, \widehat{y}_m)$, for each of the charts for M, we will see that when two points share their y-coordinates, they will also share $\widehat{y}$-coordinates since both are adapted to the action. Since G acts smoothly on M, each of the coordinate functions can then be equated by smooth maps χ, Υ such that $(x, y) = (\chi(\widehat{x}, \widehat{y}), \Upsilon(\widehat{y}))$. The transition map we are truly concerned with is $\widehat{\eta} \circ \eta^{-1}$, which is just the smooth map equating the y- and the $\widehat{y}$-coordinates, or $y = \Upsilon(\widehat{y})$. Since they are smooth by assumption, they are said to be smoothly compatible.

Now, suppose again that (U, φ) and $(\widehat{U}, \widehat{\varphi})$ are each adapted coordinate charts but centered at $p, \widehat{p} \in M$, respectively. Suppose that $\pi(p) = \pi(\widehat{p})$. Then they are in the same orbit and there is an element $g \in G$ such that $g \cdot p = \widehat{p}$. Because D is involutive, the collection of the orbits create a foliation on M, and the orbit map $\theta_g : M \longrightarrow M$ is a diffeomorphism such that it takes orbits to orbits. Then $\widehat{\varphi} \circ \theta_g = \tilde{\varphi}$ is some other adapted coordinate chart centered at p. Notice that if we define $\tilde{\sigma} = \theta_g^{-1} \circ \widehat{\sigma}$, with $\widehat{\sigma} : \widehat{V} \longrightarrow \widehat{U}$ defined similarly to σ, then

$$\tilde{\varphi} \circ \tilde{\sigma} = \widehat{\varphi} \circ \theta_g \circ \theta_g^{-1} \circ \widehat{\sigma} = \widehat{\varphi} \circ \widehat{\sigma},$$

and we can see it is the smooth local section corresponding to $\tilde{\varphi}$. But then the corresponding chart for M/G follows:

$$\tilde{\eta} = \pi_n \circ \tilde{\varphi} \circ \tilde{\sigma} = \pi_n \circ \widehat{\varphi} \circ \widehat{\sigma} = \widehat{\eta}$$

And in similar fashion as before, there is a smooth map defined about the origin equating the $\tilde{y}$- and $\widehat{y}$-coordinates equal to the transition map. Thus, any two coordinate charts in the atlas for M/G are smoothly compatible.

(2) Let us look at $\pi(v^1, \cdots, v^k, v^{k+1}, \cdots, v^{k+n}) = (v^{k+1}, \cdots, v^{k+n})$, where we are using the coordinate representation of an adapted coordinate chart defined earlier. To see it is a smooth submersion, just note that dim $M >$ dim M/G and π has full rank equal to $n = m - k$ at each point in M.

(3) Now, Suppose M/G has two different smooth structures such that $\pi : M \longrightarrow M/G$ is a smooth submersion. We take an equivalence class $[v]$ in the first structure, send the class back via corresponding local sections, and send it to the second structure; define such a map F. We get that F is the identity map of level sets, hence, smooth via Theorem 3.4 above. Similarly, sending an equivalence class from the second structure to the second, we get that F^{-1} is also smooth via Theorem 3.4. Thus, F is a diffeomorphism and is the identity map between the two smooth structures, proving that any smooth structure on the orbit space of G on M is unique. ■

References

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