An Overview of Diffeology

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Jordan Watts

Central Michigan University

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Diffeology

Definition

Let X be a set. A **parametrisation** $p\colon U_p \to X$ is a map from an open subset U_p of some $\mathbb{R}^n$ (n is not fixed). A **diffeology** $\mathcal{D}_X$ on X is a family of parametrisations satisfying

1. all constant parametrisations are in $\mathcal{D}_X$,
2. if p is a parametrisation and $\{U_\alpha\}$ an open cover of U_p such that for each α
$$p|_{U_\alpha} \in \mathcal{D}_X$$
then $p \in \mathcal{D}_X$,
3. if $p \in \mathcal{D}_X$ and $f\colon V \to U_p$ is smooth with V an open subset of some $\mathbb{R}^n$ then $p \circ f \in \mathcal{D}_X$.

Call $(X, \mathcal{D}_X)$ a **diffeological space** and each $p \in \mathcal{D}_X$ a **plot**.

Definition

A map $F\colon (X,\mathcal{D}_X) \to (Y,\mathcal{D}_Y)$ is **diffeologically smooth** if $F \circ p \in \mathcal{D}_Y$ for every $p \in \mathcal{D}_X$.

- Obtain a “complete, co-complete quasi-topos” [BH11]. In particular, we obtain a category admitting all subsets, quotients, products, coproducts, and function spaces.

Irrational Tori

Definition

Let X be a diffeological space with an equivalence relation $\sim$ and quotient map $\pi\colon X \to X/\sim$. A parametrisation $p\colon U_p \to X/\sim$ is a plot in the **quotient diffeology** if for every $u \in U_p$ there exists an open neighbourhood V of u and a plot $q\colon V \to X$ such that $p|_V = \pi \circ q$.

Example

Fix an irrational number α. Consider the action of the group $\mathbb{Z}^2$ on $\mathbb{R}$ by

$$(m, n) \cdot x = x + m + \alpha n.$$

The quotient group $T_\alpha := \mathbb{R}/\mathbb{Z}^2$ has trivial topology, but its diffeology is rich. This space is an example of an **irrational torus**.

Irrational Tori

- Recall that a prequantisation bundle on a symplectic manifold (M, ω) with integral symplectic form is a circle bundle $P \to M$ with connection whose curvature is ω.
- What happens if ω is not integral?
- We obtain an irrational torus bundle $P \to M$, in which the fibre is determined by the group of periods of ω, which in general may be a dense subgroup of $\mathbb{R}$ [I95, IZ13].

Function Spaces

Definition

Let X and Y be diffeological spaces. A parametrisation p of $C^\infty(X,Y)$ is a plot of the **(standard) functional diffeology** if the map

$$U_p \times X \to Y\colon (u,x) \mapsto p(u)(x)$$

is smooth.

- This leads to the Exponential Law for diffeological spaces:

$$C^\infty(X, C^\infty(Y,Z)) \cong C^\infty(X \times Y, Z).$$

Function Spaces

- The category of Fréchet spaces with infinitely-differentiable maps between them forms a full subcategory of diffeological spaces [L92].
- The convenient setting of Kriegl-Michor [KM97] sits naturally within the diffeological framework.
- Diffeomorphism groups, spaces of sections of bundles, etc., come equipped with functional diffeologies.
- We have diffeological homotopy theory, classifying spaces of diffeological groups [MW17,CW21], etc.
- Effective Lie group actions on a manifold M correspond exactly to diffeological subgroups of $\mathrm{Diff}(M)$ [IZK12].

Diffeological Groupoids

Definition

A smooth map $f\colon X \to Y$ is a **subduction** if for every plot p of Y and for every $u \in U_p$, there is an open neighbourhood V of u and a plot $q\colon V \to X$ such that $p|_V = f \circ q$.

Definition

A **diffeological groupoid** is a groupoid $\mathcal{G} = (\mathcal{G}_1 \rightrightarrows \mathcal{G}_0)$ in which $\mathcal{G}_1$ and $\mathcal{G}_0$ are diffeological spaces, and all structure maps are diffeologically smooth.

- The source and target maps are automatically subductions.
- Smooth functors and smooth natural transformations are defined analogously to the Lie case, which gives us a strict 2-category containing Lie groupoids, relation groupoids, inertia groupoids, integrations of Lie algebroids, etc.

Diffeological Groupoids

- Similar to the 2-category of Lie groupoids, the 2-category of diffeological groupoids can be localised (using the calculus of fractions à la Pronk [P96,PS22], anafunctors à la Roberts [R21], bibundles [vdS], or stacks [W22b]), yielding a bicategory which contains the corresponding bicategory of Lie groupoids as a full sub-bicategory [W22b].
- In particular, diffeological Morita equivalence between Lie groupoids is equivalent to Lie Morita equivalence.

Differential Forms

Definition

Let X be a diffeological space.

- A **(differential)** k**-form** α is an assignment to each plot p of X a differential k-form $\alpha_p \in \Omega^k(U_p)$ such that for any plot p and smooth function $f\colon V \to U_p$,

$$f^*\alpha_p = \alpha_{p\circ f}.$$

  (We often may denote α_p by $p^*\alpha$ for this reason.)
- Given a k-form α, the **differential** or **exterior derivative** of α is the $(k+1)$-form $d\alpha$ defined by

$$(d\alpha)_p = d(\alpha_p)$$

  for each plot p of X.

Differential Forms

Definition

- Denote the set of all k-forms of X by $\Omega^k(X)$. Then $(\Omega^*(X), d)$ is the **de Rham complex** of X.
- One can equip $\Omega^k(X)$ with a functional diffeology as well; see [IZ11] for details.

Differential Forms

- Let G be a compact Lie group acting on a manifold M. The quotient map $\pi\colon M \to M/G$ induces an isomorphism of complexes

  $$\pi^*\colon (\Omega^*(M/G), d) \to (\Omega^*(M)_{\text{basic}}, d)$$

  from the de Rham complex of diffeological forms on M/G to the subcomplex of “basic” differential forms on M (*i.e.* G-invariant forms that vanish on vectors tangent to G-orbits) [W12].
- This was generalised to proper actions (and a diffeomorphic isomorphism) in [KW16], and to proper Lie groupoids in [W22a].
- This was further generalised to various foliation groupoids in [Miy23].

Symplectic Quotients

Definition

Given a diffeological space X and a subset $Y \subseteq X$, the **subset diffeology** on Y is the subset of plots of X whose image is in Y.

Definition

Let G be a Lie group acting properly on a symplectic manifold (M, ω) in a hamiltonian fashion with (equivariant) momentum map $\mu \colon M \to \mathfrak{g}^*$. The **(Marsden-Weinstein-(Meyer)) reduced space** or **symplectic quotient** at 0 is the subquotient

$$M/\!/_0 G := \mu^{-1}(0)/G$$

equipped with the subquotient diffeology (*i.e.* the quotient diffeology of the subset diffeology on $\mu^{-1}(0)$).

Symplectic Quotients

- It is known that Sjamaar differential forms on $M//_0G$ (*i.e.* forms defined on the open dense stratum of $M//_0G$ that lift and extend to forms of M) extend to diffeological forms of $M//_0G$; however, it is currently unknown whether these extensions are unique, nor whether all diffeological forms of $M//_0G$ can be obtained in this way [W12].

Sikorski Spaces

Definition

Let X be a set. A **Sikorski (differential) structure** on X is a family of real-valued functions $\mathcal{F}$ on X satisfying

1. if $g \in C^\infty(\mathbb{R}^n)$ and $f_1, \dots, f_n \in \mathcal{F}$, then $g(f_1, \dots, f_n) \in \mathcal{F}$; and
2. with respect to the initial topology on X generated by $\mathcal{F}$, if $f\colon X \to \mathbb{R}$ admits a function $f_x \in \mathcal{F}$ for each $x \in X$ satisfying
$$f|_{U_x} = f_x|_{U_x}$$
on an open neighbourhood U_x of x, then $f \in \mathcal{F}$.

$(X, \mathcal{F})$ is called a **Sikorski (differential) space.**

Definition

A map $\varphi\colon (X, \mathcal{F}_X) \to (Y, \mathcal{F}_Y)$ is **Sikorski smooth** if $\varphi^* f \in \mathcal{F}_X$ for every $f \in \mathcal{F}_Y$.

Sikorski Spaces

- Sikorski spaces form a category admitting subspaces, products, coproducts, and quotients.
- Given a diffeological space X, the set of diffeologically smooth real-valued functions $C^\infty(X)$ is a Sikorski structure on the underlying set of X.
- In fact, these spaces $(X, C^\infty(X))$ in which $C^\infty(X)$ is the ring of diffeologically smooth functions coming from a diffeological space X form a subcategory of Sikorski spaces isomorphic to the category of Frölicher spaces.

Sikorski Spaces

- Given a proper action of a Lie group G on a manifold M, the diffeologically smooth real-valued functions on the orbit space $C^\infty(M/G)$ is the set of functions that pullback to invariant smooth functions on M.
- $(M/G, C^\infty(M/G))$, in turn, is a “subcartesian space” in the sense of Aronszajn/Śniatycki, in that it locally is Sikorski diffeomorphic to orbit spaces V/H of linear representations of compact Lie groups $H \circlearrowright V$, which by Schwarz [Sch74] are Sikorski diffeomorphic to semi-algebraic varieties sitting in Euclidean spaces.

Orbifolds

- Viewing (effective) orbifolds as the full sub-bicategory of proper effective étale Lie groupoids, there is an “essentially injective” functor from these to diffeological spaces, whose image is known as the category of “diffeological orbifolds” [IZKZ10]; that is, two diffeological orbifolds are diffeomorphic if and only if the corresponding orbifold groupoids are Morita equivalent.
- The orbit type strata of diffeological orbifolds are exactly the so-called Klein strata, up to connectivity; the Klein strata are the orbits of the pseudo-group of local diffeomorphisms of the orbifold [GIZ23].
- Passing to the corresponding Sikorski spaces as above, we obtain a functor from orbifold groupoids to Sikorski spaces that remains essentially injective [W17].

Orbit Type Stratifications

- Let G be a Lie group acting on a manifold M properly. The orbit type stratification on M is exactly the collection of accessible sets of invariant vector fields; that is, two points are in the same orbit type stratum if and only if one can get from one to the other by moving along a finite number of flows of these vector fields.
- This fact, combined with the results of Bierstone [B80], Schwarz [Sch80], and Śniatycki [Ś03], show that the orbit type stratification of M/G is given exactly by the accessible sets of vector fields on $(M/G, C^\infty(M/G))$ as a Sikorski space.

Orbit Type Stratifications

- If (M,ω) is a symplectic manifold and the G-action is symplectic, then the G-invariant smooth functions admit a Poisson structure that descends to a Poisson structure on $C^\infty(M/G)$; the "Poisson reduced space" [CŚ01].
- If the G-action is furthermore hamiltonian, then the pullback of the Poisson structure to the symplectic quotient $M/\!/_0G$ yields the Poisson structure of Sjammar-Lerman [SjL91].
- They show that the orbit type strata of this space are symplectic manifolds; moreover, some folklore theory allows one to arrive at the result that the accessible sets of vector fields on $(M/\!/_0G, C^\infty(M/\!/_0G))$ are exactly these strata.
- It remains open whether accesssible sets of hamiltonian vector fields on $(M/\!/_0G, C^\infty(M/\!/_0G))$ induce these strata.

Čech-de Rham Complex

- For manifolds, it is well-known that Čech cohomology and de Rham cohomology are isomorphic.
- For diffeological spaces, this is no longer true. In [IZ24], it is shown that on the level of degree one, the obstruction includes the space of (isomorphism classes of) real-line bundles with connection.

$$0 \to H^1_{dR}(X) \to \check{H}^1_{IZ}(X,\mathbb{R}) \to E_2^{1,0}(X) \to H^2_{dR}(X) \to \check{H}^2_{IZ}(X,\mathbb{R})$$

- These bundles can be non-trivial (the irrational flow in $\mathbb{T}^2$ yields a non-trivial real-line bundle over an irrational torus).
- For higher degrees, connections on real-line bundle gerbes show up as part of the obstruction [Min24].

Thank you!

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