Bicategories of Diffeological Groupoids

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Jordan Watts, “Bicategories of diffeological groupoids”,
https://arxiv.org/abs/2206.12730

Why Study Diffeological Groupoids?

- They appear in the work of Blohmann, Fernandes, and Weinstein on general relativity, in which a diffeological groupoid describes the choices of embeddings of an initial space-like hypersurface in a lorentzian spacetime, up to a prescribed equivalence [BFW13].
- They show up naturally in the study of singular subalgebroids of Lie algebroids, and related, the holonomy and fundamental groupoids of singular foliations [AZ20,GV21].
- Presentations of "higher geometric" loop spaces (*i.e.* loop stacks) are presented by diffeological groupoids [RV18]; in fact, these groupoids are Fréchet-Lie groupoids.
- Inertia groupoids of Lie groupoids and relation groupoids of equivalence relations are typically not Lie groupoids, but they are diffeological groupoids.

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What Has Been Done So Far?

- A rigorous foundation for diffeological groupoids using so-called “bibundles” was written by Nesta van der Schaaf in his Master’s thesis [vdS20] and subsequent paper [vdS21].
- An open question from his work was whether a so-called diffeological Morita equivalence between Lie groupoids was in fact a Lie Morita equivalence. (*i.e.* Do groupoids coming from the realm of smooth manifolds fit into the diffeological theory as one would hope?)
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Fix a set X.

Definition

A **parametrisation of** X is a function $p\colon U \to X$ where U is a Euclidean open set (no particular dimension).

A **diffeology on** X is a family $\mathcal{D}$ of parametrisations of X satisfying:

1. all constants maps $U \to X$ are in $\mathcal{D}$;
2. given $p\colon U \to X$ for which there is an open cover $\{U_\alpha\}$ of U, and for each α a parametrisation $p_\alpha \in \mathcal{D}$ such that $p_\alpha = p|_{U_\alpha}$, then $p \in \mathcal{D}$;
3. if $f\colon V \to U$ is smooth with V a Euclidean open set and $(p\colon U \to X) \in \mathcal{D}$, then $p \circ f \in \mathcal{D}$.

We call $(X, \mathcal{D})$ a **diffeological space**, and parametrisations $p \in \mathcal{D}$ **plots.**

Diffeological Spaces

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A function $F\colon (X, \mathcal{D}_X) \to (Y, \mathcal{D}_Y)$ is **(diffeologically) smooth** if for every $p \in \mathcal{D}_X$,

$$F \circ p \in \mathcal{D}_Y.$$

- Diffeological spaces with smooth maps form a category $\mathbf{Diffeol}$ which is a complete and cocomplete quasi-topos. In particular, subsets, quotients, products, coproducts, function spaces, etc., all have natural diffeological structures.
- $\mathbf{Diffeol}$ contains the category of smooth manifolds (as a full subcategory), effective orbifolds, orbit spaces, etc.

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Subductions

Definition

A smooth surjection $F\colon (X,\mathcal{D}_X) \to (Y,\mathcal{D}_Y)$ is a **subduction** if for every plot $p\colon U \to Y$ of Y, there is an open cover $\{U_\alpha\}$ of U and for each α a plot $q_\alpha\colon U_\alpha \to X$ such that

$$p|_{U_\alpha} = F \circ q_\alpha.$$

Definition

A subduction $F\colon (X,\mathcal{D}_X) \to (Y,\mathcal{D}_Y)$ is a **local subduction** if for every plot $p\colon U \to Y$, every $u \in U$, and every $x \in F^{-1}(p(u))$, there is an open neighbourhood $V \subseteq U$ of u and a plot $q\colon V \to Y$ of Y so that $p|_V = F \circ q$ and $q(u) = x$.

Exercise

A map between manifolds is a local subduction if and only if it is a surjective submersion.

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A map is a diffeomorphism if and only if it is an injective subduction.

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The smooth map $\mathbb{R} \amalg \mathbb{R} \to \mathbb{R}$ which is constant (say, equal to 0) on the first copy of $\mathbb{R}$, but equal to the identity on the second copy of $\mathbb{R}$, is a subduction. However, it is not a local subduction.

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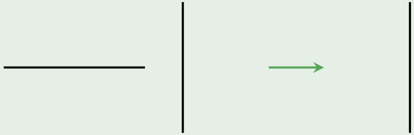

Groupoids

Definition

A groupoid $\mathcal{G}$ (or $\mathcal{G}_1 \rightrightarrows \mathcal{G}_0$) is a (small) category in which all of the morphisms are invertible.

- The **objects** are denoted $\mathcal{G}_0$;
- the **arrows** are denoted $\mathcal{G}_1$;
- the **source map**, denoted s, sends an arrow $x \underset{g}{\longrightarrow} x'$ to object x;
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Examples

- A group G is a groupoid with one object, and whose arrows are exactly G.
- A set X can be perceived as a **trivial groupoid** whose objects are the points of X and arrows are exactly the corresponding units.
- Given a set X, the **pair groupoid** is the groupoid whose objects are points of X and arrows are all pairs (x, x') in $X \times X$.
- Given an equivalence relation $\sim$ on a set X, the **relation groupoid** R has objects points of X, with an arrow between x and x' if $x \sim x'$.

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- Groupoids, functors, and natural transformations form a (strict) 2-category.

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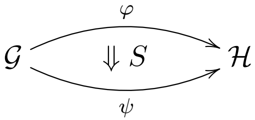

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- Groupoids, functors, and natural transformations form a (strict) 2-category.

Lie Groupoids

Definition

A **Lie groupoid** is a groupoid in which the object and arrow sets are smooth manifolds, and all structure maps are smooth. Additionally, we require s and t to be surjective submersions (*i.e.* local subductions). (For the purposes of this talk, we also require arrow spaces to be Hausdorff.)

Definition

A functor $\varphi\colon \mathcal{G} \to \mathcal{H}$ is **smooth** if $\varphi_0\colon \mathcal{G}_0 \to \mathcal{H}_0$ and $\varphi_1\colon \mathcal{G}_1 \to \mathcal{H}_1$ are smooth. A natural transformation $S\colon \varphi \Rightarrow \psi$ between two smooth functors $\varphi, \psi\colon \mathcal{G} \to \mathcal{H}$ is **smooth** if the corresponding map $\mathcal{G}_0 \to \mathcal{H}_1$ is smooth.

- Lie groupoids, smooth functors, and smooth natural transformations form a 2-category, denoted $\mathbf{LieGpoid}$.

Lie Groupoids

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A **Lie groupoid** is a groupoid in which the object and arrow sets are smooth manifolds, and all structure maps are smooth. Additionally, we require s and t to be surjective submersions (*i.e.* local subductions). (For the purposes of this talk, we also require arrow spaces to be Hausdorff.)

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A functor $\varphi\colon \mathcal{G} \to \mathcal{H}$ is **smooth** if $\varphi_0\colon \mathcal{G}_0 \to \mathcal{H}_0$ and $\varphi_1\colon \mathcal{G}_1 \to \mathcal{H}_1$ are smooth. A natural transformation $S\colon \varphi \Rightarrow \psi$ between two smooth functors $\varphi, \psi\colon \mathcal{G} \to \mathcal{H}$ is **smooth** if the corresponding map $\mathcal{G}_0 \to \mathcal{H}_1$ is smooth.

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Example

- A Lie group is a Lie groupoid.
- A manifold is a Lie groupoid.
- Given a Lie group G acting on a manifold M, the **action groupoid** $G \ltimes M$ is a Lie groupoid whose objects are points of M and arrows are pairs (g, x) in $G \times M$. The source map sends (g, x) to x, and the target sends (g, x) to $g \cdot x$.
- Given a vector field on a manifold M, its local flow induces a Lie groupoid whose object space is M and arrow space is a submanifold of $\mathbb{R} \times M$.

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A **diffeological groupoid** is a groupoid $\mathcal{G} = (\mathcal{G}_1 \rightrightarrows \mathcal{G}_0)$ in which $\mathcal{G}_1$ and $\mathcal{G}_0$ are diffeological spaces, and all structure maps are diffeologically smooth.

- The source and target maps are automatically subductions, but unlike the Lie category, *they are not local subductions*.
- Smooth functors and smooth natural transformations are defined analogously to the Lie case, which gives us a strict 2-category, $\mathbf{DGpoid}$.
- $\mathbf{DGpoid}$ contains Lie groupoids, relation groupoids, inertia groupoids, integrations of Lie algebroids, etc.

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Equivalences of Categories

- In mathematics, we love equivalences of categories:
  - $\mathbb{N} \hookrightarrow \mathbf{FiniteSets}$ is an equivalence of categories.
  - The category of covering spaces of a (nice) path-connected topological space X forms a category equivalent to the category of permutation representations of $\pi_1(X)$.
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Given categories $\mathcal{C}$ and $\mathcal{D}$, an equivalence of categories $f\colon \mathcal{C} \to \mathcal{D}$ is a functor equipped with another functor $g\colon \mathcal{D} \to \mathcal{C}$ and two natural transformations $S\colon \mathrm{id}_{\mathcal{C}} \Rightarrow g \circ f$ and $T\colon f \circ g \Rightarrow \mathrm{id}_{\mathcal{D}}$. Sometimes, these are also required to satisfying the Triangle Identities:

$$(Tf) \circ (fS) = \mathrm{id}_{\varphi} \quad \text{and} \quad (gT) \circ (Sg) = \mathrm{id}_{\psi}.$$

Example

Suppose $\rho\colon P \to M$ is a principal G-bundle. If the action groupoid $G \ltimes P$ admits an equivalence to the groupoid M, then P must be the trivial bundle.

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- A solution is to consider an "equivalent definition of equivalence" in the case of small categories: we can ask for the functor to be **essentially surjective** on objects (*i.e.* surjective up to isomorphism), and **fully faithful** (*i.e.* bijective on hom sets). With this in mind...

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Weak Equivalences

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A **weak equivalence** $\varphi\colon \mathcal{G} \to \mathcal{H}$ in $\mathbf{DGpoid}$ is a smooth functor that satisfies

1. **(Smooth Essential Surjectivity)** $\Psi_\varphi\colon \mathcal{G}_0{}_\varphi\times_t \mathcal{H}_1 \to \mathcal{H}_0\colon (x,h) \mapsto s(h)$ is a *subduction*,
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The action groupoid of any principal G-bundle over a manifold M is weakly equivalent to M.

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The weak equivalences in the previous example are invertible if and only if the bundle is trivial.

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- Let W be the class of all weak equivalences between diffeological groupoids.
- Define a bicategory $\mathbf{DGpoid}[W^{-1}]$, the **bicategory of fractions**, as follows:
  0. Objects are diffeological groupoids.
  1. A 1-cell from $\mathcal{G}$ to $\mathcal{H}$ is a **generalised morphism**:

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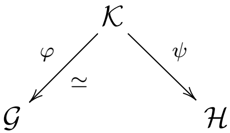

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- The equivalence relation is given by yet another generalised morphism between the centres of the diagrams, and some identities involving the natural transformations. (See Pronk [P96] or [W] for details.)

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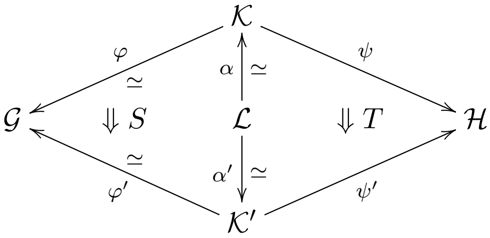

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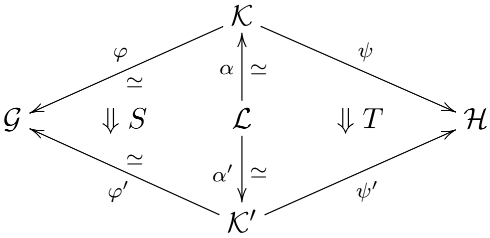

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- Let M be a manifold.
- Let $\{U_\alpha\}$ and $\{V_\beta\}$ be two open covers of M. Define

$$\mathcal{U}_0 := \coprod U_\alpha,$$
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- Let

$$\mathcal{U}_1 := \coprod U_\alpha \cap U_{\alpha'},$$
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- If $\{W_\gamma\}$ is a refinement of the two open covers, then we construct a Čech groupoid $\mathcal{W}$ similarly.
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- Through the work of Moerdijk and Pronk [MP97], we can do a similar procedure with orbifolds atlases:
- Two orbifold atlases (viewed as Lie groupoids) $\mathcal{A}$ and $\mathcal{B}$ are equivalent if their charts are all compatible, leading to a larger orbifold atlas.
- More precisely, if there is another orbifold atlas $\mathcal{C}$ and a Morita equivalence

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$$\mathcal{A} \xleftarrow{\ \simeq\ } \mathcal{C} \xrightarrow{\ \simeq\ } \mathcal{B}$$

then the orbifold altases are equivalent; they describe the same orbifold.

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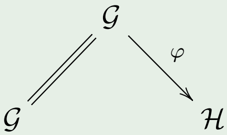

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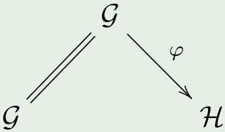

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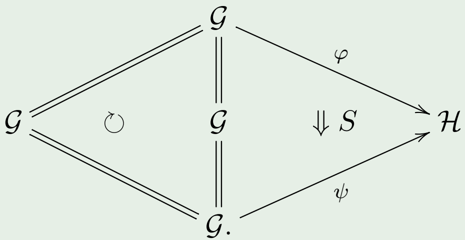

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The bicategory whose objects are diffeological groupoids, 1-cells are generalised morphisms, and 2-cells are as described above, denoted $\mathbf{DGpoid}[W^{-1}]$, admits an inclusion pseudofunctor $\mathfrak{S}\colon \mathbf{DGpoid} \to \mathbf{DGpoid}[W^{-1}]$ given by spanisation, and admits a pseudo-inverse for every weak equivalence $\varphi \in W$.

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$$\mathcal{G} \xleftarrow[\simeq]{\ \varphi\ } \mathcal{K} \xrightarrow[\simeq]{\ \psi\ } \mathcal{H} \qquad \mathcal{H} \xleftarrow[\simeq]{\ \psi\ } \mathcal{K} \xrightarrow[\simeq]{\ \varphi\ } \mathcal{G}.$$

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  0. Objects are diffeological groupoids.
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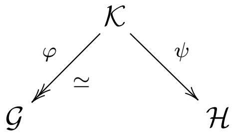

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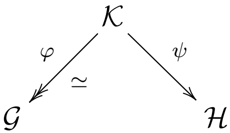

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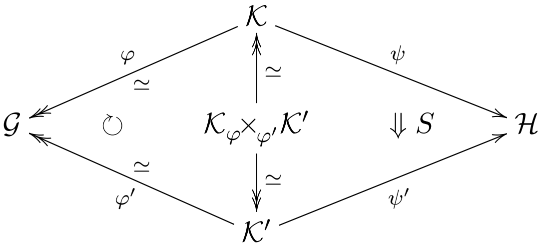

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Definition

A **right principal bibundle from $\mathcal{G}$ to $\mathcal{H}$** is a right principal $\mathcal{H}$-bundle $a_L \colon P \to \mathcal{G}_0$ with anchor map $a_R \colon P \to \mathcal{H}_0$ equipped with a $\mathcal{G}$-action with anchor map a_L that commutes with the $\mathcal{H}$-action, and so that a_R is $\mathcal{G}$-invariant.

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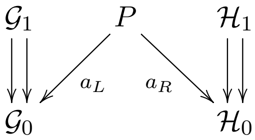

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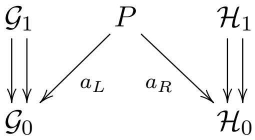

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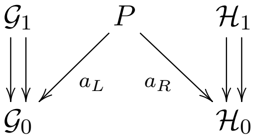

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Theorem (continued)

Moreover, we have the following 2-commutative diagram

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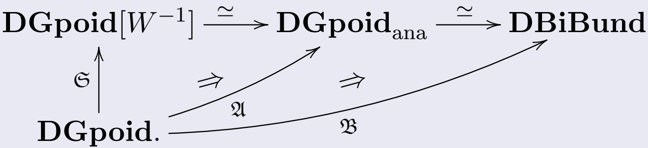

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In fact, we can repeat the entire story above using the 2-category of Lie groupoids $\mathbf{LieGpoid}$, and obtain three equivalent bicategories

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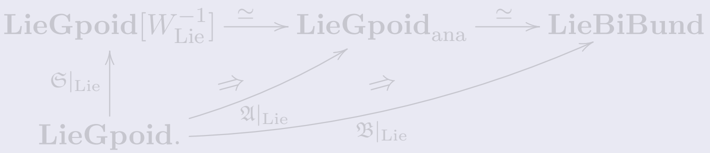

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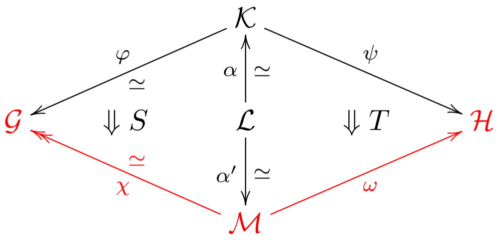

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- Passing from $\mathbf{DGpoid}[W^{-1}]$ to $\mathbf{DGpoid}_{\mathrm{ana}}$ chooses an "optimal" representative of each 2-cell from $\mathbf{DGpoid}[W^{-1}]$: the (unique) transformation in the equivalence class.
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We are in the realm of diffeology: can we make these "optimisation" processes true optimisation processes? More precisely, can we create a reasonable diffeological space of representatives of a 2-cell from $\mathbf{DGpoid}[W^{-1}]$, and a reasonable diffeological space of anafunctors (or generalised morphisms) between two fixed diffeological groupoids, and make this a true optimisation problem?

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Work in Progress

- Fix two diffeological groupoids $\mathcal{G}$ and $\mathcal{H}$. Work in progress indicates that there is a natural diffeology on the groupoid whose objects are a subset of the class of anafunctors from $\mathcal{G}$ to $\mathcal{H}$, and whose arrows are a subset of the class of 2-cells between these anafunctors (as generalised morphisms).

Work in Progress

- In a joint project with Carla Farsi and Laura Scull (soon to be put on the arXiv), we focus on action groupoids of Lie group actions on manifolds, and show that any generalised morphism between two such groupoids admits a 2-cell to an anafunctor

$$G \ltimes M \xleftarrow[\simeq]{\varphi} \mathcal{K} \xrightarrow{\psi} H \ltimes N$$

  in which $\mathcal{K}$ is the action groupoid of a $(G \times H)$-action and φ and ψ are induced by equivariant maps.

  - This is accomplished by passing from $\mathbf{LieGpoid}[W^{-1}]$ to $\mathbf{LieGpoid}_{\mathrm{ana}}$ to $\mathbf{LieBiBund}$ and then back.
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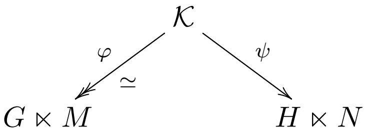

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Applications

1. $\mathbf{DGpoid}[W^{-1}]$, $\mathbf{DGpoid}_{\text{ana}}$, and $\mathbf{DBiBund}$ are all equivalent as bicategories to the 2-category of geometric stacks over diffeological spaces.
2. Given a diffeological space X and an abelian diffeological group G, the category of generalised morphisms from X to G is equivalent to $\check{\mathcal{H}}^1(X;G)$, the category of diffeological Čech 1-cocycles with 0-chains in between. This, in turn, is equivalent to principal G-bundles over X with bundle isomorphisms in between. (See [KWW21].)
3. If X is such that all principal G-bundles are D-numerable, then this category is also equivalent to $[X, BG]$, comprising smooth maps $X \to BG$ with smooth homotopies in between. (See [CW21, MW17].)

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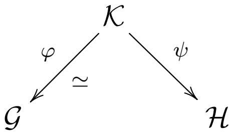
Construction: Generalised Morphism to Bibundle



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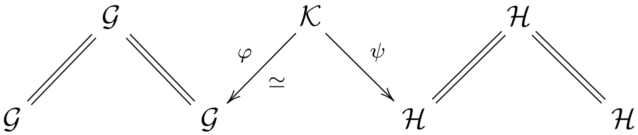

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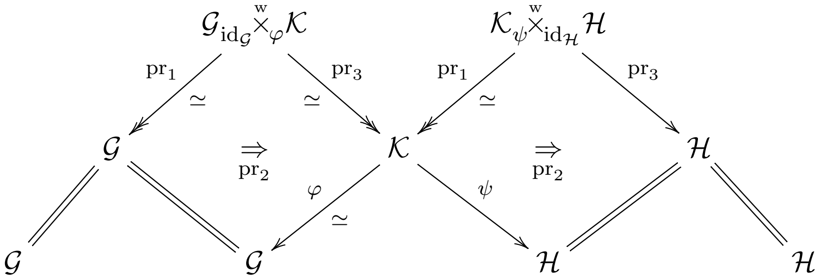

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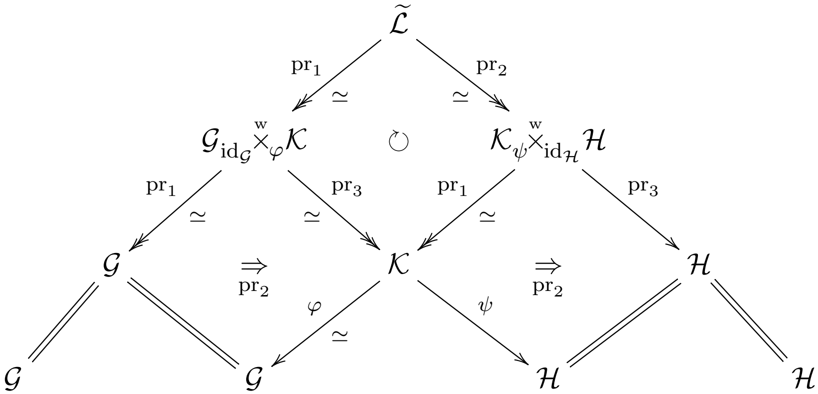

where $\widetilde{\mathcal{L}} = (\mathcal{G}_{\mathrm{id}_\mathcal{G}}\overset{\mathrm{w}}{\times}_\varphi\mathcal{K})_{\mathrm{pr}_3}\times_{\mathrm{pr}_1}(\mathcal{K}_\psi\overset{\mathrm{w}}{\times}_{\mathrm{id}_\mathcal{H}}\mathcal{H})$.

Construction: Generalised Morphism to Bibundle

$\widetilde{\mathcal{L}}$ comes equipped with a left and a right $\mathcal{K}$-action. Let $\mathcal{L} := \mathcal{K}\backslash\widetilde{\mathcal{L}}/\mathcal{K}$.

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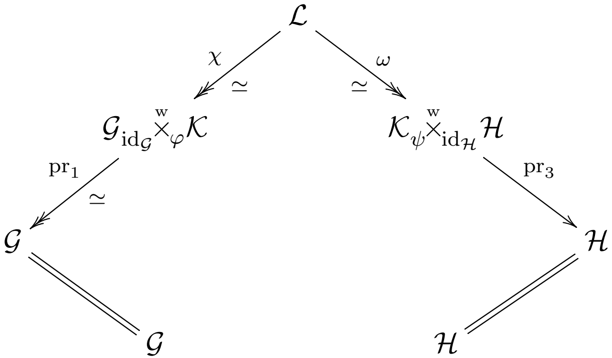

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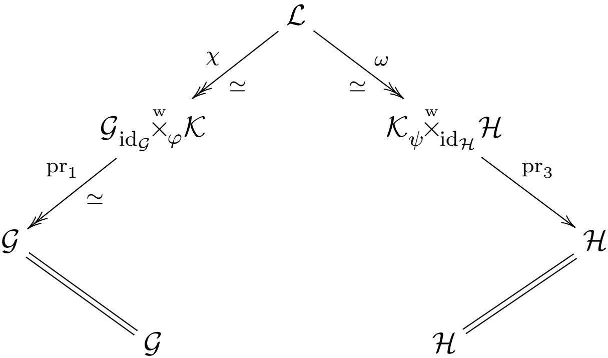

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Thank you!

References

AZ20 Iakovos Androulidakis and Marco Zambon, “Integration of singular subalgebroids”, 2020 (preprint).

BFW13 Christian Blohmann, Marco Cezar Barbosa Fernandes, and Alan Weinstein, “Groupoid symmetry and constraints in general relativity”, *Commun. Contemp. Math.*, 1.1250061 (2013).

CW21 J. Dan Christensen and Enxin Wu, “Smooth classifying spaces”, *Israel J. Math.* **241** (2021), 911–954.

GV21 Alfonso Garmendia and Joel Villatoro, “Integration of singular foliations via paths”, 2021 (preprint).

KWW21 Derek Krepski, Jordan Watts, and Seth Wolbert, “Sheaves, principal bundles, and Čech cohomology for diffeological spaces, 2021 (preprint).

MW17 Jean-Pierre Magnot and Jordan Watts, “The diffeology of Milnor’s classifying space”, *Topol. Appl.* **232** (2017), 189–213.

MP97 Ieke Moerdijk and Dorette Pronk, “Orbifolds, sheaves and groupoids”, *K-Theory* **12** (1997), 3–21.

P96 Dorette Pronk, “Etendues and stacks as bicategories of fractions”, *Compos. Math.*, **102** (1996), 243–303.

PS10 Dorette Pronk and Laura Scull, “Translation groupoids and orbifold cohomology”, *Canad. J. Math.* **62** (2010), 614–645.

R21 David Michael Roberts, “The elementary construction of formal anafunctors”, *Categ. Gen. Algebr. Struct.*, **15** (2021), 183–229.

RV18 David Michael Roberts and Raymond F. Vozzo, “Smooth loop stacks of differentiable stacks and gerbes”,

Cah. Topol. Géom. Différ. Catég., **59** (2018), 95–141.

References

PS21 Dorette Pronk and Laura Scull, "Bicategories of fractions revisited: towards small Homs and canonical 2-cells", 2021 (preprint).

vdS20 Nesta van der Schaaf, *Diffeology, Groupoids & Morita Equivalence*, Master's thesis, Radboud University, 2020.

vdS21 Nesta van der Schaaf, "Diffeological Morita Equivalence", *Cah. Topol. Géom. Différ. Catég.*, **LXII-2** (2021), 177–238.

W Jordan Watts, "Bicategories of Diffeological Groupoids", 2022 (preprint).