Sheaves, principal bundles, and Cech cohomology for diffeological spaces

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~joint with Derek Krepski and Seth Wolbert

CLASSICAL SHEAVES

- X top sp
- $\mathrm{Open}(X)$ – category of open sets of X with inclusions
- A presheaf $P : \mathrm{Open}(X)^{op} \longrightarrow \mathcal{C}$

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$$\begin{tikzcd} V \arrow[r, hook, "i"] \arrow[d, squiggly] & U \arrow[d, squiggly] \\ P(V) & P(U) \arrow[l, "i^*"'] \end{tikzcd}$$

i^* ← "restriction"

- a sheaf $S : \mathrm{Open}(X)^{op} \longrightarrow \mathcal{C}$ is a presheaf

s.t. $\underbrace{0 \longrightarrow S(U) \longrightarrow \bigoplus_{\alpha \in A} S(U_\alpha) \longrightarrow \bigoplus_{\alpha, \beta \in A} S(U_\alpha \cap U_\beta)}_{\text{exact}}$

where $U \subseteq X$ is any open set, $\{U_\alpha\}$ open cover of U,

and $S(\emptyset) = 0$.

- Let $(X, \mathcal{D}_X)$ be a diffeological space

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- Let $Plot(X)$ be the category with plots as objects and

$$
\begin{tikzcd}
U_p \arrow[rr, "f_p^q"] \arrow[dr, "p"'] & & U_q \arrow[dl, "q"] \\
& X &
\end{tikzcd}
$$

smooth

as arrows.

- A presheaf P on X is a functor

$$P : Plot(X)^{op} \longrightarrow \mathcal{C}$$

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- A sheaf S on X is a presheaf s.t. $\forall$ plot p of X, $S|_{U_p}$ is a sheaf in the classical sense above.

e.g. 1) $\Omega^k : Plot(X)^{op} \longrightarrow \underline{Vect}_{\mathbb{R}}$

$$p \longmapsto \Omega^k(U_p)$$
$$f_p^q \longmapsto (f_p^q)^*$$

2) $C^\infty(\cdot, Y) : Plot(X)^{op} \longrightarrow \underline{Set}$

$$: p \longmapsto C^\infty(U_p, Y)$$
$$: f_p^q \longmapsto (f_p^q)^*$$

3) Let $\varphi : X \to Y$ be smooth,

$\Gamma(\ \ ) : Plot(X)^{op} \longrightarrow Set$

not determined by the plot-

3) Let $\varphi : X \to Y$ be smooth,

$$\Gamma(\cdot, \varphi) : \mathrm{Plot}(Y)^{op} \longrightarrow \underline{\mathrm{Set}}$$
$$\Gamma(p, \varphi) = \{ \tilde{p} \in C^\infty(U_p, X) \mid \varphi \circ \tilde{p} = p \}$$
$$\Gamma(f_p^q, \varphi) = (f_p^q)^*$$

not determined by the plot-domain!

//

- Let $P : \mathrm{Plot}(X)^{op} \to \mathcal{C}$ be a presheaf.

$P(X)$ = <u>global sections</u> whose elements are "coherent families" $\{\eta_p\}_{p \in \mathcal{D}_X}$.

That is, $\quad (f_p^q)^* \eta_q = \eta_p \quad \forall\, p \xrightarrow{f_p^q} q$.

e.g. 1) The global sections of Ω^k are k-forms on X.

2) —— " —— of $C^\infty(\cdot, Y)$ are smooth maps $C^\infty(X, Y)$.

3) —— " —— $\Gamma(\cdot, \varphi)$ are global lifts $Y \to X$, which need not exist.

$$\begin{tikzcd} Y \arrow[r, "\text{global lift}"'] \arrow[rr, bend right, "\mathrm{id}_Y"'] & X \arrow[r, "\varphi"] & Y \end{tikzcd}$$

- $\varphi : X \to Y$, $\quad P : \mathrm{Plot}(Y)^{op} \to \mathcal{C}$ presheaf, define the <u>pullback</u>

<u>presheaf</u> $\quad \varphi^* P : \mathrm{Plot}(X)^{op} \to \mathcal{C} : p \mapsto P(\varphi \circ p),$

$$f_p^q \longmapsto (f_p^q)^*$$

N.B. If P is a sheaf, then so is $\varphi^* P$.

A (covering) generating family of X is a collection $\mathcal{Q} \subset \mathcal{D}_X$ so that $\forall p \in \mathcal{D}_X$ $\exists$ open cover $\{U_\alpha\}$ of U_p, $\{q_\alpha\} \subset \mathcal{Q}$, $\{f_\alpha\}$ so that $p|_{U_\alpha} = q_\alpha \circ f_\alpha$ $\forall \alpha$.

The nebula of $\mathcal{Q}$, denoted $\mathcal{N}(\mathcal{Q})$ is $\coprod_{q \in \mathcal{Q}} U_q$. It has an evaluation map

$$\begin{aligned} ev : \mathcal{N}(\mathcal{Q}) &\longrightarrow X \\ u \in U_q &\longmapsto q(u) \end{aligned}$$

e.g. M = smooth mfld, $\mathcal{Q}$ induced by an atlas, then $\mathcal{N}(\mathcal{Q}) = \coprod$ (inverted) chart domains. //

$$\mathcal{N}(\mathcal{Q})_k = \underbrace{\mathcal{N}(\mathcal{Q}) {}_{ev}\!\times_{ev} \mathcal{N}(\mathcal{Q}) {}_{ev}\!\times_{ev} \cdots {}_{ev}\!\times_{ev} \mathcal{N}(\mathcal{Q})}_{k+1}$$

$$= \{(u_0, .., u_k) \mid ev(u_i) = ev(u_j)\ \forall i,j\}$$

Evaluation map $ev_k : \mathcal{N}(\mathcal{Q})_k \longrightarrow X$

<u>Evaluation map</u> $ev_k : \mathcal{N}(\mathcal{Q})_k \longrightarrow X$
$: (u_0, .., u_k) \longmapsto ev(u_k)$

<u>Degeneracy map</u> $d_i : \mathcal{N}(\mathcal{Q})_k \longrightarrow \mathcal{N}(\mathcal{Q})_{k-1}$
$: (u_0, .., u_k) \longmapsto (u_0, .., \hat{u}_i, .., u_k)$

P be a presheaf of <u>abelian groups</u> on X.
$ev_k^* P$ is a —— " ———— on $\mathcal{N}(\mathcal{Q})_k$.

The degeneracy maps induce

$$\tilde{d}_i : ev_k^* P(\mathcal{N}(\mathcal{Q})_k) \longrightarrow ev_{k+1}^* P(\mathcal{N}(\mathcal{Q})_{k+1})$$
$$\{\eta_{(f_0, .., f_k)}\} \longmapsto \{\eta_{(f_0, .., \hat{f}_i, .., f_{k+1})}\}$$

Define $\check{C}^k(\mathcal{Q}, P) := ev_k^* P(\mathcal{N}(\mathcal{Q})_k)$ and
$\partial = \sum_{i=0}^{k+1} (-1)^i \tilde{d}_i$.

Get $\partial^2 = 0$, get complex $(\check{C}^\bullet(\mathcal{Q}, P), \partial)$.
Its homology $\check{H}^k(\mathcal{Q}, P)$ is the k^{th} <u>Čech cohom</u>. w.r.t. $\mathcal{Q}$.

If $\mathcal{Q}$ is a gen. family, a <u>refinement</u> is

a gen. family $\mathcal{R}$ where if $\mathcal{Q} = \{q_\alpha\}_{\alpha \in A}$ and

$\mathcal{R} = \{r_\beta\}_{\beta \in B}$, then $\exists$ a function $f : B \to A$ s.t. $\forall r_\beta$

$\exists f_\beta : U_{r_\beta} \to U_{q_{f(\beta)}}$ s.t. $q_{f(\beta)} \circ f_\beta = r_\beta$.

$\exists\, f_\beta : U_{\underline{r_\beta}} \to U_{\underline{q_{\psi(\beta)}}}$ s.t. $q_{\psi(\beta)} \circ f_\beta = r_\beta$.

- every two gen families have a common ref.

- If $\mathcal{R}$ refines $\mathcal{Q}$, then $\exists$ an induced map $f : \mathcal{N}(\mathcal{R}) \longrightarrow \mathcal{N}(\mathcal{Q})$.
$$u \in U_{r_\beta} \longmapsto f_\beta(u) \in U_{q_{\psi(\beta)}}$$

- Get $f^k : \mathcal{N}(\mathcal{R})_k \longrightarrow \mathcal{N}(\mathcal{Q})_k$
$$f^{k-1} \circ d_i = d_i \circ f^k.$$

- $f^\# : (\check{C}^\bullet(\mathcal{Q}, P), \partial) \longrightarrow (\check{C}^\bullet(\mathcal{R}, P), \partial)$ and
$$\check{f} : \check{H}^\bullet(\mathcal{Q}, P) \longrightarrow \check{H}^\bullet(\mathcal{R}, P)$$

- Take directed limits (i.e. colimits) over refinements, get $\check{H}^\bullet(X, P)$

natural in X.

Results

- G – a diffeolgical abelian group, $\check{H}^1(X, C^\infty(\cdot, G)) =: \check{H}^1(X, G)$

classifies all (diffeol) principal G-bundles over X.

- If X is smoothly paracompact, then

[illegible]

$$\check{H}^1(X, G) \cong \underset{G\text{-bundles}}{\text{Principal}} \underset{\text{Magnot-W. / Christensen-Wu}}{\cong} [X, BG] = \text{homotopy classes of smooth maps from } X \text{ to } BG.$$

• If $0 \Rightarrow P_1 \Rightarrow P_2 \Rightarrow P_3 \Rightarrow 0$ is a short exact sequence of sheaves (defined similarly to classical case),

then get a long exact seq

$$0 \to P_1(X) \to P_2(X) \to P_3(X) \to \check{H}^1(X, P_1) \to \check{H}^1(X, P_2)$$
$$\to \check{H}^1(X, P_3) \to \cdots$$

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e.g. $\check{H}^1(T_\alpha, \mathbb{R}) \neq 0$ $\qquad \left.\begin{array}{c} \mathbb{T}^2 \to T_\alpha \\ \mathbb{R}\text{-bundle} \end{array}\right\}$ represents non-trivial class.

e.g. $\overset{\text{Lie group}}{G} \curvearrowright M$ proper action $\qquad \check{H}^k(M/G, \mathbb{R}) = 0$ $\forall k > 0$.

(Holds for orbit spaces of proper groupoids, too.)