

~ joint with Derek Krepski and Seth Wolbert

CLASSICAL SHEAVES

• X top sp

• $\text{Open}(X)$ — category of open sets of X with inclusions

• A presheaf $P: \text{Open}(X)^{\text{op}} \rightarrow \mathcal{C}$

$$\begin{array}{ccc} V & \xrightarrow{i} & U \\ \{ & & \{ \end{array}$$

$$P(V) \xleftarrow{i^*} P(U)$$

i^* — "restriction"

↑
sets / groups / vector spaces

• a sheaf $S: \text{Open}(X)^{\text{op}} \rightarrow \mathcal{C}$ is a presheaf

$$\text{s.t. } 0 \longrightarrow S(U) \xrightarrow{\text{exact}} \bigoplus_{\alpha \in A} S(U_\alpha) \longrightarrow \bigoplus_{\alpha, \beta \in A} S(U_\alpha \cap U_\beta)$$

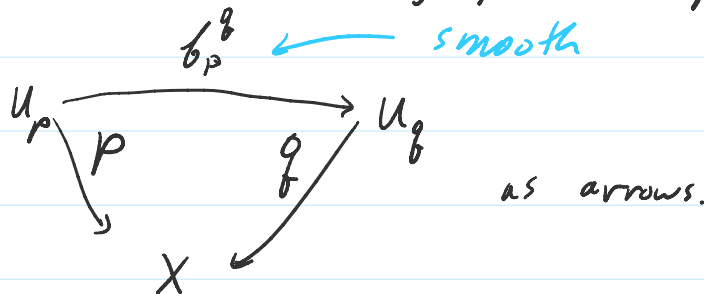
where $U \subseteq X$ is any open set, $\{U_\alpha\}$ open cover of U ,

and $S(\emptyset) = 0$.

• Let (X, \mathcal{D}_X) be a diffeological space

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• Let $\text{Plot}(X)$ be the category with plots as objects and



• A presheaf P on X is a functor

$$P: \text{Plot}(X)^{op} \longrightarrow \mathcal{C} \leftarrow \begin{array}{l} \text{sets / groups / vector spaces} \end{array}$$

• A sheaf S on X is a presheaf s.t. \forall plot p of X , $S|_{U_p}$ is a sheaf in the classical sense above.

e.g. 1) $\Omega^k: \text{Plot}(X)^{op} \longrightarrow \underline{\text{Vect}}_{\mathbb{R}}$

$$\begin{array}{l} p \longmapsto \Omega^k(U_p) \\ b_p \longmapsto (b_p)^* \end{array}$$

2) $C^\infty(\cdot, Y): \text{Plot}(X)^{op} \longrightarrow \underline{\text{Set}}$

$$\begin{array}{l} : p \longmapsto C^\infty(U_p, Y) \\ : b_p \longmapsto (b_p)^* \end{array}$$

3) Let $\varphi: X \rightarrow Y$ be smooth,

} not determined by the plot,

3) Let $\varphi: X \rightarrow Y$ be smooth,
 $\Gamma(\cdot, \varphi): \text{Plot}(Y)^{\text{op}} \rightarrow \text{Set}$
 $\Gamma(p, \varphi) = \{ \tilde{p} \in C^\infty(U_p, X) \mid \varphi \circ \tilde{p} = p \}$
 $\Gamma(\tilde{b}_p^q, \varphi) = (\tilde{b}_p^q)^*$

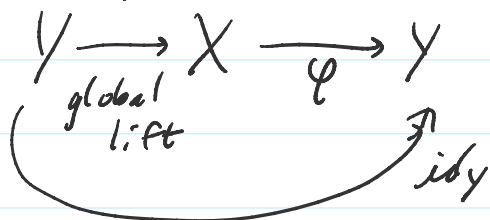
} not determined by the plot-domain!
 //

• Let $P: \text{Plot}(X)^{\text{op}} \rightarrow \mathcal{C}$ be a presheaf.
 $P(X) =$ global sections whose elements are "coherent families" $\{\eta_p\}_{p \in \mathcal{D}_X}$.
 That is, $(\tilde{b}_p^q)^* \eta_q = \eta_p \quad \forall p \xrightarrow{\tilde{b}_p^q} q$.

e.g. 1) The global sections of Ω^k are k -forms on X .

2) $\text{---} \parallel \text{---}$ of $C^\infty(\cdot, Y)$ are smooth maps $C^\infty(X, Y)$.

3) $\text{---} \parallel \text{---}$ $\Gamma(\cdot, \varphi)$ are global lifts $Y \rightarrow X$, which need not exist.



• $\varphi: X \rightarrow Y$, $P: \text{Plot}(Y)^{\text{op}} \rightarrow \mathcal{C}$, define the pullback presheaf

presheaf $\varphi^* P: \text{Plot}(X)^{\text{op}} \rightarrow \mathcal{C} : p \mapsto P(\varphi \circ p),$
 $\tilde{b}_p^q \mapsto (\tilde{b}_p^q)^*$

N.B. If P is a sheaf, then so is q^*P .

A (covering) generating family of X is a collection $\mathcal{Q} \subset \mathcal{D}_X$ so that $\forall p \in \mathcal{D}_X$

\exists open cover $\{U_\alpha\}$ of U_p , $\{f_\alpha\} \subset \mathcal{Q}$, $\{g_\alpha\}$ so that $p|_{U_\alpha} = g_\alpha \circ f_\alpha \quad \forall \alpha$.

The nebula of \mathcal{Q} , denoted $\mathcal{N}(\mathcal{Q})$ is

$\coprod_{f \in \mathcal{Q}} U_f$. It has an evaluation map

$$\begin{aligned} \text{ev}: \mathcal{N}(\mathcal{Q}) &\rightarrow X \\ u \in U_f &\mapsto f(u) \end{aligned}$$

e.g. $M = \text{smooth mfld}$, \mathcal{Q} induced by an atlas, then $\mathcal{N}(\mathcal{Q}) = \coprod$ (inverted) chart domains. //

$$\begin{aligned} \mathcal{N}(\mathcal{Q})_k &= \underbrace{\mathcal{N}(\mathcal{Q}) \overset{x}{\text{ev}} \mathcal{N}(\mathcal{Q}) \overset{x}{\text{ev}} \dots \overset{x}{\text{ev}} \mathcal{N}(\mathcal{Q})}_{k+1} \\ &= \{(u_0, \dots, u_k) \mid \text{ev}(u_i) = \text{ev}(u_j) \quad \forall i, j\} \end{aligned}$$

Evaluation map $\text{ev}_k: \mathcal{N}(\mathcal{Q})_k \rightarrow X$

Evaluation map $\omega_k: \mathcal{N}(2)_k \rightarrow X$

$$: (u_0, \dots, u_k) \mapsto \omega(u_k)$$

Degeneracy map $d_i: \mathcal{N}(2)_k \rightarrow \mathcal{N}(2)_{k-1}$

$$: (u_0, \dots, u_k) \mapsto (u_0, \dots, \hat{u}_i, \dots, u_k)$$

P be a presheaf of abelian groups on X .

$\omega_k^* P$ is a " " " " on $\mathcal{N}(2)_k$.

The degeneracy maps induce

$$\tilde{d}_i: \omega_k^* P(\mathcal{N}(2)_k) \rightarrow \omega_{k+1}^* P(\mathcal{N}(2)_{k+1})$$
$$\left\{ \eta_{(u_0, \dots, u_k)} \right\} \mapsto \left\{ \eta_{(u_0, \dots, \hat{u}_i, \dots, u_{k+1})} \right\}$$

Define $\check{C}^k(2, P) := \omega_k^* P(\mathcal{N}(2)_k)$ and

$$\partial = \sum_{i=0}^{k+1} (-1)^i \tilde{d}_i.$$

Get $\partial^2 = 0$, get complex $(\check{C}^\bullet(2, P), \partial)$.

Its homology $\check{H}^k(2, P)$ is the k^{th} Čech cohom. w.r.t. ∂ .

If 2 is a gen. family, a refinement is

a gen. family \mathcal{R} where if $2 = \{g_\alpha\}_{\alpha \in A}$ and

$\mathcal{R} = \{r_\beta\}_{\beta \in B}$, then \exists a function $f: B \rightarrow A$ s.t. $\forall r_\beta$

$\exists f_\beta: U_{r_\beta} \rightarrow U_{g_{f(\beta)}}$ s.t. $g_{f(\beta)} \circ f_\beta = r_\beta$.

$\exists f_\beta: U_{\mathcal{R}_\beta} \rightarrow U_{\mathcal{R}_{\beta'}} \text{ s.t. } f_{\beta'} \circ f_\beta = \tau_\beta.$

- every two gen families have a common ref.

- If \mathcal{R} refines \mathcal{Q} , then \exists an induced map $f: \mathcal{N}(\mathcal{R}) \rightarrow \mathcal{N}(\mathcal{Q})$.

$$u \in U_{\mathcal{R}_\beta} \mapsto f_\beta(u) \in U_{\mathcal{R}_{\beta'}}$$

- Get $f^k: \mathcal{N}(\mathcal{R})_k \rightarrow \mathcal{N}(\mathcal{Q})_k$

$$f^{k+1} \circ d_i = d_i \circ f^k.$$

- $f^\#: (\check{C}^\bullet(\mathcal{Q}, P), \partial) \rightarrow (\check{C}^\bullet(\mathcal{R}, P), \partial)$ and

$$\check{f}: \check{H}^\bullet(\mathcal{Q}, P) \rightarrow \check{H}^\bullet(\mathcal{R}, P)$$

- Take directed limits (i.e. colimits) over refinements, get $\check{H}^\bullet(X, P)$

↳ natural in X .

Results

- G - a diffeological abelian group, $\check{H}^1(X, C^\infty(\cdot, G)) =: \check{H}^1(X, G)$

classifies all (diffeol) principal G -bundles over X .

- If X is smoothly paracompact, then

...

$$\check{H}^1(X, G) \cong \text{Principal } G\text{-bundles} \cong [X, BG] = \text{homotopy classes of smooth maps from } X \text{ to } BG.$$

\uparrow
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• If $0 \Rightarrow P_1 \Rightarrow P_2 \Rightarrow P_3 \Rightarrow 0$ is a short exact sequence of sheaves (defined similarly to classical case),

then get a long exact seq

$$0 \rightarrow P_1(X) \rightarrow P_2(X) \rightarrow P_3(X) \rightarrow \check{H}^1(X, P_1) \rightarrow \check{H}^1(X, P_2) \rightarrow \check{H}^1(X, P_3) \rightarrow \dots$$

e.g. $\check{H}^1(T_x, \mathbb{R}) \neq 0$ $\left. \begin{array}{l} \mathbb{T}^2 \rightarrow T_x \\ \mathbb{R}\text{-bundle} \end{array} \right\} \text{represents non-trivial class.}$

e.g. G Lie group $G \curvearrowright M$ proper action $\check{H}^k(M/G, \mathbb{R}) = 0$ $\forall k > 0$.

(Holds for orbit spaces of proper groupoids, too.)