Differential Forms on Symplectic Quotients

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Setup

$$(M, \omega)$$

- (M, ω) - a connected symplectic manifold,

Setup

$$\begin{array}{ccc} (M,\omega) & \xrightarrow{\Phi} & \mathfrak{g}^* \\ \Big\downarrow{\scriptstyle \pi} & & \\ M/G & & \end{array}$$

- G - a compact Lie group acting in a Hamiltonian fashion on M with (equivariant) momentum map Φ,

Setup

$$
\begin{array}{ccccc}
Z & \xrightarrow{\ i\ } & (M,\omega) & \xrightarrow{\ \Phi\ } & \mathfrak{g}^* \\
{\scriptstyle \pi_Z}\downarrow & \circlearrowright & \downarrow{\scriptstyle \pi} & & \\
M/\!\!/_0\, G & \xrightarrow[\ j\ ]{} & M/G & &
\end{array}
$$

- Z - the level set $\Phi^{-1}(0)$.
- If 0 is a regular value of Φ, then Z is a closed submanifold of M on which G acts locally freely.
- In this case, $M/\!\!/_0\, G$ is a symplectic orbifold.

Setup

$$\begin{array}{ccccc}
Z & \xrightarrow{\;i\;} & (M,\omega) & \xrightarrow{\;\Phi\;} & \mathfrak{g}^* \\
\downarrow{\scriptstyle \pi_Z} & \circlearrowright & \downarrow{\scriptstyle \pi} & & \\
M/\!\!/_0\, G & \xrightarrow[\;j\;]{} & M/G & &
\end{array}$$

- If 0 is a critical value of Φ, then Z is a (closed) Whitney stratified subspace of M on which G acts.
- In this case, $M/\!\!/_0\, G$ is a symplectic stratified space [SjL91].

Orbit-Type Stratifications

- $M_{(H)} := \{x \in M \mid \mathrm{Stab}(x)$ is conjugate to $H\}$
- Together, the connected components of each (non-empty) $M_{(H)}$ form a Whitney stratification, called the **orbit-type stratification on M.**
- This induces a Whitney stratification on M/G whose strata are given by connected components of each (non-empty) $(M/G)_{(H)} := \pi(M_{(H)})$, also called the **orbit-type stratification on M/G.**

Orbit-Type Stratifications

- Also induced is a Whitney stratification on Z whose strata are given by connected components of each (non-empty) $Z_{(H)} := Z \cap M_{(H)}$, also called the **orbit-type stratification on** Z.
- This, in turn, induces a Whitney stratification on $Z/G =: M/\!\!/_0\, G$ whose strata are given by connected components of each (non-empty) $(Z/G)_{(H)} := \pi_Z(Z_{(H)})$, also called the **orbit-type stratification on** $M/\!\!/_0\, G$.
- Each stratum of the orbit-type stratification on $M/\!\!/_0\, G$ is a symplectic manifold, with each symplectic structure induced by one global Poisson structure on $M/\!\!/_0\, G$.

Sjamaar Forms

- Let Z_{prin} and $(M/\!/_0\,G)_{\text{prin}}$ be the principal strata of the orbit-type stratifications on Z and $M/\!/_0\,G$, resp., which are open and dense in Z and $M/\!/_0\,G$, resp.
- Denote by I and J the inclusions $Z_{\text{prin}} \hookrightarrow Z$ and $(M/\!/_0\,G)_{\text{prin}} \hookrightarrow M/\!/_0\,G$, resp.
- Denote by π_{prin} the restriction $\pi|_{Z_{\text{prin}}}$.

$$\begin{array}{ccccccc}
Z_{\text{prin}} & \xrightarrow{\ I\ } & Z & \xrightarrow{\ i\ } & (M,\omega) & \xrightarrow{\ \Phi\ } & \mathfrak{g}^* \\
\downarrow{\scriptstyle \pi_{\text{prin}}} & \circlearrowright & \downarrow{\scriptstyle \pi_Z} & \circlearrowright & \downarrow{\scriptstyle \pi} & & \\
(M/\!/_0\,G)_{\text{prin}} & \xrightarrow[\ J\ ]{} & M/\!/_0\,G & \xrightarrow[\ j\ ]{} & M/G & &
\end{array}$$

Sjamaar Forms

Definition ([Sj05])

A **Sjamaar** k**-form** σ on $M/\!/_0\,G$ is a k-form on $(M/\!/_0\,G)_{\text{prin}}$ for which there exists $\widetilde{\alpha}\in\Omega^k(M)$ satisfying $(i\circ I)^*\widetilde{\alpha}=\pi^*_{\text{prin}}\sigma$.

- Without loss of generality, we may assume $\widetilde{\alpha}$ is G-invariant.
- Obtain a de Rham complex $(\Omega^\bullet_{\text{Sj}}(M/\!/_0\,G),d)$.
- Obtain an associated Poincaré Lemma, Stokes' Theorem, and a de Rham Theorem.

Sjamaar Forms

Question

Is $(\Omega^\bullet_{\mathrm{Sj}}(M/\!/_0\,G), d)$ intrinsic? That is, is it independent of how we obtain $M/\!/_0\,G$? (For instance, if doing reduction in stages, there are multiple ways of presenting the symplectic quotient, all of which are "symplectomorphic".)

- If we could show that $(\Omega^\bullet_{\mathrm{Sj}}(M/\!/_0\,G), d) \cong (\Omega^\bullet(M/\!/_0\,G), d)$, where the latter is the diffeological de Rham complex, then the answer would be "yes".

Diffeology

Definition

Let X be a set. A **parametrisation** $p\colon U_p \to X$ is a map from an open subset U_p of some $\mathbb{R}^n$ (n is not fixed). A **diffeology** $\mathcal{D}_X$ on X is a family of parametrisations satisfying

1. all constant parametrisations are in $\mathcal{D}_X$,
2. if p is a parametrisation and $\{U_\alpha\}$ an open cover of U_p such that for each α
$$p|_{U_\alpha} \in \mathcal{D}_X$$
then $p \in \mathcal{D}_X$,
3. if $p \in \mathcal{D}_X$ and $f\colon V \to U_p$ smooth with V an open subset of some $\mathbb{R}^n$ then $p \circ f \in \mathcal{D}_X$.

Call $(X, \mathcal{D}_X)$ a **diffeological space** and each $p \in \mathcal{D}_X$ a **plot**.

Diffeology

Definition

A map $F\colon (X, \mathcal{D}_X) \to (Y, \mathcal{D}_Y)$ is **diffeologically smooth** if $F \circ p \in \mathcal{D}_Y$ for every $p \in \mathcal{D}_X$.

- Obtain a "complete, co-complete quasi-topos" [BH11]. In particular, we obtain a category admitting all subsets, quotients, products, coproducts, and function spaces.

Diffeology

Definition

- A **diffeological k-form** η on $(X, \mathcal{D}_X)$ is an assignment to each $p \in \mathcal{D}_X$ a k-form $\eta_p \in \Omega^k(U_p)$ such that for any $f\colon V \to U_p$ smooth with V an open subset of some $\mathbb{R}^n$,

$$\eta_{p\circ f} = f^*\eta_p.$$

- Given a diffeological from η, define $d\eta$ to be the assignment $p \mapsto d\eta_p$.

- Obtain a de Rham complex $(\Omega^\bullet(X), d)$.

Examples

Example

Given a smooth manifold, it has a natural diffeology consisting of all smooth maps into it from open subsets of cartesian spaces. The diffeological de Rham complex is (isomorphic to) the standard one.

Example ([KW16], [W12] for compact case)

If $G \circlearrowright M$ is a proper Lie group action, then

$$\pi^*\colon (\Omega^\bullet(M/G), d) \to (\Omega^\bullet_{\text{basic}}(M), d)$$

is an isomorphism. (In fact, a diffeomorphism.)

Examples

Example ([W22] ([W13?]))

Let $\mathcal{G}_1 \rightrightarrows \mathcal{G}_0$ be a proper Lie groupoid. Then

$$\pi^*\colon (\Omega^\bullet(\mathcal{G}_0/\mathcal{G}_1), d) \to (\Omega^\bullet_{\mathrm{basic}}(M), d)$$

is an isomorphism. (In fact, a diffeomorphism.) Here, π is the quotient map to the orbit space, and $\mu \in \Omega^k(\mathcal{G}_0)$ is **basic** if $s^*\mu = t^*\mu$. (Definition due to Eugene Lerman.)

Example ([M22])

If $\mathcal{F}$ is a foliation on a manifold M which is regular, or singular but whose leaves of the same dimension assemble into diffeological submanifolds of M, then $(\Omega^\bullet(M/\mathcal{F}), d)$ is similarly isomorphic to $(\Omega^\bullet_{\mathrm{basic}}(M), d)$. Here, $\mu \in \Omega^k(M)$ is **basic** if for every local section X of the distribution associated to $\mathcal{F}$,

$$X \lrcorner \mu = 0 \quad \text{and} \quad \pounds_X \mu = 0.$$

Hamiltonian Action Case

$$
\begin{array}{ccccccc}
Z_{\text{prin}} & \xrightarrow{I} & Z & \xrightarrow{i} & (M,\omega) & \xrightarrow{\Phi} & \mathfrak{g}^* \\
\downarrow{\scriptstyle \pi_{\text{prin}}} & \circlearrowright & \downarrow{\scriptstyle \pi_Z} & \circlearrowright & \downarrow{\scriptstyle \pi} & & \\
(M/\!/_0 G)_{\text{prin}} & \xrightarrow[J]{} & M/\!/_0 G & \xrightarrow[j]{} & M/G & &
\end{array}
$$

- The goal is to show that $J^*\colon (\Omega^\bullet(M/\!/_0 G), d) \to (\Omega^\bullet_{\text{Sj}}(M/\!/_0 G), d)$ is a (well-defined) isomorphism.

Proposition ([W12])

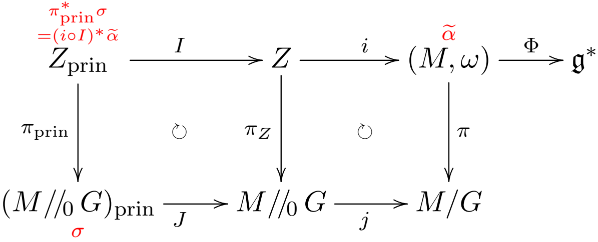
- *If 0 is a regular value of Φ, then J^* is a (well-defined) isomorphism.*
- *If 0 is a critical value of Φ, then $(\Omega^\bullet_{\text{Sj}}(M/\!/_0 G), d) \subseteq J^*(\Omega^\bullet(M/\!/_0 G), d))$.*

Idea of Proof

$$
\begin{array}{ccccccc}
Z_{\mathrm{prin}} & \xrightarrow{\;I\;} & Z & \xrightarrow{\;i\;} & (M,\omega) & \xrightarrow{\;\Phi\;} & \mathfrak{g}^* \\
\Big\downarrow \pi_{\mathrm{prin}} & \circlearrowright & \Big\downarrow \pi_Z & \circlearrowright & \Big\downarrow \pi & & \\
(M/\!/_0\, G)_{\mathrm{prin}} & \xrightarrow[\;J\;]{} & M/\!/_0\, G & \xrightarrow[\;j\;]{} & M/G & & \\
{\color{red}\sigma} & & & & & &
\end{array}
$$

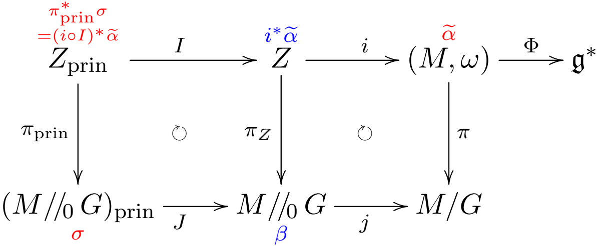
- If σ is a Sjamaar form,

Idea of Proof



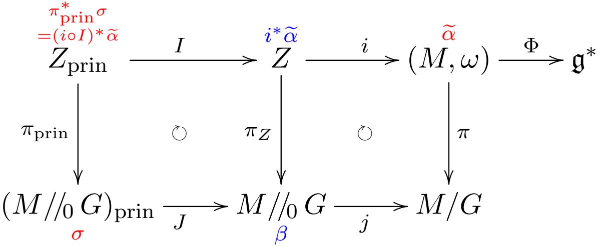
- If σ is a Sjamaar form, then there exists a G-invariant $\widetilde{\alpha}$ on M such that $(i \circ I)^*(\widetilde{\alpha}) = \pi^*_{\text{prin}}\sigma$.

Idea of Proof



- If σ is a Sjamaar form, then there exists a G-invariant $\widetilde{\alpha}$ on M such that $(i\circ I)^*(\widetilde{\alpha}) = \pi^*_{\text{prin}}\sigma$.
- If 0 is a regular value, then since being horizontal is a closed condition, $i^*\alpha$ is a basic form on Z. Obtain a form β on $M/\!/_0\, G$ such that $J^*\beta = \sigma$.

Idea of Proof



- If 0 is a critical value, then we can use the local finiteness of the stratification on Z, as well as the fact that $\widetilde{\alpha}$ restricts to a basic form on each stratum of Z [Sj05], to obtain the second statement of the proposition.

Idea of Proof

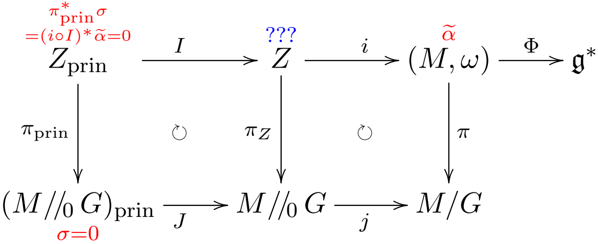
$$
\begin{array}{ccccccc}
Z_{\text{prin}} & \xrightarrow{I} & Z & \xrightarrow{i} & (M,\omega) & \xrightarrow{\Phi} & \mathfrak{g}^* \\
\downarrow{\scriptstyle \pi_{\text{prin}}} & \circlearrowright & \downarrow{\scriptstyle \pi_Z} & \circlearrowright & \downarrow{\scriptstyle \pi} & & \\
\underset{\color{red}{\sigma=0}}{(M/\!/_0\, G)_{\text{prin}}} & \xrightarrow[J]{} & M/\!/_0\, G & \xrightarrow[j]{} & M/G & &
\end{array}
$$

- If $\sigma = 0$ and $0 \in \mathfrak{g}^*$ is a regular value of Φ,

Idea of Proof

$$
\begin{array}{ccccccc}
\overset{\pi^*_{\text{prin}}\sigma = (i\circ I)^*\widetilde{\alpha}=0}{Z_{\text{prin}}} & \xrightarrow{\;I\;} & \overset{i^*\widetilde{\alpha}=0}{Z} & \xrightarrow{\;i\;} & \overset{\widetilde{\alpha}}{(M,\omega)} & \xrightarrow{\;\Phi\;} & \mathfrak{g}^* \\
\Big\downarrow {\scriptstyle \pi_{\text{prin}}} & \circlearrowright & \Big\downarrow {\scriptstyle \pi_Z} & \circlearrowright & \Big\downarrow {\scriptstyle \pi} & & \\
\underset{\sigma=0}{(M/\!/_0\,G)_{\text{prin}}} & \xrightarrow[\;J\;]{} & \underset{\beta=0}{M/\!/_0\,G} & \xrightarrow[\;j\;]{} & M/G & &
\end{array}
$$

- If $\sigma = 0$ and $0 \in \mathfrak{g}^*$ is a regular value of Φ, then $i^*\alpha = 0$ since Z_{prin} is open and dense in Z.
- π^*_Z is injective, and so $\beta = 0$. Thus J^* is injective.



- If 0 is a critical value of Φ, then the continuity argument going from Z_{prin} to Z no longer is clear.
- In particular, the relationship between tangent vectors and differential forms has yet to be explored in the diffeological world. Work by Christensen-Wu [CW16,CW22] in recent years may be a starting point.

Idea of Proof

$$
\begin{array}{ccccccc}
Z_{\mathrm{prin}} & \xrightarrow{\;I\;} & Z & \xrightarrow{\;i\;} & (M,\omega) & \xrightarrow{\;\Phi\;} & \mathfrak{g}^* \\
\Big\downarrow{\scriptstyle \pi_{\mathrm{prin}}} & \circlearrowright & \Big\downarrow{\scriptstyle \pi_Z} & \circlearrowright & \Big\downarrow{\scriptstyle \pi} & & \\
(M/\!/_0\, G)_{\mathrm{prin}} & \xrightarrow[\;J\;]{} & \underset{\color{red}{\beta}}{M/\!/_0\, G} & \xrightarrow[\;j\;]{} & M/G & &
\end{array}
$$

- If β is a form on $M/\!/_0\, G$, then to obtain that $J^*\beta$ is Sjamaar,

Idea of Proof

$$\begin{array}{ccccccc}
Z_{\text{prin}} & \xrightarrow{\ I\ } & \overset{\pi_Z^*\beta}{Z} & \xrightarrow{\ i\ } & \overset{\widetilde{\alpha}?}{(M,\omega)} & \xrightarrow{\ \Phi\ } & \mathfrak{g}^* \\
{\scriptstyle \pi_{\text{prin}}}\downarrow & \circlearrowright & {\scriptstyle \pi_Z}\downarrow & \circlearrowright & \downarrow{\scriptstyle \pi} & & \\
(M/\!/_0\, G)_{\text{prin}} & \xrightarrow[\ J\ ]{} & \underset{\beta}{M/\!/_0\, G} & \xrightarrow[\ j\ ]{} & M/G & &
\end{array}$$

- If β is a form on $M/\!/_0\, G$, then to obtain that $J^*\beta$ is Sjamaar, we require $\pi_Z^*\beta$ to extend to some form $\widetilde{\alpha}$ on M.
- If $0 \in \mathfrak{g}$ is a regular value, then Z is a closed submanifold, and so this occurs.

Idea of Proof

$$\begin{array}{ccccccc}
Z_{\text{prin}} & \xrightarrow{I} & \overset{\pi_Z^*\beta}{Z} & \xrightarrow{i} & \overset{\widetilde{\alpha}?}{(M,\omega)} & \xrightarrow{\Phi} & \mathfrak{g}^* \\
\Big\downarrow{\scriptstyle \pi_{\text{prin}}} & \circlearrowright & \Big\downarrow{\scriptstyle \pi_Z} & \circlearrowright & \Big\downarrow{\scriptstyle \pi} & & \\
(M/\!/_0\, G)_{\text{prin}} & \xrightarrow[J]{} & \underset{\beta}{M/\!/_0\, G} & \xrightarrow[j]{} & M/G & &
\end{array}$$

- If $0 \in \mathfrak{g}$ is a critical value, then we are left with an extension problem. (We also still need to understand the relationship between tangent vectors and forms in the diffeological world.)

An Example

Example

- Suppose $G = \mathbb{S}^1$ acts on $M = \mathbb{C}^2 \cong \mathbb{R}^4$ linearly with weights ± 1, and

$$\Phi(z_1, z_2) = |z_2|^2 - |z_1|^2.$$

- Then Z is a quadratic cone over a torus (homeomorphic to $(\mathbb{T}^2 \times [0,1])/(\mathbb{T}^2 \times \{0\})$).
- In the case of 0-forms, the extension problem becomes: does every diffeologically smooth function $f\colon Z \to \mathbb{R}$ extend to a smooth function on M?

- This extension problems brings us to another facet of my research.

Definition

Let X be a set. A **Sikorski structure** on X is a family of $\mathbb{R}$-valued functions $\mathcal{F}_X$ such that

1. if $f_1, \dots, f_n \in \mathcal{F}_X$ and $g \in C^\infty(\mathbb{R}^n)$, then $g(f_1, \dots, f_n) \in \mathcal{F}_X$;
2. if given $f \colon X \to \mathbb{R}$ there is an open cover $\{U_\alpha\}$ of X and for each α there exists $g_\alpha \in \mathcal{F}_X$ such that
$$f|_{U_\alpha} = g_\alpha|_{U_\alpha},$$
then $f \in \mathcal{F}_X$. (This is with respect to the initial topology on X induced by $\mathcal{F}_X$.)

$(X, \mathcal{F}_X)$ is called a **Sikorski space**.

Examples

Examples

- Any smooth manifold M has a natural Sikorski structure: $C^\infty(M)$.
- Any subset Z of a Sikorski space $(X, \mathcal{F}_X)$ has a natural Sikorski structure: the set of all $\mathbb{R}$-valued functions that locally extend to X.
- If Z is closed, then this structure is just the restrictions of $\mathcal{F}_X$ to Z.
- If $\sim$ is an equivalent relation on X, then $X/\sim$ has a natural Sikorski structure: the set of all $\mathbb{R}$-valued functions on $X\sim$ that lift to $\mathcal{F}_X$.

Triples

Definition

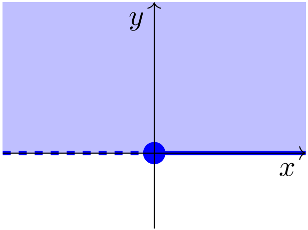
- Given a set X, one can equip it with a diffeology $\mathcal{D}_X$ and a Sikorski structure $\mathcal{F}_X$, obtaining a triple $(\mathcal{D}_X, X, \mathcal{F}_X)$.
- If $f \circ p$ is smooth for every $p \in \mathcal{D}_X$ and $f \in \mathcal{F}_X$, then we call the triple **compatible**.
- If $\mathcal{D}_X$ (resp. $\mathcal{F}_X$) contains *all* parametrisations p (resp. functions f) such that $p \circ f$ is smooth for all $f \in \mathcal{F}_X$ (resp. plots $p \in \mathcal{D}_X$), we say that $\mathcal{D}_X$ is **determined by** $\mathcal{F}_X$ (resp. $\mathcal{F}_X$ is **determined by** $\mathcal{D}_X$).
- If $\mathcal{D}_X$ and $\mathcal{F}_X$ determine each other, we call the triple **reflexive**.
- The category of reflexive triples is isomorphic to the category of Frölicher spaces. [W12,BIZKW]

Convex Sets

Let $K \subseteq \mathbb{R}^n$ be convex, equipped with the induced diffeology $\mathcal{D}_K$ and Sikorski structure $\mathcal{F}_K$.

Theorem ([KW22])

1. *If K is locally closed, then $(\mathcal{D}_K, K, \mathcal{F}_K)$ is a reflexive triple.*
2. *Let $K \subseteq \mathbb{R}^2$ be the open upper half plane along with the non-negative x-axis. There exists a diffeologically smooth $f : K \to \mathbb{R}$ that does not locally extend to a smooth function of $\mathbb{R}^2$ about the origin. Thus $(\mathcal{D}_K, K, \mathcal{F}_K)$ is not reflexive.*



The Hamiltonian Case

$$\begin{array}{ccccccc}
Z_{\text{prin}} & \xrightarrow{\;I\;} & Z & \xrightarrow{\;i\;} & (M,\omega) & \xrightarrow{\;\Phi\;} & \mathfrak{g}^* \\
\downarrow{\scriptstyle \pi_{\text{prin}}} & \circlearrowright & \downarrow{\scriptstyle \pi_Z} & \circlearrowright & \downarrow{\scriptstyle \pi} & & \\
(M/\!/_0\,G)_{\text{prin}} & \xrightarrow[\;J\;]{} & M/\!/_0\,G & \xrightarrow[\;j\;]{} & M/G & &
\end{array}$$

- Back to whether Sjamaar 0-forms correspond to diffeological 0-forms, this is true if $(\mathcal{D}_Z, Z, C^\infty(M)|_Z)$ is a reflexive triple.

Thank you!

References

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