Coarse Moduli Spaces of Stacks over Manifolds

(joint work with Seth Wolbert)

Jordan Watts

`(CMS Summer Meeting 2014)`

University of Illinois at Urbana-Champaign

June 8, 2014

Introduction

Let G be a Lie group, and let M be a manifold admitting a proper G-action.

If the action is free, then the orbit space M/G is a manifold.

All of the equivariant information upstairs descends to smooth information downstairs.

Non-Free Actions

In the case of a non-free action, M/G is typically not a manifold.

We would like a category in which to take this quotient that (1) remembers as much information about the action as possible but (2) treats the quotient as a manifold in the free case.

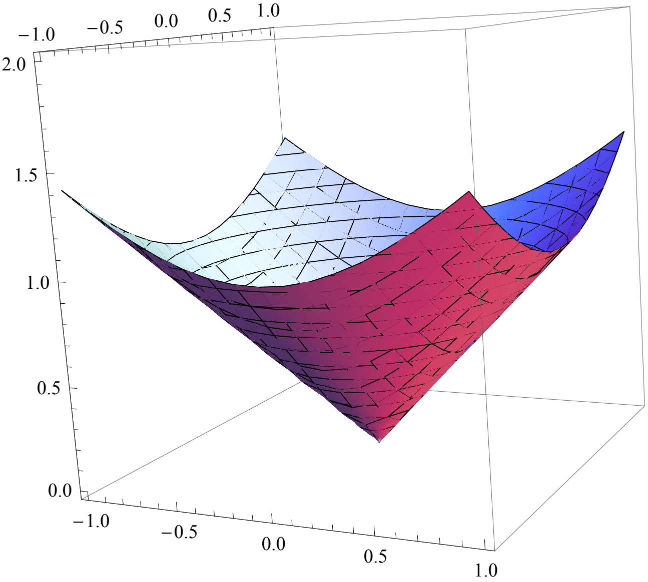
The *quotient topology* obviously is a bad candidate, and cannot tell the difference between the group actions of $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ ($n > 0$) on the plane $\mathbb{R}^2$ by rotations. That is, all topological quotients $\mathbb{R}^2/\mathbb{Z}_n$ are homeomorphic.

Differential Structure

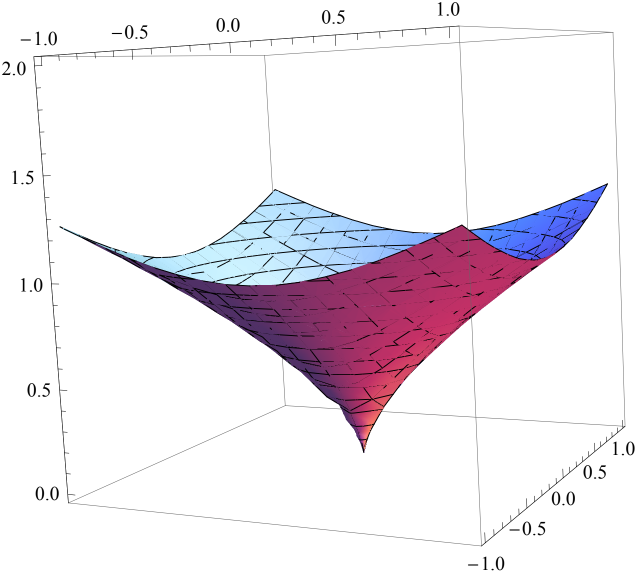
The "differential structure" on M/G is the ring of G-invariant smooth functions on M.

- It induces the quotient topology.
- It yields vector fields that match left-invariant vector fields upstairs.
- These vector fields yield the orbit-type stratification on M/G.
- The differential structures on $\mathbb{R}^2/\mathbb{Z}_n$ are not isomorphic for different n.

$\mathbb{R}^2/\mathbb{Z}_2$



$\mathbb{R}^2/\mathbb{Z}_3$



However...

The differential structures on the orbit spaces $\mathbb{R}^n/\mathrm{SO}(n)$ are all diffeomorphic to the natural differential structure on the manifold with boundary $[0, \infty)$.

We can do better.

Sheaves of Sets and Diffeology

Before moving onto another type of structure to put on M/G, a definition.

Let **Mfld** be the category of smooth manifolds with smooth maps between them, and **Set** the category of sets. A **sheaf of sets over Mfld** is a functor $S : \mathbf{Mfld}^{op} \to \mathbf{Set}$ that satisfies the sheaf condition over open covers of manifolds.

Let X be a set. A **diffeology** $\mathcal{D}$ on X is a sheaf of sets $\mathbf{Mfld}^{op} \to \mathbf{Set}$ such that

- $\mathcal{D}(*) = X$,
- For any manifold N, the set $\mathcal{D}(N)$ is a set of maps $p : N \to X$.

Diffeological Spaces - Some Examples

Example

Manifolds, manifolds with boundary, and manifolds with corners are examples of diffeological spaces.

Example

Fix a diffeological space $(X, \mathcal{D})$.

Let $\pi : X \to X/\sim$ be the quotient map where $\sim$ is an equivalence relation on X.

$X/\sim$ acquires the **quotient diffeology** $\mathcal{D}_\sim$, where the set $\mathcal{D}_\sim(N)$ consists of maps that locally look like $\pi \circ p$ where $(p : U \to X) \in \mathcal{D}(U)$ and $U \subseteq N$ is open.

Smooth Maps

A map $F : (X, \mathcal{D}_X) \to (Y, \mathcal{D}_Y)$ is **diffeologically smooth** if it is a map of sheaves (*i.e.* a natural transformation).

This yields a map of sets $F : X \to Y$ such that for any manifold N and $p \in \mathcal{D}_X(N)$, we have $F \circ p \in \mathcal{D}_Y(N)$. (And conversely.)

Properties on M/G

- $\mathcal{D}_{M/G}$ induces the differential structure on M/G.
- Diffeology yields a de Rham complex that, in the case of M/G, is isomorphic to the basic differential form subcomplex on M.
- The quotient diffeologies on $\mathbb{R}^n/SO(n)$ remember which n we started with.
- Given a Lie group G acting on a point $\{*\}$, the quotient diffeology does not see G.

Stacks

Think of a **stack over Mfld** $\mathcal{X}$ as a sheaf of *groupoids* over **Mfld**.

Example (Geometric Stacks)

Let $\mathcal{G} = (G_1 \rightrightarrows G_0)$ be a Lie groupoid. Then the stack $B\mathcal{G}$ assigns to each manifold N the groupoid of principal $\mathcal{G}$-bundles, with isomorphisms (equivariant bundle diffeomorphisms) as arrows.

BG

Example

If G is a Lie group acting on a point $\{*\}$, then stack BG of the corresponding action groupoid $G \times \{*\} \rightrightarrows \{*\}$ assigns to a manifold N all principal G-bundles over N.

So, stacks can see which G is acting on $\{*\}$. But do they induce the quotient diffeology on M/G?

Coarse Moduli Space of a Stack

Theorem: (W.–Wolbert, 2014)
There is a (2-)functor Coarse from the (2-)category of stacks over manifolds to diffeological spaces, taking any stack to an "underlying" diffeological space, called the **coarse moduli space** of the stack.

Geometric Stacks

Theorem: (W.–Wolbert, 2014)
If $\mathcal{X}$ is a *geometric* stack, and $\mathcal{G} = (G_1 \rightrightarrows G_0)$ is a Lie groupoid such that $B\mathcal{G} \simeq \mathcal{X}$,

then Coarse($\mathcal{X}$) is diffeomorphic to the orbit space G_0/G_1 equipped with its quotient diffeology.

In particular, the quotient diffeology on G_0/G_1 only depends on the isomorphism class of $\mathcal{X}$.

Thank you!

References

- J. C. Baez, A. E. Hoffnung, “Convenient categories of smooth spaces”, *Trans. Amer. Math. Soc.*, **363** (2011), no. 11, 5789–5825.
- Eugene Lerman, “Orbifolds as stacks?”, *L’Enseign. Math. (2)* **56** (2010), no. 3-4, 315–363.
- J. Watts, “The orbit space and basic forms of a proper Lie groupoid”, (submitted). `http://arxiv.org/abs/1309.3001`
- J. Watts and S. Wolbert, “Diffeology: a concrete foundation for stacks”, (preprint). `http://arxiv.org/abs/1406.1392`

All images were made using *Mathematica* 9.0.