Classifying Spaces of Diffeological Groups

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Joint Work

This talk is based on joint work with Jean-Pierre Magnot (Université D'Angers).

Motivation

- Fix $\alpha \in \mathbb{R} \smallsetminus \mathbb{Q}$.
- Let $\mathbb{Z}^2$ act on $\mathbb{R}$ by

$$(m, n) \cdot x = x + m + n\alpha.$$

- The orbits are dense, and so the orbit space T has trivial topology.
- T is a group, but would not typically be considered a topological group.
- Is there a "differentiable structure" that we could equip T with?

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Definition

Fix a set X. A **parametrisation** is a (set-theoretical) map $p\colon U \to X$ where U is some open subset of some Euclidean space.

A **diffeology** $\mathcal{D}$ on X is a family of parametrisations, called **plots**, satisfying:

- (Covering Axiom) Every constant parametrisation is in $\mathcal{D}$,
- (Locality Axiom) If $p\colon U \to X$ is a parametrisation that locally is equal to a plot in $\mathcal{D}$, then p is in $\mathcal{D}$.
- (Smoothness Axiom) If $p\colon U \to X$ is in $\mathcal{D}$ and $f\colon V \to U$ is a smooth map between open subsets of Euclidean spaces, then $p \circ f$ is in $\mathcal{D}$.

Call $(X, \mathcal{D})$ a **diffeological space**.

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Definition

A map $F\colon (X, \mathcal{D}_X) \to (Y, \mathcal{D}_Y)$ is **(diffeologically) smooth** if for every plot $p \in \mathcal{D}_X$, the composition $F \circ p$ is in $\mathcal{D}_Y$.

The resulting category of diffeological spaces is a “complete, co-complete quasi-topos”, meaning you can do almost anything you would ever want to do in this category without leaving it: subsets, quotients, function spaces, etc.

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D-Topology

Definition

Let $(X, \mathcal{D}_X)$ be a diffeological space. The **D-topology** on X is the strongest topology making all of the plots of $\mathcal{D}$ continuous.

Example

Smooth manifolds, smooth manifolds with boundary, and smooth manifolds with corners are all full subcategories of diffeological spaces.

Effective orbifolds are “almost” a subcategory, in that there is a natural functor from the orbifolds to diffeological spaces that is essentially injective on objects.

Examples

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Diffeological Groups

Definition

A **diffeological group** G is a diffeological space equipped with smooth multiplication and smooth inverse maps.

Example

Given a diffeological space $(X, \mathcal{D}_X)$, the group $\mathrm{Diff}(X)$ of diffeomorphisms of X is a diffeological group.

Example

Our group $T = \mathbb{R}/\mathbb{Z}^2$ equipped with the quotient diffeology is a diffeological group, an example of what is called an **irrational torus** (Iglesias-Zemmour), or an **infra-circle** (Weinstein), or a **quasi-torus** (Prato).

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Prequantisation

Let (M, ω) be a symplectic manifold.

The first step in the programme of geometric quantisation is to form a **prequantisation bundle**, a circle bundle (or complex line bundle) with connection over M whose curvature is ω.

To do this, one requires that the cohomology class $[\omega]$ sit in the image of $H^2(M; \mathbb{Z}) \to H^2(M; \mathbb{R})$; that is, ω is required to be **integral**.

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Question: What happens if ω is not integral?

Answer: One can still construct a principal bundle over M; however, the structure group is no longer a circle, but some irrational torus – which one depends on ω.

It thus makes sense to study such principal bundles, and so in turn it makes sense to study classifying spaces of such groups.

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Milnor's Construction

Fix a diffeological group G.

Definition

Define EG to be the quotient

$$EG := \left\{ (t_i, g_i) \in [0,1]^{\mathbb{N}} \times G^{\mathbb{N}} \;\middle|\; \sum t_i = 1, \text{ only finitely many } t_i \text{ are non-zero} \right\} \Big/ \sim$$

where $(t_i, g_i) \sim (t'_i, g'_i)$ if

1. $t_i = t'_i$ for all i, and
2. for each i, $t_i = t'_i \neq 0$ implies $g_i = g'_i$.

Denote elements of EG by $(t_i g_i)$.

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EG comes equipped with a smooth free action of G: for $h \in G$,

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The resulting quotient $BG := EG/G$ is the **classifying space** of G.

Example

If G is a Lie group, then the underlying topological space of BG is (homotopy equivalent to) the usual classifying space of G.

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Definition

A **D-numerable principal G-bundle** $\pi\colon P \to X$ is a diffeological space P with a smooth surjection π such that

1. each fibre $\pi^{-1}(x)$ admits a transitive free action of G,
2. there is an open cover $\{U_\alpha\}$ of X admitting a subordinate smooth partition of unity,
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Results

Theorem (Magnot-W., Christensen-Wu)

Fix a diffeological group G.

1. *$EG \to BG$ is a principal G-bundle.*
2. *There is a natural isomorphism between $\mathcal{B}_G(\cdot)$ and $[\cdot, BG]$, where $\mathcal{B}_G(X)$ is the set of all D-numerable principal G-bundles and $[X, BG]$ is the set of smooth homotopy classes of maps $X \to BG$.*
3. *EG is smoothly contractible.*
4. *$\pi_k(BG) \cong \pi_{k-1}(G)$ for all $k > 0$, where $\pi_k(\cdot)$ is the kth smooth homotopy group.*
5. *Given a smooth homomorphism $\varphi\colon G \to H$ between diffeological groups, we obtain a smooth map $\Phi\colon BG \to BH$.*

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Connection 1-Forms

Definition

A **connection** 1**-form** $\omega\colon TP \to T_eG$ of a principal G-bundle $P \to X$ is a 1-form taking values in T_eG such that

1. ω is G-equivariant with respect to the adjoint action,
2. for every $y \in P$ and $\xi \in T_eG$, we have

$$\omega(\xi_P|_y) = \xi$$

   where ξ_P is induced by the infinitesimal action.

Example

The Maurer-Cartan form of a diffeological group G is a connection 1-form on $G \to \{*\}$.

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Fix a diffeological group G.

1. *$EG \to BG$ admits a connection 1-form.*
2. *Any D-numerable principal G-bundle $P \to X$ in $\mathcal{B}_G(X)$ admits a connection 1-form.*
3. *With mild conditions on G, connection 1-forms induce unique liftings of paths from X to P.*

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Back to T

Corollary

If T is an irrational torus, and ω a connection 1-form on $ET \to BT$, then

- *$d\omega$ descends to a curvature form Ω on BT,*
- *for any diffeological space X and smooth map $F\colon X \to BT$, $F^*\Omega$ is the curvature of $F^*ET \to X$ with connection 1-form $F^*\omega$,*
- *any D-numerable principal T-bundle over X is uniquely characterised by a class in $H^1(\mathrm{Path(X)}, T)$,*
- *if X is also (smoothly) simply-connected then any non-zero closed 2-form μ on X admits a principal T-bundle for some (possibly irrational) torus T whose curvature is μ; if this bundle is D-numerable, then $\mu = F^*\Omega$.*

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Back to T

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If T is an irrational torus, and ω a connection 1-form on $ET \to BT$, then

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Shortly after our paper was published, a paper of Dan Christensen and Enxin Wu appeared with similar results (minus the results involving the connection 1-form), in which a different diffeology on EG was used.

Consequently, they are able to show that $EG \to BG$ is unique up to smooth homotopy equivalence; that is, it is truly universal.

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Thank you!

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