Bredon Cohomology for Transitive Groupoids

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June 9, 2019

Joint Work

This talk is based on joint work with Carla Farsi (University of Colorado Boulder) and Laura Scull (Fort Lewis College).

Motivation

CW-structures are excellent tools for computing the homology of a CW-complex.

Fixing, say, a compact Lie group G, can we find a "G-equivariant" version of this?

Ideally, such a theory should give the homology of the orbit space of a G-space, as well as information about fixed point sets, etc.

Such a theory already exists, namely Bredon (co)homology for a fixed topological group G.

We will briefly review how this theory works; in particular, we will define the orbit category, G-CW-complexes, and Bredon cohomology.

We will then move onto an extension of the theory to transitive topological groupoids.

The Orbit Category

Definition

Fix a topological group G. Define the **orbit category for** G, denoted OG, as follows:

- **objects:** homogeneous G-spaces; that is, topological spaces X admitting a transitive G-action (*i.e.* there is only one orbit),
- **arrows:** G-equivariant maps.

This category is equivalent to the one whose objects are the spaces G/H for closed subgroups $H \leq G$, with arrows G-equivariant maps.

Remark

A map $G/H \to G/K$ is G-equivariant if and only if $gHg^{-1} \subseteq K$ for some $g \in G$.

Definition

Let D^{n+1} be the $(n+1)$-ball with boundary S^n, the n-sphere.

An $(n+1)$**-cell** is a product $G/H \times D^{n+1}$ with **boundary** $G/H \times S^n$, where H is a closed subgroup of G.

Note that G acts on these cells, and the inclusion $j\colon G/H \times S^n \hookrightarrow G/H \times D^{n+1}$ is G-equivariant.

G-CW-Complexes

Definition

A G**-CW-complex** is a right G-space X equal to the union

$$X^0 \subseteq X^1 \subseteq \cdots \subseteq X^n \subseteq X^{n+1} \subseteq \cdots \subseteq \bigcup_{i\in\mathbb{N}} X^i = X,$$

where each X^i is defined recursively as follows:

- The 0**-cells**, X^0, is a disjoint union of **canonical orbits**, $\coprod_{i\in I_0} G/H_i$;

G-CW-Complexes

Definition

- Given a collection of $(n+1)$-cells

$$\{G/H_i \times D^{n+1}\}_{i \in I_{n+1}}$$

and a collection of G-equivariant **attaching maps**

$$\{q_i^{n+1} \colon G/H_i \times S^n \to X^n\}_{i \in I_{n+1}},$$

the $(n+1)$**-skeleton** X^{n+1} of X is the pushout of G-spaces:

$$\begin{array}{ccc} \coprod_{i \in I_{n+1}} G \times H_i \times S^n & \xrightarrow{\coprod_i q_i^{n+1}} & X^n \\ \downarrow & & \downarrow \\ \coprod_{i \in I_{n+1}} G \times H_i \times D^{n+1} & \xrightarrow{\coprod_i Q_i^{n+1}} & X^{n+1} \end{array}$$

G-CW-Complexes

Definition

- X is the colimit of spaces of the X_i, and inherits a G-equivariant structure from its n-skeleta and the equivariant inclusion maps.

One sees that this definition allows the orbit-type strata of a G-space to be detected by the G-CW-structure.

Definition

A contravariant (resp. covariant) G**-coefficient system** is a contravariant (resp. covariant) functor from OG to Abelian groups.

For G-spaces X and Y, let $\mathrm{Hom}_G(Y, X)$ be all G-equivariant maps from Y to X.

Let $\Phi_X : OG \to$ **Top** be the contravariant functor $\Phi_X(Y) := \mathrm{Hom}_G(Y, X)$.

Remark

Given a G-space X and a closed subgroup $H \leq G$,

$$\mathrm{Hom}_G(G/H, X) \cong X^H,$$

where X^H is the subspace of all H-fixed points in X.

Bredon Cohomology

Let X be a G-CW-complex. Let $\underline{C}_n(X)$ be the contravariant G-coefficient system

$$\underline{C}_n(X) := \underline{H}_n(\Phi_{X^n}(\cdot), \Phi_{X^{n-1}}(\cdot); \mathbb{Z}).$$

By the previous remark, we have

$$\underline{C}_n(G/H) = H_n((X^n)^H, (X^{n-1})^H; \mathbb{Z}).$$

The connecting homomorphisms for the long exact sequence of relative homology groups coming from triples $\big((X^n)^H, (X^{n-1})^H, (X^{n-2})^H\big)$ induces natural transformations $d\colon \underline{C}_n(X) \to \underline{C}_{n-1}(X)$.

Definition

Let X be a G-CW-complex, and fix a contravariant G-coefficient system M. Define

$$C^n_G(X, M) := \mathrm{Hom}_{OG}\left(\underline{C}_n(X), M\right), \quad \delta := \mathrm{Hom}_{OG}(d, \mathrm{id}_M),$$

where Hom_{OG} denotes natural transformations between the coefficient systems. This induces a complex, whose homology is the **Bredon cohomology** of X with coefficients in M.

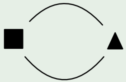
An Example

Example

Let $G = \mathbb{Z}/2$. Construct a G-CW-complex X as follows: let X^0 be two 0-cells,

$$X^0 = (G/G \times D^0) \amalg (G/G \times D^0),$$

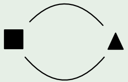
which is isomorphic to two discrete points $\{\blacksquare, \blacktriangle\}$ with a trivial G-action. Let X^1 be given by one 1-cell $G/1 \times D^1$ attached to X^0:



The result is isomorphic to $\mathbb{Z}/2$ acting on the circle by reflection through a fixed axis.

Example

Computing the Bredon cohomology:

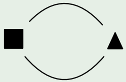


$\mathbb{Z}/2$ acting on $\underline{C}_0(X)(G/1)$; $\mathbb{Z}/2$ acting on $H_0(X^0;\mathbb{Z})$; $\mathbb{Z}/2$ (trivial) acting on $\mathbb{Z}\oplus\mathbb{Z}$

$$\begin{array}{ccccc}
\underline{C}_0(X)(G/1) & = & H_0(X^0;\mathbb{Z}) & = & \mathbb{Z}\oplus\mathbb{Z} \\
\uparrow & & \uparrow & & \uparrow \scriptstyle{\mathrm{id}} \\
\underline{C}_0(X)(G/G) & = & H_0((X^0)^G;\mathbb{Z}) & = & \mathbb{Z}\oplus\mathbb{Z}
\end{array}$$

An Example

Example



$$
\begin{array}{ccccc}
\mathbb{Z}/2 \circlearrowright & & \mathbb{Z}/2 \circlearrowright & & \mathbb{Z}/2 \text{ non-trivial} \circlearrowright \\
\underline{C}_1(X)(G/1) & = & H_1(X^1, X^0; \mathbb{Z}) & = & \mathbb{Z} \oplus \mathbb{Z} \\
\uparrow & & \uparrow & & \uparrow \\
\underline{C}_1(X)(G/G) & = & H_1((X^1)^G, (X^0)^G; \mathbb{Z}) & = & 0
\end{array}
$$

An Example

Example

Take $\underline{A}$ to be the coefficient system

$$\Big(\underline{A}(G/G) \to \underline{A}(G/1)\Big) = \Big(\mathbb{Z} \to 0\Big).$$

$$C^0_G(X, A) = \left\{ \begin{array}{ccc} \mathbb{Z} \oplus \mathbb{Z} & \longrightarrow & 0 \\ \mathrm{id} \uparrow & \circlearrowleft & \uparrow \\ \mathbb{Z} \oplus \mathbb{Z} & \longrightarrow & \mathbb{Z} \end{array} \right\} \cong \mathbb{Z}^2$$

$$C^1_G(X, A) = \left\{ \begin{array}{ccc} \mathbb{Z}/2 \ \circlearrowright & & \\ \mathbb{Z} \oplus \mathbb{Z} & \longrightarrow & 0 \\ \uparrow & \circlearrowleft & \uparrow \\ 0 & \longrightarrow & \mathbb{Z} \end{array} \right\} \cong 0$$

An Example

Example

The corresponding Bredon cohomology is $\mathbb{Z}^2$ at degree 0, and vanishes otherwise.

Note that this is the singular cohomology of X^G.

Groupoids

Definition

A **groupoid** is a small category $\mathcal{G} = (\mathcal{G}_1 \rightrightarrows \mathcal{G}_0)$ in which every arrow is invertible. In particular, there is

- a set of objects $\mathcal{G}_0$,
- a set of arrows $\mathcal{G}_1$,
- a **source map** $s\colon \mathcal{G}_1 \to \mathcal{G}_0$,
- a **target map** $t\colon \mathcal{G}_1 \to \mathcal{G}_0$,
- a **unit map** $u\colon \mathcal{G}_0 \to \mathcal{G}_1$,
- a **multiplication map** $\circ\colon \mathcal{G}_1 \times_{\mathcal{G}_0} \mathcal{G}_1 \to \mathcal{G}_1$,
- and an **inversion** map $\mathrm{inv}\colon \mathcal{G}_1 \to \mathcal{G}_1$;

all of these satisfying the expected relations.

A **topological groupoid** is a groupoid $\mathcal{G}$ in which $\mathcal{G}_0$ and $\mathcal{G}_1$ are Hausdorff spaces, and all structure maps are continuous, with s and t open. Henceforth, assume all groupoids are topological.

Groupoids

Example

- If G is a topological group, then it is a groupoid whose object space is a single point.
- If X is a space, we have its **pair groupoid** $X \times X \rightrightarrows X$ with projections as source and target.
- If X is a space, then there is its **fundamental groupoid**, whose arrows are homotopy classes of paths, and objects are points of X.
- If X is a G-space, then we have the **action groupoid** $G \times X \rightrightarrows X$.

Definition

A groupoid $\mathcal{G}$ is **transitive** if given any two objects $x, y \in \mathcal{G}_0$, there is some arrow $g \in \mathcal{G}_1$ with $s(g) = x$ and $t(g) = y$.

Groupoid Actions

Definition

Given a groupoid $\mathcal{G}$, a **right $\mathcal{G}$-space** is a space X, along with an **anchor map** $a\colon X \to \mathcal{G}_0$ and an **action** $\mathrm{act}\colon X\,{}_a\!\times_t \mathcal{G}_1 \to X\colon (x,g) \mapsto xg$ satisfying

1. $a(xg) = s(g)$ for all $(x,g) \in X\,{}_a\!\times_t \mathcal{G}_1$,
2. $xu(a(x)) = x$ for all $x \in X$, and
3. $(xg)g' = x(gg')$ for all $(x,g) \in X\,{}_a\!\times_t \mathcal{G}_1$ and $g' \in \mathcal{G}_1$ such that $t(g') = s(g)$.

We will assume that a is proper. A groupoid action is **transitive** if for every $x, y \in X$, there is some $g \in \mathcal{G}_1$ so that $y = xg$.

Definition

Fix a transitive groupoid $\mathcal{G}$. Define the **orbit category** $O\mathcal{G}$ to be the category whose objects are transitive $\mathcal{G}$-spaces with $\mathcal{G}$-equivariant maps between them (*i.e.* continuous functors that intertwine the anchor maps and actions).

Notation: $\mathcal{G}^b := t^{-1}(b)$, $\mathcal{G}_b := s^{-1}(b)$, and $\mathcal{G}_b^b := \mathcal{G}^b \cap \mathcal{G}_b$.

Lemma (Farsi-Scull-W.)

Fix $b \in \mathcal{G}_0$. The orbit category $O\mathcal{G}$ is equivalent to the full subcategory of spaces $\mathcal{G}^b/H$ with $\mathcal{G}$-equivariant maps between them, where H runs over closed subgroups of $\mathcal{G}_b^b$.

Bredon Cohomology for Transitive Groupoids

Definition

We define $\mathcal{G}$-CW-complexes analogously to the group case, but where the canonical orbits G/H are replaced with $\mathcal{G}^b/H$.

Remark

Note that to do this properly, we need to define what a “trivial” groupoid action is. We found that the appropriate definition is an action in which $\mathcal{G}_b^b$ acts trivially on the $\mathcal{G}$-space (from which it follows that all stabilisers $\mathcal{G}_c^c$, $c \in \mathcal{G}_0$, act trivially).

Bredon Cohomology for Transitive Groupoids

Definition

We define Bredon cohomology for $\mathcal{G}$-spaces again in precisely the same way.

Notation: if X is a $\mathcal{G}$-space with anchor map a, and $b \in \mathcal{G}_0$, let $X_b := a^{-1}(b)$. This is a $\mathcal{G}_b^b$-space.

Theorem (Farsi-Scull-W.)

$$\mathrm{Hom}_{\mathcal{G}}(\mathcal{G}^b/H, X) \cong \mathrm{Hom}_{\mathcal{G}_b^b}(\mathcal{G}_b^b/H, X_b) \cong X_b^H$$

Consequently, the $\mathcal{G}$-Bredon cohomology of a $\mathcal{G}$-space X reduces to the $\mathcal{G}_b^b$-Bredon cohomology of X_b.

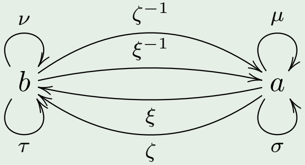
An Example

Example

Consider the groupoid $\mathcal{G}$ with objects $\{a, b\}$ and arrows

$$\{\xi, \zeta, \xi^{-1}, \zeta^{-1}, \mu = u(a), \nu = u(b), \sigma = \zeta^{-1}\xi, \tau = \zeta\xi^{-1}\}$$

with relations $\sigma^2 = \mu$ and $\tau^2 = \nu$.



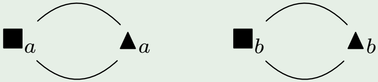
Then $\mathcal{G}_b^b = \{\nu, \tau\}$ and $\mathcal{G}^b = \{\nu, \tau, \xi, \zeta\}$.

An Example

Example

Construct a $\mathcal{G}$-CW-structure as follows:

- $X^0 := (\mathcal{G}^b/\mathcal{G}^b_b \times D^0) \amalg (\mathcal{G}^b/\mathcal{G}^b_b \times D^0) = \{\blacksquare_a, \blacktriangle_a, \blacksquare_b, \blacktriangle_b\}$,
- Attach a 1-cell $\mathcal{G}^b/1 \times D^1$ via the attaching maps $\mathcal{G}^b/1 \times \{-1\} \to \{\blacksquare_a, \blacksquare_b\}$ and $\mathcal{G}^b/1 \times \{1\} \to \{\blacksquare_b, \blacktriangle_b\}$.
- The result looks like:



Since the $\mathcal{G}$-Bredon cohomology is isomorphic to the $\mathcal{G}^b_b$-Bredon cohomology on the fibre X_b, we are reduced to the previous example.

Thank you!

References

- Carla Farsi, Laura Scull, and Jordan Watts, “Classifying spaces and Bredon (co)homology for transitive groupoids”, 2019 (submitted).
  https://arxiv.org/abs/1809.00272