Basic Forms on Geometric Stacks

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Introduction

Let G be a Lie group, and let M be a manifold admitting a proper G-action.

If the action is free, then the orbit space M/G is a manifold.

Otherwise, M/G may not be a manifold.

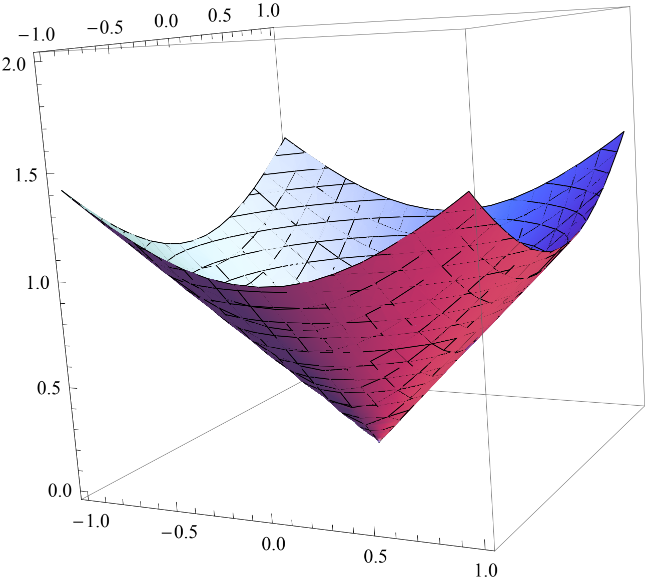
Structures on the Orbit Space

Geometers often want to equip M/G with some form of structure that "remembers" some set of invariants obtained from the action.

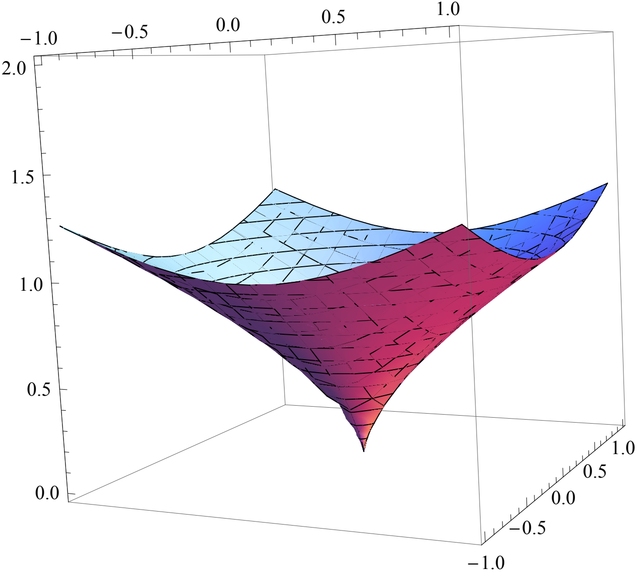
The *quotient topology* cannot tell the difference between the group actions of $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ on the plane $\mathbb{R}^2$ by rotations ($n > 0$). That is, all $\mathbb{R}^2/\mathbb{Z}_n$ are homeomorphic.

The "differential structure" on $\mathbb{R}^2/\mathbb{Z}_n$ (isomorphic to the ring of smooth $\mathbb{Z}_n$-invariant functions on $\mathbb{R}^2$) can tell the difference, however.

$\mathbb{R}^2/\mathbb{Z}_2$



$\mathbb{R}^2/\mathbb{Z}_3$



However...

The differential structures on the orbit spaces $\mathbb{R}^n/\mathrm{SO}(n)$ are all diffeomorphic to the usual smooth structure on the manifold with boundary $[0, \infty)$.

There is another category of smooth structures, called "diffeological spaces", which can tell the difference between all of the above example orbit spaces.

It cannot see the difference between, for example, the orbit spaces obtained from the adjoint actions of SO(3) and SU(2).

Diffeological Spaces

Let X be a set. A **diffeology** $\mathcal{D}$ on X is a family of maps p into X, called **plots**, whose domains are open subsets of Euclidean spaces.

$\mathcal{D}$ is required to satisfy three conditions.

1. (Covering) $\mathcal{D}$ contains all the constant maps.
2. (Locality) If $p : U \to X$ is a map satisfying: there exist an open cover $\{U_\alpha\}$ and plots $\{p_\alpha : U_\alpha \to X\}$ such that

$$p|_{U_\alpha} = p_\alpha$$

   for each α, then p is a plot.
3. (Smooth Compatibility) If $p : U \to X$ is in $\mathcal{D}$ and $F : V \to U$ is a smooth map, then $p \circ F \in \mathcal{D}$.

Diffeological Spaces - Some Examples

Example

Manifolds, manifolds with boundary, and manifolds with corners are examples of diffeological spaces.

Example

Fix a diffeological space $(X, \mathcal{D})$.

Let $\pi : X \to X/\sim$ be the quotient map where $\sim$ is an equivalence relation on X.

$X/\sim$ acquires the **quotient diffeology**, whose plots are locally of the form $\pi \circ p$ for $p \in \mathcal{D}$.

Smooth Maps

A map $F : (X, \mathcal{D}_X) \to (Y, \mathcal{D}_Y)$ is **(diffeologically) smooth** if for each p in $\mathcal{D}_X$, the composition $F \circ p$ is in $\mathcal{D}_Y$.

Diffeologies as Stacks

(Baez–Hoffnung 2011) There is a fully faithful functor I from the category of diffeological spaces to stacks (over the category of manifolds).

Given a diffeological space $(X, \mathcal{D})$, the fibre over any manifold M is the *set* of (diffeologically) smooth maps $M \to X$.

Coarse Moduli Space of a Stack

(W.–Wolbert) There is a (2-)functor Coarse from the (2-)category of stacks over manifolds to itself, taking any stack to an “underlying” diffeological space (viewed as a stack via *I*).

Moreover, given any stack $\mathcal{X}$, there is a morphism of stacks $\pi_{\mathcal{X}}$ from $\mathcal{X}$ to Coarse($\mathcal{X}$).

Universal Property

If $\mathcal{X}$ is a stack, and $(Y, \mathcal{D})$ a diffeological space, and $F : \mathcal{X} \to I(Y, \mathcal{D})$ a morphism of stacks,

then there is a unique morphism of stacks $\overline{F} : \text{Coarse}(\mathcal{X}) \to I(Y, \mathcal{D})$ making the following diagram commute.

$$\begin{tikzcd} \mathcal{X} \arrow[r, "F"] \arrow[d, "\pi_{\mathcal{X}}"'] & I(Y, \mathcal{D}) \\ \text{Coarse}(\mathcal{X}) \arrow[ur, dashed, "\overline{F}"'] & \end{tikzcd}$$

Geometric Stacks

If $\mathcal{X}$ is a *geometric* stack, and $G = (G_1 \rightrightarrows G_0)$ is a Lie groupoid such that $BG \simeq \mathcal{X}$,

then Coarse($\mathcal{X}$) is diffeomorphic to the orbit space G_1/G_0 equipped with its quotient diffeology.

Diffeological Forms

Fix a diffeological space $(X, \mathcal{D})$.

A **(diffeological) *k*-form** α on X is an assignment to each plot $(p : U \to X) \in \mathcal{D}$ a k-form

$$\alpha_p \in \Omega^k(U)$$

such that for any smooth map $F : V \to U$,

$$F^*(\alpha_p) = \alpha_{p \circ F}.$$

Denote the set of k-forms by $\Omega^k(X)$.

Define the **exterior derivative** of a k-form α on X by

$$(d\alpha)_p = d(\alpha_p)$$

for all plots $p \in \mathcal{D}$.

$(\Omega^\bullet(X), d)$ is a differential complex.

Basic Forms of a Proper Lie Groupoid

(W. 2013) If $G = (G_1 \rightrightarrows G_0)$ is a proper Lie groupoid, and $\pi : G_0 \to G_0/G_1$ is the quotient map to the orbit space,

then π^* is an isomorphism of de Rham complexes from $(\Omega^\bullet(G_0/G_1), d)$ to the complex of *basic differential forms* on G_0.

(Recall that a form α on G_0 is **basic** if $s^*\alpha = t^*\alpha$.)

Back to Stacks

If we define a differential k-form α on a stack $\mathcal{X}$ to be a map of stacks

$$\alpha : \mathcal{X} \to \Omega^k$$

and $\mathcal{X}$ is geometric, with an atlas given by a proper Lie groupoid,
then α factors as a composition of the map

$$\pi_{\mathcal{X}} : \mathcal{X} \to \mathrm{Coarse}(\mathcal{X})$$

and a unique differential form $\overline{\alpha}$ on Coarse($\mathcal{X}$) such that

$$\alpha = \overline{\alpha} \circ \pi_{\mathcal{X}}.$$

$$\begin{tikzcd}
\mathcal{X} \arrow[r, "\alpha"] \arrow[d, "\pi_{\mathcal{X}}"'] & \Omega^k \\
\mathrm{Coarse}(\mathcal{X}) \arrow[ur, dashed, "\overline{\alpha}"'] &
\end{tikzcd}$$

Thank you!

References

- J. C. Baez, A. E. Hoffnung, “Convenient categories of smooth spaces”, *Trans. Amer. Math. Soc.*, **363** (2011), no. 11, 5789–5825.
- J. Watts, “The orbit space and basic forms of a proper Lie groupoid”, (preprint). `http://arxiv.org/abs/1309.3001`
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