Weak Equivalences between Action Groupoids

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June 5, 2023

Joint Work

- This is joint work with Carla Farsi and Laura Scull, “Bicategories of Action Groupoids”.
- https://arxiv.org/abs/2208.06281

Why Morita Equivalence?

- Recall the non-trivial irreducible real representations of $\mathbb{Z}/5$:
$$\mathbb{Z}/5 \times \mathbb{C} \to \mathbb{C} \colon \left(e^{2\pi ik/5}, z\right) \mapsto e^{2\pi i\alpha k/5} \cdot z$$
where $k \in \{0, 1, 2, 3, 4\}$ and the weight α is 1 or 2.
- These representations (or linear actions) are *not* isomorphic. In fact, there is no equivariant diffeomorphism between them.
- However, as a geometer, I care about the symmetry arising in this situation, and not so much the “speed” at which the symmetry is being produced. So, these two representations should be “equivalent” in some sense.
- This is precisely the Morita equivalence that is the partial topic of this talk.

Setting

Definition

Let $\mathcal{G} = (\mathcal{G}_1 \rightrightarrows \mathcal{G}_0)$ be a Lie groupoid:

1. a small category in which all arrows are invertible;
2. the set of objects $\mathcal{G}_0$ is a smooth manifold;
3. the set of arrows $\mathcal{G}_1$ is a smooth manifold;
4. the structure maps are all smooth: source s, target t, unit u, multiplication m, and inverse inv;
5. and the source and target are also required to be submersions.

Example

The action of a Lie group G on a manifold M (called a G-manifold) induces a Lie groupoid, $G \ltimes M$, called the **action groupoid**:

$$G \times M \rightrightarrows M.$$

Setting

Definition

A morphism of Lie groupoids $F\colon \mathcal{G} \to \mathcal{H}$ is a **smooth functor**

$$\begin{array}{ccc} \mathcal{G}_1 & \xrightarrow{F_1} & \mathcal{H}_1 \\ \downarrow\downarrow & & \downarrow\downarrow \\ \mathcal{G}_0 & \xrightarrow[F_0]{} & \mathcal{H}_0; \end{array}$$

that is, a functor in which the map between objects and the map between arrows are both smooth.

Setting

Example

Given a Lie group homomorphism $\varphi\colon G \to H$, an equivariant map $f\colon M \to N$ from a G-manifold M to an H-manifold N,

$$\textit{i.e.} \quad f(g \cdot x) = \varphi(g) \cdot f(x),$$

induces a functor

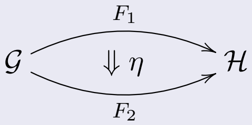
$$(g, x) \mapsto (\varphi(g), f(x)).$$

We call such a functor between action groupoids an **equivariant functor.**

Setting

Definition

A 2-morphism/2-arrow $\eta\colon F_1 \Rightarrow F_2$ is a smooth natural transformation



smoothly sending an object $x \in \mathcal{G}_0$ to an arrow $\eta_x \in \mathcal{H}_1$ in a natural way.

Example

Given a smooth homotopy $H\colon M \times I \to N$ between smooth maps $f_0, f_1\colon M \to N$, there is a corresponding natural transformation between the induced functors between fundamental groupoids:

$$\Pi_1 M \underset{\Pi_1 f_1}{\overset{\Pi_1 f_0}{\rightrightarrows}} \Pi_1 N. \qquad \Downarrow \Pi_1 H$$

Drawback

- Groupoids are categories, and one can have a smooth functor that is an equivalence of categories that is *not* (smoothly) invertible.

Example

Squaring $z \mapsto z^2$ induces a functor between our two representations of $\mathbb{Z}/5$ of weights 1 and 2, restricted to $\mathbb{C} \smallsetminus \{0\}$, that is surjective on objects and fully faithful on arrows. However, squaring does not admit a global smooth (or even continuous) inverse.

Weak Equivalences

Definition

A **weak equivalence** $\varphi\colon \mathcal{G} \xrightarrow{\simeq} \mathcal{H}$ is a smooth functor that satisfies

1. **(Smooth Essential Surjectivity)** $\Psi_\varphi\colon \mathcal{G}_0{}_\varphi\times_{\mathrm{t}}\mathcal{H}_1 \to \mathcal{H}_0\colon (x,h)\mapsto \mathrm{s}(h)$ is a *surjective submersion*,
2. **(Smooth Fully Faithfulness)** $\Phi_\varphi\colon \mathcal{G}_1 \to \mathcal{G}_0^2{}_{\varphi^2}\times_{(\mathrm{s},\mathrm{t})}\mathcal{H}_1\colon g\mapsto (\mathrm{s}(g),\mathrm{t}(g),\varphi(g))$ is a diffeomorphism.

Example

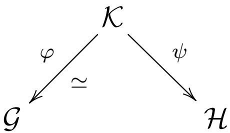
The action groupoid of any principal G-bundle over a manifold M is weakly equivalent to M.

Question

How do we (formally) invert these weak equivalences?

Bicategory of Fractions - Pronk [P96]

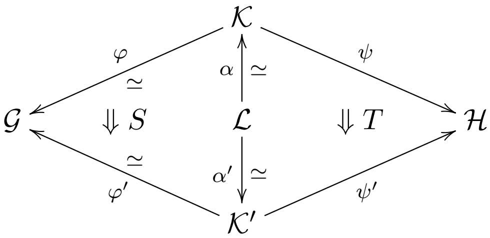
- Let W be the class of all weak equivalences in the 2-category $\mathbf{LieGpoid}$.
- Define a bicategory $\mathbf{LieGpoid}[W^{-1}]$, the **bicategory of fractions**, as follows:
  0. Objects are Lie groupoids.
  1. A 1-cell from $\mathcal{G}$ to $\mathcal{H}$ is a **generalised morphism**:



- A generalised morphism is a **Morita equivalence** if both functors are weak equivalences.

Bicategory of Fractions - Pronk [P96]

2. A 2-cell between two generalised morphisms is an equivalence class of 2-commutative diagrams



- The equivalence relation is given by yet another generalised morphism between the centres of the diagrams, and some identities involving the natural transformations. (See Pronk [P96] details.)

Example: Open Covers

Example

- Let M be a manifold.
- Let $\{U_\alpha\}$ and $\{V_\beta\}$ be two open covers of M. Define

$$\mathcal{U}_0 := \coprod U_\alpha,$$
$$\mathcal{V}_0 := \coprod V_\beta.$$

- Let

$$\mathcal{U}_1 := \coprod U_\alpha \cap U_{\alpha'},$$
$$\mathcal{V}_1 := \coprod V_\beta \cap V_{\beta'}.$$

- $\mathcal{U}$ and $\mathcal{V}$ are so-called Čech groupoids of M.

Example: Open Covers

Example

- If $\{W_\gamma\}$ is a refinement of the two open covers, then we construct a Čech groupoid $\mathcal{W}$ similarly.
- We have the generalised morphism

$$\mathcal{U} \xleftarrow{\ \simeq\ } \mathcal{W} \xrightarrow{\ \simeq\ } \mathcal{V}$$

where the two functors are inclusions.

Example: Effective Orbifold

Example

- Through the work of Moerdijk and Pronk [MP97], we can do a similar procedure with orbifolds atlases:
- Two orbifold atlases (viewed as Lie groupoids) $\mathcal{A}$ and $\mathcal{B}$ are equivalent if their charts are all compatible, leading to a larger orbifold atlas.
- More precisely, if there is another orbifold atlas $\mathcal{C}$ and a Morita equivalence

$$\mathcal{A} \xleftarrow{\ \simeq\ } \mathcal{C} \xrightarrow{\ \simeq\ } \mathcal{B}$$

then the orbifold altases are equivalent; they describe the same orbifold.

Example: Spanisation

Example

- Let $\varphi\colon \mathcal{G} \to \mathcal{H}$ be a functor. There is a corresponding generalised morphism:

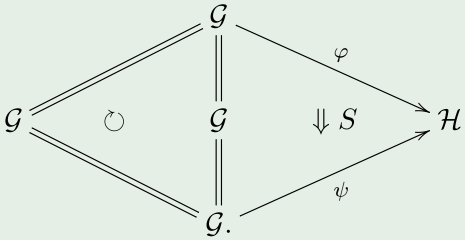
$$\mathcal{G} \xleftarrow{\;=\;} \mathcal{G} \xrightarrow{\;\varphi\;} \mathcal{H}$$

called the **spanisation** of φ.

Example: Spanisation

Example

- Let $\psi\colon \mathcal{G} \to \mathcal{H}$ be another functor, and $S\colon \varphi \Rightarrow \psi$ a natural transformation.
- The **spanisation** of S is given by the 2-cell represented by the diagram



$\mathbf{LieGpoid}[W^{-1}]$

Theorem

The bicategory whose objects are Lie groupoids, 1-cells are generalised morphisms, and 2-cells are as described above, denoted $\mathbf{LieGpoid}[W^{-1}]$, admits an inclusion pseudofunctor $\mathfrak{S}\colon \mathbf{LieGpoid} \to \mathbf{LieGpoid}[W^{-1}]$ given by spanisation, and admits a pseudo-inverse for every weak equivalence $\varphi \in W$.

- The following generalised morphisms are pseudo-inverses of each other:

$$\mathcal{G} \xleftarrow[\simeq]{\varphi} \mathcal{K} \xrightarrow[\simeq]{\psi} \mathcal{H} \qquad\qquad \mathcal{H} \xleftarrow[\simeq]{\psi} \mathcal{K} \xrightarrow[\simeq]{\varphi} \mathcal{G}.$$

Exercise!

Exercise

Show that any two non-trivial irreducible real representations of $\mathbb{Z}/5$ yield Morita equivalent Lie groupoids. (Better yet, do this for $\mathbb{Z}/p$ for prime p.)

Goal

- Let G and H be Lie groups. We want to show that any generalised morphism between action groupoids $G \ltimes M$ and $H \ltimes N$ admits a 2-arrow to a generalised morphism made up of equivariant functors.

History

- Pronk and Scull prove this for action groupoids that are orbifolds [PS10]. They use this to prove a decomposition theorem, and use this to go on and prove some Morita-invariance results for theories like K-theory and Bredon cohomology.

Results

Theorem (Farsi-Scull-W.)

Let $G \ltimes M$ and $H \ltimes N$ satisfy any subset of the following properties:

$$\left\{ \begin{array}{c} \textit{compact, effective, free, locally free, transitive,} \\ \textit{proper, discrete, is an orbifold groupoid} \end{array} \right\}.$$

Let

$$G \ltimes M \xleftarrow{\simeq} \mathcal{K} \longrightarrow H \ltimes N$$

be a generalised morphism. Then there exist a second generalised morphism

$$G \ltimes M \xleftarrow[\varphi]{\simeq} G \ltimes \mathcal{L} \rtimes H \xrightarrow[\psi]{} H \ltimes N$$

in which φ and ψ are equivariant functors, and a 2-arrow between the two generalised morphisms.

Results

Theorem (Farsi-Scull-W.)

Let $G \ltimes M$ *and* $H \ltimes N$ *satisfy any subset of the following properties:*

$$\left\{ \begin{array}{c} \textit{compact, free, locally free, transitive,} \\ \textit{proper, discrete, is an orbifold groupoid} \end{array} \right\}.$$

Let $\varphi\colon G \ltimes M \xrightarrow{\simeq} H \ltimes N$ *be a weak equivalence. Then there exist equivariant weak equivalences*

$$\pi\colon G \ltimes M \xrightarrow{\simeq} K\backslash G \ltimes K\backslash M$$

and

$$i\colon K\backslash G \ltimes K\backslash M \xrightarrow{\simeq} H \ltimes N$$

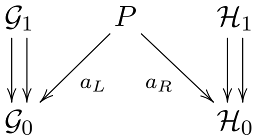
where $K \trianglelefteq G$ *acts freely on* M*, and a* 2*-arrow* $\varphi \Rightarrow i \circ \pi$*.*

Corollary

- Combining the previous two theorems allows us, when working with action groupoids, to restrict completely to the equivariant setting, and to decompose the corresponding weak equivalences into equivariant weak equivalences induced by epimorphisms and monomorphisms.
- This will allow us to generalise the work of Pronk-Scull from representable orbifolds to action groupoids satisfying any of the properties listed previously.

Definition

A **right principal bibundle from $\mathcal{G}$ to $\mathcal{H}$** is a right principal $\mathcal{H}$-bundle $a_L\colon P \to \mathcal{G}_0$ with anchor map $a_R\colon P \to \mathcal{H}_0$ equipped with a $\mathcal{G}$-action with anchor map a_L that commutes with the $\mathcal{H}$-action, and so that a_R is $\mathcal{G}$-invariant.



- If a_R is a left principal $\mathcal{G}$-bundle as well, then P is also called a **Morita equivalence**.

Bibundles

- We form a bicategory $\mathbf{LieBiBund}$ with 2-cells as follows: given bibundles P and Q between $\mathcal{G}$ and $\mathcal{H}$, a 2-cell is a ($\mathcal{G}$-$\mathcal{H}$)-biequivariant diffeomorphism between P and Q.

Theorem (Hilsum-Skandalis HS[87])

The bicategory whose objects are Lie groupoids, 1-cells are right principal bibundles, and 2-cells are bi-equivariant diffeomorphisms, denoted $\mathbf{LieBiBund}$, admits an inclusion functor $\mathfrak{B}\colon \mathbf{LieGpoid} \to \mathbf{LieBiBund}$ (called "bibundlisation"), and admits a pseudo-inverse for every weak equivalence $\varphi \in W$.

Theorem (continued - PS[96])

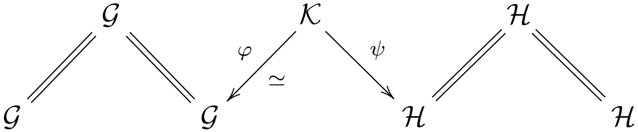
Moreover, we have the following 2-commutative diagram

$$\begin{array}{ccc} \mathbf{LieGpoid}[W^{-1}] & \xrightarrow{\simeq} & \mathbf{LieBiBund} \\ \mathfrak{S}\uparrow & \Rightarrow & \nearrow \mathfrak{B} \\ \mathbf{LieGpoid}. & & \end{array}$$

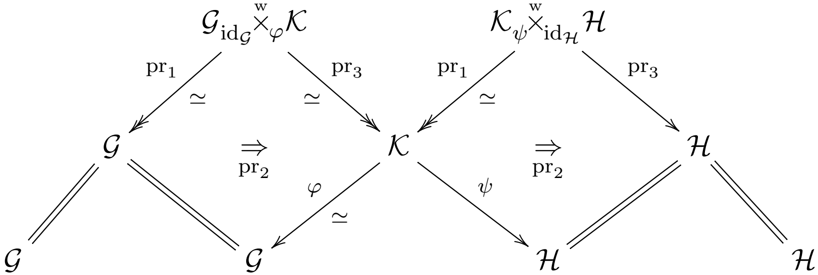
Construction: Generalised Morphism to Bibundle

$$\mathcal{G} \xleftarrow[\simeq]{\ \varphi\ } \mathcal{K} \xrightarrow{\ \psi\ } \mathcal{H}$$

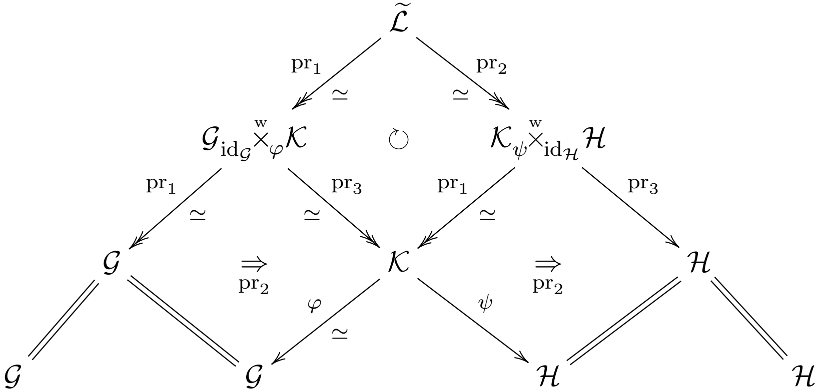
Construction: Generalised Morphism to Bibundle



Construction: Generalised Morphism to Bibundle



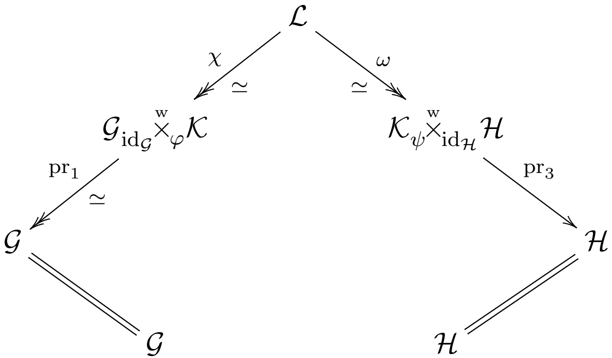
Construction: Generalised Morphism to Bibundle



where $\widetilde{\mathcal{L}} = (\mathcal{G}\,{}_{\mathrm{id}_\mathcal{G}}\!\overset{\mathrm{w}}{\times}_{\varphi}\mathcal{K})\,{}_{\mathrm{pr}_3}\!\times_{\mathrm{pr}_1}(\mathcal{K}\,{}_{\psi}\!\overset{\mathrm{w}}{\times}_{\mathrm{id}_\mathcal{H}}\mathcal{H})$.

Construction: Generalised Morphism to Bibundle

$\widetilde{\mathcal{L}}$ comes equipped with a left and a right $\mathcal{K}$-action. Let $\mathcal{L} := \mathcal{K}\backslash\widetilde{\mathcal{L}}/\mathcal{K}$.



$\mathcal{L}$ is isomorphic as a groupoid to the action groupoid of a bibundle from $\mathcal{G}$ to $\mathcal{H}$.

Thank you!

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