Homework #07 or Take Home Exam (in case it's required)

Math 6230 - Section 001

Due: Wednesday, May 3, 2017 – **absolutely no extensions**.

Instructions. Prove the following statements. **All of your solutions must be typed up using LaTeX.** Your solutions must be legible, and the grader must be able to follow the logic. (It may be helpful to write out a rough draft of a proof first, and then make a good copy.) Utter nonsense will receive negative points, and so if you do not know how to prove a problem, do not just make things up and pass it in. Finally, while you are encouraged to work together, each person must pass in their own work. If you copy a solution off of the internet, this is pretty easy to figure out, is considered cheating, and will be treated as such.

1. **Read:** - Chapter 16 (pages 400–415), Chapter 17.
2. Problem 16-1 from the text.
3. Problem 16-2 from the text.
4. **(Integration on Manifolds)** Let M be a compact, oriented m-dimensional manifold. Let $\omega \in \Omega^p(M)$ and $\eta \in \Omega^q(M)$ such that $p+q=m-1$. Prove:

$$\int_M d\omega \wedge \eta = (-1)^{p+1} \int_M \omega \wedge d\eta.$$

5. **(Symplectic Forms)** Let M be a smooth manifold of dimension $2m$. A **symplectic form** ω on M is a *closed* 2-form satisfying the following *nondegeneracy* condition: for any $x \in M$ and any $u \in T_xM$,

$$\omega(u,v) = 0 \text{ for all } v \in T_xM \iff u = 0.$$

We call (M, ω) a **symplectic manifold**.

   (a) Show that $\omega^m := \omega \wedge \cdots \wedge \omega \in \Omega^{2m}(M)$ is a volume form.

   (b) Show that if ω is exact, so is the corresponding volume form.

   (c) For which n does $\mathbb{S}^n$ admit a symplectic form ($n > 0$)? Justify your answer.

   (d) Let $\omega = dx^1 \wedge dx^2 + dx^3 \wedge dx^4 + dx^5 \wedge dx^6$ on $\mathbb{R}^6$. Show that there does not exist a diffeomorphism $\varphi\colon \mathbb{R}^6 \to \mathbb{R}^6$ such that $\varphi^*\omega = \omega$ and such that $\varphi(\mathbb{S}^5)$ is a sphere of radius $r \neq 1$.

6. **(Cohomology of Spaces)** Let p, q, r be distinct points of $\mathbb{S}^2$. Find the de Rham cohomology groups of the following spaces.

   (a) $\mathbb{S}^2 \setminus \{p\}$.

   (b) $\mathbb{S}^2 \setminus \{p, q\}$.

   (c) $\mathbb{S}^2 \setminus \{p, q, r\}$.