Homework #02

Math 6230 - Section 001

Due: Wednesday, February 15, 2017.

Instructions. Prove the following statements. **All of your assignments must be typed up using LaTeX.** Either way, your solutions must be legible, and the grader must be able to follow the logic. (It may be helpful to write out a rough draft of a proof first, and then make a good copy.) Utter nonsense will receive negative points, and so if you do not know how to prove a problem, do not just make things up and pass it in. Finally, while you are encouraged to work together, each person must pass in their own work. If you copy a solution off of the internet, this is pretty easy to figure out, is considered cheating, and will be treated as such.

1. **Read:** Chapter 2 (especially pages 32–42), and Chapter 3 (especially pages 50–71) of the text.
2. **(Smoothness is Local)**
   (a) Prove the following: Given a map $F\colon M \to N$ between smooth manifolds M and N, F is smooth if and only if for every $x \in M$ there exists an open neighbourhood U of x such that the restriction $F|_U\colon U \to N$ is smooth.
   (b) Prove the corollary: Let M and N be smooth manifolds, and let $\{U_\alpha\}_{\alpha\in A}$ be an open cover of M and $\{F_\alpha\colon U_\alpha \to N\}_{\alpha\in A}$ a family of smooth maps satisfying: for each pair α, β, if $U_\alpha \cap U_\beta \neq \emptyset$, then
   $$F_\alpha|_{U_\alpha\cap U_\beta} = F_\beta|_{U_\alpha\cap U_\beta}.$$
   Then the family $\{F_\alpha\}$ "glues together" to form a global smooth function; that is, there exists a smooth map $F\colon M \to N$ such that for any $\alpha \in A$, we have $F|_{U_\alpha} = F_\alpha$.
3. #2-6 from the text.
4. #2-14 from the text.
5. **(Different Definitions of Tangent Space)** Let M be a smooth manifold of dimension m, and fix $x \in M$. Consider the following three sets:
   (a) Derivations of $C^\infty(M)$ at x,
   (b) $\mathcal{C}_x/\sim$ where $\mathcal{C}_x$ is the collection of all smooth curves $c\colon \mathbb{R} \to M$ such that $c(0) = x$, and $c_1 \sim c_2$ if there exists a chart $\varphi\colon U \to \widetilde{U}$ about x such that
   $$\frac{d}{dt}\Big|_{t=0}\varphi\circ c_1(t) = \frac{d}{dt}\Big|_{t=0}\varphi\circ c_2(t).$$
   (c) Let $\{\varphi_\alpha\colon U_\alpha \to \widetilde{U}_\alpha\}_{\alpha\in A}$ be a smooth atlas for M, and let $B \subseteq A$ be the collection of all α such that $x \in U_\alpha$. Denote by $\underline{v}$ a family $\{v_\beta\}_{\beta\in B}$ of vectors

$v_\beta = (v^1_\beta, \dots, v^m_\beta) \in \mathbb{R}^m$ satisfying for any pair β_1 and β_2 in B and $j \in \{1, \dots, m\}$:

$$v^j_{\beta_2} = \sum_{i=1}^{m} \frac{\partial y^j}{\partial x^i}\bigg|_{\varphi_{\beta_1}(x)} v^i_{\beta_1}$$

where $\left[\frac{\partial y^j}{\partial x^i}\right]_{ij}$ is the derivative $d(\varphi_{\beta_2} \circ \varphi_{\beta_1}^{-1})$ of the transition function $\varphi_{\beta_2} \circ \varphi_{\beta_1}^{-1}$. Denote by $\mathcal{V}$ the collection of all families $\underline{v}$.

Show that there is a natural bijection between T_xM, $\mathcal{C}_x/\sim$, and $\mathcal{V}$.

6. **(Tangent Spaces on Non-Manifolds)** Consider the subset $[0, \infty) \subset \mathbb{R}$. This is *not* a manifold (it is a "manifold with boundary"). Define $C^\infty([0, \infty))$ to be all restrictions of smooth functions $f: \mathbb{R} \to \mathbb{R}$ to $[0, \infty)$. Note that "derivations of $C^\infty([0, \infty))$ at x" still makes sense on $[0, \infty)$, even at 0. Denote the set of all derivations of $C^\infty([0, \infty))$ at 0 by $T_0([0, \infty))$.

   Define $\mathcal{C}_0$ to be all smooth curves $c: \mathbb{R} \to \mathbb{R}$ with image in $[0, \infty)$ such that $c(0) = 0$, and let $\sim$ be the equivalence relation $c_1 \sim c_2$ if

$$\frac{d}{dt}\bigg|_{t=0} c_1(t) = \frac{d}{dt}\bigg|_{t=0} c_2(t).$$

   Compare $T_0([0, \infty))$ and $\mathcal{C}_0/\sim$; are they the same? Are they different? Justify your answer.

7. **(T Preserves Products)**
   (a) Let M_1 and M_2 be two smooth manifolds. Show that the product topological atlas on $M_1 \times M_2$ is smooth, and hence $M_1 \times M_2$ has a natural product smooth structure such that the projection maps are smooth.
   (b) Show that $T: \underline{C^\infty\text{-Mfld}} \to \underline{C^\infty\text{-Mfld}}$, sending a smooth manifold M to its tangent bundle TM, and a smooth map $F: M \to N$ to the pushforward map $TF = F_* = dF: TM \to TN$, is a functor. (This is the **tangent functor**.)
   (c) Show that T preserves products (up to diffeomorphism); that is, $T(M_1 \times M_2) \cong TM_1 \times TM_2$.

8. **(Natural Transformations)** Let $\mathcal{C}$ and $\mathcal{D}$ be two categories, and let $F, G: \mathcal{C} \to \mathcal{D}$ be two functors between them. A **natural transformation** $\eta: F \Rightarrow G$ is an assignment to every object $c \in \mathcal{C}_0$ an arrow $\eta_c: F(c) \to G(c)$ in $\mathcal{D}$ such that for any two objects $c_1, c_2 \in \mathcal{C}_0$ and arrow $f: c_1 \to c_2$, the following diagram commutes:

$$\begin{array}{ccc} F(c_1) & \xrightarrow{\eta_{c_1}} & G(c_1) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(c_2) & \xrightarrow[\eta_{c_2}]{} & G(c_2) \end{array}$$

   Show that there exists a natural transformation η from the tangent functor T to the identity functor I on $\underline{C^\infty\text{-Mfld}}$.