Homework #01

Math 6230 - Section 001

Due: Wednesday, February 1, 2017.

Instructions. Prove the following statements. **All of your assignments must be typed up using LaTeX.** Either way, your solutions must be legible, and the grader must be able to follow the logic. (It may be helpful to write out a rough draft of a proof first, and then make a good copy.) Utter nonsense will receive negative points, and so if you do not know how to prove a problem, do not just make things up and pass it in. Finally, while you are encouraged to work together, each person must pass in their own work. If you copy a solution off of the internet, this is pretty easy to figure out, is considered cheating, and will be treated as such.

1. **Read:** Appendix A (especially 596–611), Appendix C (especially pages 657–662), and Chapter 1 (especially 1-24) of the text.
2. **(Topology Induced by Polynomials)** Declare a set $U \subseteq \mathbb{R}$ to be "open" if its complement $\mathbb{R} \setminus U$ is the zero-set of a (real-valued) polynomial $p \in P(\mathbb{R})$:

$$U \text{ is open } \Leftrightarrow \exists p \in P(\mathbb{R}) \text{ such that } \mathbb{R} \setminus U = p^{-1}(0).$$

   Show that the collection of all "open" sets is a topology on $\mathbb{R}$ that is not equal to the standard Euclidean topology.
   Bonus: prove the same result for $\mathbb{R}^n$ (this requires some deep theorems from algebra).
   Note: If we replaced $\mathbb{R}$ with $\mathbb{C}^n$ above, and modified the definition of an open set slightly, then we would get the definition of the *Zariski topology* from classical algebraic geometry.
3. **(Conic Sections and Projective Spaces)** Let $p \colon \mathbb{R}^n \to \mathbb{R}$ be a polynomial, and let $x^1, \ldots, x^n$ be the coordinates on $\mathbb{R}^n$. Define $P(x^0, x^1, \ldots, x^n) = p(x^1/x^0, \ldots, x^n/x^0)$ for $x^0 \neq 0$.
   (a) Show that $(x^0)^d P(x^0, \ldots, x^n)$ uniquely extends to a homogeneous polynomial Q (*i.e.* each term of the polynomial is of the same degree) on $\mathbb{R}^{n+1}$ with coordinates $x^0, \ldots, x^n$, where d is the degree of p.
   (b) Show that any zero level set of Q determines a well-defined subset of $\mathbb{RP}^n$, consisting of points $[x^0, \ldots, x^n]$ such that $Q(x^0, \ldots, x^n) = 0$.
   (c) Consider the conic sections:
      i. the empty set given by $x^2 + y^2 = c$ where $c < 0$,
      ii. the point given by $x^2 + y^2 = 0$,
      iii. the ellipse given by $ax^2 + by^2 = 1$ where $a, b > 0$,
      iv. the parabola given by $y = ax^2$ where $a \neq 0$,
      v. the straight line $x^2 = 0$,
      vi. the union of two distinct straight lines $ax^2 - by^2 = 0$ where $a, b > 0$,

vii. and the hyperbola $ax^2 - by^2 = 1$ where $a, b > 0$.

For each conic section, extend the level set to $\mathbb{RP}^2$ using the procedure from Part (a), and describe what the extension looks like in each homogeneous coordinate chart.

(d) Repeat the same procedure above for the quadric surfaces sitting in $\mathbb{R}^3$: the ellipsoid, elliptic paraboloid, hyperbolic paraboloid, hyperboloids of one and two sheets, and the cone.

4. **(The 2-Sphere)**

(a) Show that the stereographic projections at the north and south poles of the 2-sphere,

$$\mathbb{S}^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\},$$

together form a smooth atlas.

(b) Use the implicit function theorem to construct another smooth atlas of $\mathbb{S}^2$.

(c) Show that the two atlases from Part (a) and Part (b) are equivalent.

5. **(Smoothness Versus Analyticity)** Define $f\colon \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x > 0, \\ 0 & \text{if } x \leq 0. \end{cases}$$

Prove that f is a smooth function, but not analytic. **Note:** This scenario is in complete contrast to what happens for complex-valued functions: a complex-valued function is (once) differentiable if and only if it is analytic. (Hence, being k-differentiable for any positive k implies smoothness for complex variables.)

6. **(The Inverse Function Theorem)** Recall the inverse function theorem:

> **Inverse Function Theorem:** Let $f\colon \mathbb{R}^m \to \mathbb{R}^m$ be a smooth function. Fix $x_0 \in \mathbb{R}^m$ and let $y_0 := f(x_0)$. Assume that the differential of f at x_0 has full rank. Then there exist open neighbourhoods A of x_0 and B of y_0 such that $f|_A$ is a diffeomorphism from A to B.

Prove that the inverse function theorem is equivalent to the implicit function theorem; that is, one implies the other. **Hint:** to show that the inverse function theorem implies the implicit function theorem, if $g\colon \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ is the smooth function from the implicit function theorem, then consider the function $F\colon \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \times \mathbb{R}^m$ sending (x, y) to $(x, g(x, y))$.

7. #1-1 from the text.

8. #1-9 from the text (recall that a space is **compact** if for any open cover, there exists a finite subcover).