INTRODUCTION TO LIE GROUPOIDS

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Lie group actions are examples of a much more general structure known as a Lie groupoid. These form a category that include the category of smooth manifolds as a full subcategory (*i.e.* the inclusion functor forms a bijection on arrows), as well as the category of Lie groups (whose intersection with the subcategory of smooth manifolds has only one object: the single point / the trivial group). For an excellent introduction to Lie groupoids, see [M].

1. Fibred Products

To begin, we need a preliminary definition: the fibred product of sets.

Definition 1.1 (Fibred Product of Sets). Let S, T, U be sets, and let $f\colon S \to U$ and $g\colon T \to U$ be functions. Define the **fibred product** of S and T with respect to f and g to be

$$S\,{}_f{\times}_g T := \{(s,t) \in S \times T \mid f(s) = g(t)\}.$$

As a commutative diagram where pr_1 and pr_2 are the projection maps restricted from $S \times T$, we have

$$\begin{array}{ccc} S\,{}_f{\times}_g T & \xrightarrow{\mathrm{pr}_2} & T \\ {\scriptstyle \mathrm{pr}_1}\downarrow & & \downarrow{\scriptstyle g} \\ S & \xrightarrow[f]{} & U \end{array}$$

Example 1.2 (Cartesian Product). Let S and T be sets, and consider the constant maps $f\colon S \to \{*\}$ and $g\colon T \to \{*\}$, where $\{*\}$ is a singleton set. Then,

$$S\,{}_f{\times}_g T = S \times T.$$

Example 1.3 (Graph). Let S and T be sets, and let $f\colon S \to T$ be a function. Then $S\,{}_f{\times}_{\mathrm{id}_T} T$ is the graph of f sitting in $S \times T$.

Example 1.4 (Pre-Image). Let S and T be sets, let $f\colon S \to T$ be a function, and fix $t_0 \in T$ with inclusion $i\colon \{t_0\} \to T$. Then pr_1 is a bijection from $S\,{}_f{\times}_i \{t_0\}$ onto $f^{-1}(t_0)$.

Example 1.5 (Intersections). Let S and T be subsets of a set U, and let $i\colon S \to U$ and $j\colon T \to U$ be the inclusions. Then both pr_1 and pr_2 are bijections from $S\,{}_i{\times}_j T$ onto $S \cap T$.

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2. Lie Groupoids

Definition 2.1 (Groupoid). A **groupoid** is a small category $\mathcal{G}$ in which every arrow is invertible. Denote by $\mathcal{G}_0$ the set of objects of $\mathcal{G}$, and by $\mathcal{G}_1$ the set of arrows of $\mathcal{G}$. We have the following **structure maps**:

- two maps $s\colon \mathcal{G}_1 \to \mathcal{G}_0$ and $t\colon \mathcal{G}_1 \to \mathcal{G}_0$, called the **source map** and the **target map** respectively, defined as:
$$s(g\colon c_1 \to c_2) = c_1,$$
$$t(g\colon c_1 \to c_2) = c_2;$$
- the **unit map** $u\colon \mathcal{G}_0 \to \mathcal{G}_1$ sending each $x \in \mathcal{G}_0$ to the identity map id_x;
- the **multiplication map** $m\colon \mathcal{G}_1{}_s{\times}_t\mathcal{G}_1 \to \mathcal{G}_1$, sending pairs (h, g) such that $t(g) = s(h)$ to the composition $h \circ g$ (often, we drop the symbol $\circ$);
- the **inversion map** $\mathrm{inv}\colon \mathcal{G}_1 \to \mathcal{G}_1$ sending g to g^{-1}.

Notation 2.2. Sometimes we will denote a groupoid $\mathcal{G}$ by $\mathcal{G}_1 \rightrightarrows \mathcal{G}_0$ for convenience.

Example 2.3 (Group). Let G be a group. Then we can construct a groupoid $\mathcal{G}$ with $\mathcal{G}_0 = \{*\}$ and $\mathcal{G}_1 = G$. Multiplication is the usual multiplication of a group.

Example 2.4 (Cover Groupoid). Let X be a topological space, and let $\{U_\alpha\}_{\alpha\in A}$ be an open cover of X. Then we construct the corresponding **covering groupoid** $\mathcal{U}$ as follows: Let $\mathcal{U}_0$ be the disjoint union $\coprod_{\alpha\in A} U_\alpha$, and let $\mathcal{U}_1 = \coprod_{(\alpha,\beta)\in A\times A} U_\alpha \cap U_\beta$. Define the source map component-wise as the inclusion map: $s\colon U_\alpha \cap U_\beta \hookrightarrow U_\alpha$; similarly, the target map on $U_\alpha \cap U_\beta$ is the inclusion into U_β. Multiplication is defined in the obvious way, and corresponds to elements of triple intersections.

Remark 2.5. The above example is in fact an example of a *topological groupoid*, but we will not need to define this here; instead, we move immediately into the smooth category.

Definition 2.6 (Lie Groupoid). A groupoid $\mathcal{G}$ is a **Lie groupoid** if both $\mathcal{G}_0$ and $\mathcal{G}_1$ are smooth manifolds, s and t are surjective submersions, and u, m, and inv are smooth maps.

Exercise 1 (Smoothness of m). For the smoothness of m to make sense, we require its domain to be a smooth manifold. But its domain is $\mathcal{G}_1{}_s{\times}_t\mathcal{G}_1$, a subset of $\mathcal{G}_1 \times \mathcal{G}_1$. Why is this a smooth manifold?

Exercise 2 (Units Form an Embedded Submanifold). The unit map $u\colon \mathcal{G}_0 \to \mathcal{G}_1$ is an embedding.

Remark 2.7. In the case that s is a proper map, we can drop the requirement that inv is smooth, as we would get this for free. Indeed, we start with **Ehresmann's Theorem:** given a smooth map $F\colon M \to N$, if F is a proper surjective submersion, then F is a **locally trivial fibration**; that is, there exists an open cover $\{U_\alpha\}$ of N such that for each α, the pre-image $F^{-1}(U_\alpha)$ is diffeomorphic to $U_\alpha \times F^{-1}(x)$ for any $x \in U_\alpha$. Fix $U = U_\alpha$ for some α. Since $u \circ s\colon \mathcal{G}_1 \to \mathcal{G}_1$ is a submersion with image $u(\mathcal{G}_0)$, by the Rank Theorem, we can choose a diffeomorphism $\varphi\colon s^{-1}(U) \to U \times s^{-1}(x)$ (for some fixed $x \in U$) such that $\varphi^{-1}(y, u(x)) = u(y)$ for all $y \in U$. Note that
$$(\mathrm{pr}_2 \circ \varphi \circ m)^{-1}(u(x)) = \{(g^{-1}, g) \in \mathcal{G}_1{}_s{\times}_t\mathcal{G}_1 \mid s(g) \in U\}.$$

Since m is a submersion in its second coordinate, local applications of the implicit function theorem imply that there exists a smooth function $g \mapsto g^{-1}$, which finishes the proof. Note how this result resembles the fact that for a Lie group, we do not require that the inversion map is smooth: we get this for free from the fact that multiplication is smooth.

Example 2.8 (Action Groupoid). Let G be a Lie group acting smoothly on a manifold M. Then we can construct the **action groupoid** $G \ltimes M$, where the arrows are the pairs in the product $G \times M$, the objects are the points of M, the source map is $s(g, x) = x$, and the target map is $t(g, x) = g \cdot x$. Multiplication is given by

$$(h, y)(g, x) = (hg, x)$$

where $y = g \cdot x$.

Example 2.9 (Cover Groupoid - Part II). Let M be a manifold, and let $\{U_\alpha\}_{\alpha \in A}$ be an open cover. Then the cover groupoid constructed in Example 2.4 is a Lie groupoid.

Example 2.10 ($\mathrm{GL}(TM)$). Let M be an m-manifold, and consider its tangent bundle TM. Define $\mathrm{GL}(TM)$ to be the groupoid whose objects are the points of M, and whose arrows form the set

$$\mathrm{GL}(TM)_1 := \{\xi \colon T_xM \to T_yM \mid x, y \in M \text{ and } \xi \text{ is a linear isomorphism}\}.$$

The source map sends each arrow to the foot-point of its domain, and the target map sends each arrow to the foot-point of its codomain.

$\mathrm{GL}(TM)$ is a Lie groupoid: for every pair of charts $\varphi \colon U \to \widetilde{U} \subseteq \mathbb{R}^m$ and $\psi \colon V \to \widetilde{V} \subseteq \mathbb{R}^m$ of M, define

$$\Phi_{\varphi,\psi} \colon s^{-1}(U) \cap t^{-1}(V) \to \widetilde{U} \times \mathrm{GL}(m; \mathbb{R}) \times \widetilde{V} \subseteq \mathbb{R}^{2m+m^2}$$

sending an arrow ξ to $(s(\xi), X, t(\xi))$ where X is the map $\psi_* \circ \xi \circ \varphi_*^{-1}|_{s(\xi)}$. The collection of all such $\Phi_{\varphi,\psi}$ is a topological atlas. Smooth compatibility is easy to check.

3. Lie Groupoid Morphisms

Definition 3.1 (Lie Groupoid Morphisms). Let $\mathcal{G}$ and $\mathcal{H}$ be Lie groupoids. A **Lie groupoid morphism** is a functor $F \colon \mathcal{G} \to \mathcal{H}$ such that the following diagram is commutative and consists of smooth maps:

$$\begin{array}{ccc} \mathcal{G}_1 & \xrightarrow{F_1} & \mathcal{H}_1 \\ {\scriptstyle (s,t)}\downarrow & & \downarrow{\scriptstyle (s,t)} \\ \mathcal{G}_0 \times \mathcal{G}_0 & \xrightarrow[(F_0,F_0)]{} & \mathcal{H}_0 \times \mathcal{H}_0 \end{array}$$

It follows that an **isomorphism** of Lie groupoids is a functor F as above, where F_0 and F_1 are diffeomorphisms.

For many purposes, isomorphisms of Lie groupoids is too strict an equivalence. For example, often we only care about the orbit spaces of Lie groupoids, and perhaps stabiliser information (definitions below). If two non-isomorphic Lie groupoids yield the same information in this sense, then we wish to have a form of equivalence between these.

Example 3.2 (Non-Isomorphic Lie Groupoids that should be Equivalent). Consider two manifolds: SO(3) and $\mathbb{S}^2 \times \mathbb{S}^1$. Note that SO(3) acts transitively on $\mathbb{S}^2$ by rotations, where at any fixed point $x \in \mathbb{S}^2$ the stabiliser H is isomorphic to $\mathbb{S}^1$. Thus, it follows from the Orbit-Stabiliser Theorem that $\mathbb{S}^2 \cong \mathrm{SO}(3)/\mathbb{S}^1$, where $\mathbb{S}^1$ acts on SO(3) via multiplication, identifying $\mathbb{S}^1$ with H.

On the other hand, $\mathbb{S}^1$ acts on $\mathbb{S}^2 \times \mathbb{S}^1$ by multiplication on $\mathbb{S}^1 \cong \{x\} \times \mathbb{S}^1$ for each $x \in \mathbb{S}^2$. Both actions yield the same orbit space: $\mathbb{S}^2$. Both actions of $\mathbb{S}^1$ are free, and so have trivial stabilisers at every point. And yet, SO(3) is not diffeomorphic to $\mathbb{S}^2 \times \mathbb{S}^1$ (think about what the stabilizers of the SO(3)-action on $\mathbb{S}^2$ are doing – Exercise!).

It follows that the action groupoids $(\mathbb{S}^1 \times \mathrm{SO}(3)) \rightrightarrows \mathrm{SO}(3)$ and $(\mathbb{S}^1 \times \mathbb{S}^2 \times \mathbb{S}^1) \rightrightarrows (\mathbb{S}^2 \times \mathbb{S}^1)$ are not isomorphic Lie groupoids, even though they yield the same orbit spaces and stabilisers everywhere.

Definition 3.3 (Stabilisers, Orbits, and Orbit Spaces). Let $\mathcal{G}$ be a Lie groupoid. Fix $x \in \mathcal{G}_0$.

(1) The **stabiliser** $\mathrm{Stab}(x)$, or $\mathcal{G}_x^x$, is defined to be the set $s^{-1}(x) \cap t^{-1}(x)$. Note that this forms a group.
(2) The set $t(s^{-1}(x))$ is the **orbit** of x. The partition of $\mathcal{G}_0$ into orbits induces an equivalence relation on $\mathcal{G}_0$.
(3) The set of orbits equipped with the quotient topology, denoted $\mathcal{G}_0/\mathcal{G}_1$, is the **orbit space** of $\mathcal{G}$.

Definition 3.4 (Morita Equivalence). Let $F\colon \mathcal{G} \to \mathcal{H}$ be a morphism of Lie groupoids. Then F is a **weak equivalence** if

(1) the map $\mathcal{G}_1 \to (\mathcal{G}_0 \times \mathcal{G}_0)_{(F_0,F_0)}\times_{(s,t)}\mathcal{H}_1$ sending g to $(s(g), t(g), F_1(g))$ is a diffeomorphism, and
(2) the map $\mathcal{G}_0{}_{F_0}\times_t\mathcal{H}_1 \to \mathcal{H}_0$ sending (x, h) to $s(h)$ is a surjective submersion.

We say that two Lie groupoids $\mathcal{G}$ and $\mathcal{H}$ are **Morita equivalent** if there exists another Lie groupoid $\mathcal{U}$ and weak equivalences $F\colon \mathcal{U} \to \mathcal{G}$ and $F'\colon \mathcal{U} \to \mathcal{H}$.

Condition 2 in the definition of a weak equivalence implies that for any $y \in \mathcal{H}_0$, there is some $x \in \mathcal{G}_0$ such that $F(x)$ and y are in the same orbit. Condition 1 implies that $\mathrm{Hom}_{\mathcal{G}}(x_1, x_2) \cong \mathrm{Hom}_{\mathcal{H}}(F(x_1), F(x_2))$. For those who are familiar with equivalences of categories, this yields exactly such an equivalence: the two groupoids (as categories) are "the same up to isomorphisms". Moreover, this equivalence respects the smooth structures on the objects and arrows; it is not just a set-theoretic equivalence.

Exercise 3 (Morita Equivalence, Stabilisers, and Orbit Spaces). Let $\mathcal{G}$ and $\mathcal{H}$ be Morita equivalence Lie groupoids, and let $\pi_{\mathcal{G}}$ and $\pi_{\mathcal{H}}$ be the quotient maps onto their orbit spaces. Show that there is a homeomorphism $\Psi\colon \mathcal{G}_0/\mathcal{G}_1 \to \mathcal{H}_0/\mathcal{H}_1$, and that for any $x \in \mathcal{G}_0/\mathcal{G}_1$, the stabiliser $\mathrm{Stab}(y)$ is isomorphic as a group to $\mathrm{Stab}(z)$ for any $y \in \pi_{\mathcal{G}}^{-1}(x)$ and $z \in \pi_{\mathcal{H}}^{-1}(\Psi(x))$.

Remark 3.5. If the orbit spaces happen to be smooth manifolds, then the homeomorphism in the exercise above can be promoted to a diffeomorphism. In fact, even if the orbit spaces are not manifolds, Morita equivalence still induces a diffeomorphism in either the diffeological or the Sikorski sense between the orbit spaces. See [W13], [W15].

References

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