Final Exam Review

Math 3430 - Section 002

Instructions. Be sure to show your work and explain your reasoning.

1. Solve the following IVPs. For parts (a) and (b), answer the following: What is the domain of the solution? What is the behaviour of the solution as t approaches the endpoints of this domain?
   (a) $ty' + y = \cos(t), \quad y(1) = 2$
   (b) $y' = (y^2 + 1)t^2, \quad y(0) = 1$
   (c) $2xy + (1 + x^2 + 3y^2)\dfrac{dy}{dx} = 0, \quad y(0) = 1$

2. Compute the first two Picard iterates for the IVP $y' = t^2 + y^2$ with $y(0) = 1$.

3. Show that a (unique) solution of $y' = t + y^2, y(0) = 0$ exists for $t \in \left[0, \left(\frac{1}{2}\right)^{2/3}\right]$.

4. Solve the following IVP using the method listed below.

$$\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 4y = te^{2t} \quad y(0) = y'(0) = 1$$

   (a) Variation of parameters
   (b) Laplace transforms

5. Find the general solution of $t^2y'' + ty' - y = 0$.

6. Find the following.
   (a) $\mathcal{L}\left\{t^3\right\}$
   (b) $\mathcal{L}\left\{t\sin(t)\right\}$
   (c) $\mathcal{L}\left\{tH_2(t)\right\}$
   (d) $\mathcal{L}\left\{\delta(t-3)\right\}$
   (e) $\mathcal{L}^{-1}\left\{\dfrac{-1}{(s-3)^2}\right\}$
   (f) $\mathcal{L}^{-1}\left\{\dfrac{2}{(s-3)^2+4}\right\}$
   (g) $\mathcal{L}^{-1}\left\{\dfrac{s}{(s^2+1)^2}\right\}$
   (h) $\mathcal{L}^{-1}\left\{\dfrac{e^{-s}}{s-5}\right\}$

7. Find the general solutions to the following systems.

(a) $\dot{X} = \begin{bmatrix} -7 & 2 \\ 4 & -5 \end{bmatrix} X$

(b) $\dot{X} = \begin{bmatrix} -1 & -2 \\ 2 & 0 \end{bmatrix} X$

(c) $\dot{X} = \begin{bmatrix} -4 & 2 \\ -2 & 0 \end{bmatrix} X$

8. Find the solution to the following IVP (do not solve the integral).

$$\dot{X} = \begin{bmatrix} -7 & 2 \\ 4 & -5 \end{bmatrix} X + \begin{bmatrix} te^t \\ t \end{bmatrix}, \quad X(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

9. Determine the stability (if possible) of the following systems at the specified solutions.

(a) $\dot{X} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix} X$, all solutions.

(b) $\dot{X} = \begin{bmatrix} 20x_1 - 2x_1^2 - 4x_1x_2 \\ 40x_2 - 4x_2^2 - 80x_1x_2 \end{bmatrix}$, all equilibrium solutions.

Formulas

$$
\begin{aligned}
\sin^2(x) + \cos^2(x) &= 1 \\
\sin(a \pm b) &= \sin(a)\cos(b) \pm \sin(b)\cos(a) \\
\cos(a \pm b) &= \cos(a)\cos(b) \mp \sin(a)\sin(b) \\
\sin(2x) &= 2\sin(x)\cos(x) \\
\cos(2x) &= \cos^2(x) - \sin^2(x) \\
&= 1 - 2\sin^2(x) \\
&= 2\cos^2(x) - 1 \\
\int u\,dv &= uv - \int v\,du
\end{aligned}
$$

Table of Laplace Transforms

$f(t)$	$F(s)$	
$t^n \ (n \geq 0)$	$\frac{n!}{s^{n+1}}$	$s > 0$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$	$s > 0$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$	$s > 0$
e^{kt}	$\frac{1}{s-k}$	$s > k$
$f'(t)$	$sF(s) - f(0)$	
$f''(t)$	$s^2F(s) - sf(0) - f'(0)$	
$(-1)^n t^n f(t)$	$F^{(n)}(s)$	
$e^{kt} f(t)$	$F(s-k)$	
$H_k(t) f(t-k)$	$e^{-sk}F(s)$	
$\delta(t - t_0)$	$\begin{cases} e^{-st_0} & t_0 \geq 0 \\ 0 & t_0 < 0 \end{cases}$	
$(f * g)(t)$	$F(s)G(s)$	