MATH 2430
TAKE-HOME
QUIZ

#1. Find a series solution to $\frac{d^2y}{dt^2} + t^2 y = 0$; $y(0)=2$, $y'(0)=-1$

(#6 of §2.8) What is its interval of convergence?

SOL'N

This differential equation has no singular points.

$$y(t) = \sum_{n=0}^{\infty} a_n t^n, \quad y'(t) = \sum_{n=1}^{\infty} a_n n t^{n-1}, \quad y''(t) = \sum_{n=2}^{\infty} a_n n(n-1) t^{n-2}$$

Plugging this into the differential equation:

$$0 = \sum_{n=2}^{\infty} a_n n(n-1) t^{n-2} + t^2 \sum_{n=0}^{\infty} a_n t^n$$

$$= \sum_{n=2}^{\infty} a_n n(n-1) t^{n-2} + \sum_{n=0}^{\infty} a_n t^{n+2}$$

$$= \sum_{\substack{\cancel{n+2=2} \\ n=0}}^{\infty} a_{n+2}(n+2)(n+1) t^n + \sum_{\substack{\cancel{n-2=0} \\ n=2}}^{\infty} a_{n-2} t^n$$

$$= 2\cdot 1\cdot a_2 + 3\cdot 2\, a_3 t + \sum_{n=2}^{\infty} \left[a_{n+2}(n+2)(n+1) + a_{n-2} \right] t^n$$

$$\Rightarrow \quad a_2 = a_3 = 0 \quad \text{and} \quad \boxed{a_{n+2} = \frac{-a_{n-2}}{(n+2)(n+1)}}$$

$a_4 = \frac{-a_0}{4\cdot 3}$	$a_7 = \frac{-a_3}{7\cdot 6} = 0$	$a_{10} = \frac{-a_6}{10\cdot 9} = 0$
$a_5 = \frac{-a_1}{5\cdot 4}$	$a_8 = \frac{-a_4}{8\cdot 7} = \frac{a_0}{8\cdot 4\cdot 7\cdot 3}$	$a_{11} = \frac{-a_7}{11\cdot 10} = 0$
$a_6 = \frac{-a_2}{6\cdot 5} = 0$	$a_9 = \frac{-a_5}{9\cdot 8} = \frac{a_1}{9\cdot 5\cdot 8\cdot 4}$	$a_{12} = \frac{-a_8}{12\cdot 11} = \frac{-a_0}{12\cdot 8\cdot 4\cdot 11\cdot 7\cdot 3}$

Get: $y_1(t) = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{t^{4n}}{4^n n!(3\cdot 7\cdot 11\cdot \ldots \cdot (4n-1))}$

$y_2(t) = t + \sum_{n=1}^{\infty} (-1)^n \frac{t^{4n+1}}{4^n n!(5\cdot 9\cdot 13\cdot \ldots \cdot (4n+1))}$

The general solution is:

$y(t) = a_0 y_1(t) + a_1 y_2(t)$ where y_1 and y_2 are as above.

To find the interval of convergence, we use the ratio test: $\underline{y_1(t)}$:

$$\lim_{n\to\infty} \frac{\left|\frac{(-1)^{n+1} t^{4n+4}}{4^{n+1}(n+1)!(3\cdot 7\cdot 11\cdot \ldots \cdot (4n+3))}\right|}{\left|\frac{(-1)^n t^{4n}}{4^n n!(3\cdot 7\cdot 11\cdot \ldots \cdot 4n-1)}\right|}$$

$$= \lim_{n\to\infty} |t^4| \left|\frac{1}{4(n+1)(4n+3)}\right| = 0$$

So, the interval of convergence of y_1 is $(-\infty, \infty)$.

$\underline{y_2(t)}$:

$$\lim_{n\to\infty} \frac{\left|\frac{(-1)^{n+1} t^{4n+5}}{4^{n+1}(n+1)!(5\cdot 9\cdot 13\cdot \ldots \cdot (4n+5))}\right|}{\left|\frac{(-1)^n t^{4n+1}}{4^n n!(5\cdot 9\cdot 13\cdot \ldots \cdot (4n+1))}\right|}$$

$$= \lim_{n\to\infty} |t^4| \left|\frac{1}{4(n+1)(4n+5)}\right| = 0$$

So, the interval of convergence of y_2 is $\boxed{(-\infty, \infty)}$.

Conclusion: the interval of convergence is $(-\infty, \infty)$ for $y(t)$. Next, since $y(0) = a_0$ and $y'(0) = a_1$, we have $\boxed{y(t) = 2y_1(t) - y_2(t)}$ as the specific solution to the IVP.

#2. Find the general solution to

$$2t\frac{d^2y}{dt^2} + (1-2t)\frac{dy}{dt} - y = 0, \quad (t > 0)$$

using series. (#8 of §§2.8.2).

Sol'n

Divide the differential equation by $2t$:

$$\frac{d^2y}{dt^2} + \frac{(1-2t)}{2t}\frac{dy}{dt} - \frac{1}{2t}y = 0$$

There is a singularity at $t = 0$. Is it regular?

$$t\left(\frac{1-2t}{2t}\right) = \frac{1-2t}{2} \quad \text{(analytic)}$$

$$t^2\left(\frac{1}{2t}\right) = \frac{1}{2}t \quad \text{(analytic)}$$

Conclude: the singular point is regular, and so we can use the method of Frobenius.

④

$$y(t) = t^r \sum_{n=0}^{\infty} a_n t^n \quad (a_0 \neq 0)$$

$$y'(t) = \sum_{n=0}^{\infty} a_n (n+r) t^{n+r-1}$$

$$y''(t) = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) t^{n+r-2}$$

Plugging these into the differential equation:

$$0 = 2t \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) t^{n+r-2} + \sum_{n=0}^{\infty} a_n (n+r) t^{n+r-1}$$

$$-2t \sum_{n=0}^{\infty} a_n (n+r) t^{n+r-1} - \sum_{n=0}^{\infty} a_n t^{n+r}$$

$$= 2\sum_{n=0}^{\infty} a_n (n+r)(n+r-1) t^{n+r-1} + \sum_{n=0}^{\infty} a_n (n+r) t^{n+r-1}$$

$$-2\sum_{n=0}^{\infty} a_n (n+r) t^{n+r} - \sum_{n=0}^{\infty} a_n t^{n+r}$$

$$= 2\sum_{\substack{\cancel{n+1=0} \\ n=-1}}^{\infty} a_{n+1}(n+r+1)(n+r) t^{n+r} + \sum_{\substack{\cancel{n+1=0} \\ n=-1}}^{\infty} a_{n+1}(n+1+r) t^{n+r}$$

$$-2\sum_{n=0}^{\infty} a_n (n+r) t^{n+r} - \sum_{n=0}^{\infty} a_n t^{n+r}$$

$$= [2r(r-1) + r] a_0 t^{r-1}$$

$$+ \sum_{n=0}^{\infty} \left[2 a_{n+1} (n+r+1)(n+r) + a_{n+1}(n+1+r) - 2a_n(n+r) - a_n\right] t^{n+r}$$

$$\Rightarrow 2r^2 - r = 0 \qquad \text{and} \qquad 2a_{n+1}(n+r+1)\left(n+r+\tfrac{1}{2}\right) - 2a_n\left(n+r+\tfrac{1}{2}\right) = 0$$

$$\boxed{\Rightarrow r = 0, \tfrac{1}{2} \qquad \Rightarrow \boxed{a_{n+1} = \frac{a_n}{n+r+1} \quad \text{provided } n+r+\tfrac{1}{2} \neq 0}}$$

CASE $r=0$: $a_{n+1} = \frac{a_n}{n+1}$

$a_1 = \frac{a_0}{1}$

$a_2 = \frac{a_1}{2} = \frac{a_0}{2\cdot 1}$

$$\boxed{y_1(t) = 1 + \sum_{n=1}^{\infty} \frac{t^n}{n!}}$$

$a_3 = \frac{a_2}{3} = \frac{a_0}{3\cdot 2\cdot 1}$

$a_n = \frac{a_0}{n!}$

To find the radius of convergence, we apply the root test: $\lim_{n\to\infty} \frac{\left|\frac{t^{n+1}}{(n+1)!}\right|}{\left|\frac{t^n}{n!}\right|} = \lim_{n\to\infty} \left|\frac{1}{n+1}\right| |t| = 0$

So, the interval of convergence is $(-\infty, \infty)$.

CASE $r=\frac{1}{2}$: $a_{n+1} = \frac{a_n}{n+\frac{3}{2}}$

$a_1 = \frac{a_0}{3/2}$

$a_2 = \frac{a_1}{5/2} = \frac{a_0}{5/2 \cdot 3/2}$

$a_3 = \frac{a_2}{7/2} = \frac{a_0}{\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}}$

$a_n = \frac{2^n a_0}{3\cdot 5\cdot 7\cdot \ldots \cdot (2n+1)}$

⑥

$$\boxed{y_2(t) = t^{\frac{1}{2}} + \sum_{n=1}^{\infty} \frac{2^n t^{n+\frac{1}{2}}}{3\cdot5\cdot7\cdot\ldots\cdot(2n+1)}}$$

Applying the ratio test again:

$$\lim_{n\to\infty} \frac{\left|\dfrac{2^{n+1} t^{n+3/2}}{3\cdot5\cdot7\cdot\ldots\cdot(2n+3)}\right|}{\left|\dfrac{2^n t^{n+1/2}}{3\cdot5\cdot7\cdot\ldots\cdot(2n+1)}\right|} = \lim_{n\to\infty}\left|\frac{2t}{2n+3}\right| = 0$$

The interval of convergence is $(-\infty,\infty)$. The general solution is $y(t) = c_1 y_1(t) + c_2 y_2(t)$, where $y_1(t)$ and $y_2(t)$ are as above. The interval of convergence is $\boxed{(-\infty,\infty)}$ for the general solution. Of course, since $t>0$ to begin with, we could take $\boxed{(0,\infty)}$ as the interval of convergence.

#3. An object with mass 1 unit is attached to a spring with spring constant 64 units. Assume the system has no friction / resistance / drag. If the system is in equilibrium at $t=0$, and at rest at that point, and a force $F(t) = \frac{1}{2}t$ is applied until $\frac{7}{16}\pi$ units of time after which it is removed, then what is the position $y(t)$ of the object with respect to time (with the y-axis pointed downward, with $y=0$ the equilibrium position). (#10 of §2.6)

Sol'n: $m=1$, $k=64$, $c=0$, $F(t) = \left(H_0(t) - H_{\frac{7\pi}{16}}(t)\right)\frac{1}{2}t$

$$= \frac{1}{2}\left[H_0(t)t - H_{\frac{7\pi}{16}}(t)\left(t-\frac{7\pi}{16}\right) - H_{\frac{7\pi}{16}}(t)\frac{7\pi}{16}\right]$$

Have the differential equation:

$$\boxed{\frac{d^2y}{dt^2} + 64y = \frac{1}{2}\left[H_0(t)t - H_{\frac{7\pi}{16}}(t)\left(t-\frac{7\pi}{16}\right) - H_{\frac{7\pi}{16}}(t)\frac{7\pi}{16}\right]}$$

This is a non-homogeneous SOLODE with constant coefficients.

We will use the Laplace transform to find the general solution.

$$\mathcal{L}(LHS) = s^2Y(s) - sy(0) - y'(0) + 64Y(s),$$ where

$$Y(s) = \mathcal{L}(y(t)).$$

$$\mathcal{L}(RHS) = \frac{1}{2}\left[\mathcal{L}(H_0(t)t) - \mathcal{L}\left(H_{\frac{7\pi}{16}}(t)\left(t - \frac{7\pi}{16}\right)\right) - \frac{7\pi}{16}\mathcal{L}\left(H_{\frac{7\pi}{16}}(t)\right)\right]$$

$$= \frac{1}{2}\left[\frac{1}{s^2} - e^{-\frac{7\pi}{16}s}\cdot\frac{1}{s^2} - \frac{7\pi}{16}e^{-\frac{7\pi}{16}s}\frac{1}{s}\right]$$

Now, $y(0) = y'(0) = 0$, and so we have:

$$(s^2+64)Y(s) = \frac{1}{2}\left(1 - e^{-\frac{7\pi}{16}s} - \frac{7\pi}{16}e^{-\frac{7\pi}{16}s}s\right)\frac{1}{s^2}$$

$$\boxed{\Rightarrow Y(s) = \frac{1}{2}\left[\underbrace{\frac{1}{(s^2+64)s^2}}_{1} - e^{-\frac{7\pi}{16}s}\underbrace{\frac{\left(1+\frac{7\pi}{16}s\right)}{(s^2+64)s^2}}_{2}\right]}$$

2:
$$\frac{1+\frac{7\pi}{16}s}{(s^2+64)s^2} = \frac{As+B}{s^2+64} + \frac{Cs+D}{s^2}$$

$$\Rightarrow 1 + \frac{7\pi}{16}s = As^3 + Bs^2 + Cs^3 + 64Cs + Ds^2 + 64D$$

$$= (A+C)s^3 + (B+D)s^2 + 64Cs + 64D$$

$$\Rightarrow \begin{cases} A + C = 0 & \Rightarrow A = \frac{7\pi}{64\cdot 16} \\ B + D = 0 & \Rightarrow B = \frac{-1}{64} \\ 64C = +\frac{7\pi}{16} & \Rightarrow C = \frac{+7\pi}{64\cdot 16} \\ 64D = 1 & \Rightarrow D = \frac{1}{64} \end{cases}$$

$$\boxed{\text{Get: } \frac{1+\frac{7\pi}{16}s}{(s^2+64)s^2} = \frac{\frac{7\pi}{64\cdot 16}s - \frac{1}{64}}{s^2+64} + \frac{\frac{1}{64} + \frac{7\pi}{64\cdot 16}s}{s^2}}$$

1 $$\frac{1}{(s^2+64)s^2} = \frac{As+B}{s^2+64} + \frac{Cs+D}{s^2}$$

$$\Rightarrow \quad 1 = (A+C)s^3 + (B+D)s^2 + 64Cs + 64D$$

$$\Rightarrow \begin{cases} A+C=0 \ \Rightarrow A=0 \\ B+D=0 \ \Rightarrow B=-\frac{1}{64} \\ 64C=0 \ \Rightarrow C=0 \\ 64D=1 \ \Rightarrow D=\frac{1}{64} \end{cases}$$

$$\boxed{\text{Get: } \frac{1}{(s^2+64)s^2} = \frac{-\frac{1}{64}}{s^2+64} + \frac{\frac{1}{64}}{s^2}}$$

$$\text{So, } Y(s) = -\frac{e^{-\frac{7\pi}{16}s}}{2}\left(\frac{7\pi}{64\cdot 16}\frac{s}{s^2+64} - \frac{1}{64\cdot 8}\frac{8}{s^2+64} + \frac{1}{64s^2} + \frac{7\pi}{64\cdot 16}\frac{1}{s}\right) + \frac{1}{2}\left(\frac{1}{64\cdot 8}\frac{8}{s^2+64} - \frac{1}{64}\frac{1}{s^2}\right)$$

$$\boxed{\Rightarrow y(t) = \frac{1}{2}H_{\frac{7\pi}{16}}(t)\left(\frac{-7\pi}{64\cdot 16}\cos\left(8\left(t-\frac{7\pi}{16}\right)\right) + \frac{1}{64\cdot 8}\sin\left(8\left(t-\frac{7\pi}{16}\right)\right) - \frac{1}{64}\left(t-\frac{7\pi}{16}\right) - \frac{7\pi}{64\cdot 16}\right) - \frac{1}{64\cdot 8}\frac{\sin(8t)}{2} + \frac{t}{128}}$$