Quiz #4

Math 3430 - Section 002

Instructions. Be sure to show your work and explain your reasoning for full credit.

NAME Solutions

There are 2 problems on this quiz. Please turn the paper over to see the second question. There is a table of Laplace transforms on the back.

1. Solve the following system of differential equations.

$$\dot{X} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} X$$

Eigenvalues:

$$0 = \det \begin{bmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{bmatrix} = (1-\lambda)^2 \Rightarrow \lambda_1 = 1$$

Eigenvectors:

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow x_2 = 0,\ x_1 \text{ arbitrary, set } x_1 = 1.$$

$$X_1(t) = e^t \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Generalised Eigenvectors (rank 2):

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow x_1, x_2 \text{ arbitrary, set } x_1 = 0,\ x_2 = 1$$

$$X_2(t) = e^t\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} + t\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = e^t\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} t \\ 0 \end{bmatrix}\right)$$

$$X(t) = c_1 e^t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^t \begin{bmatrix} t \\ 1 \end{bmatrix}.$$

2. Use your solution to the first problem to solve the following IVP.

$$\dot{X} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} X + \begin{bmatrix} e^t \\ e^t \end{bmatrix}, \quad X(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Homog. Solution:

$$X_H(t) = c_1 e^t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^t \begin{bmatrix} t \\ 1 \end{bmatrix}$$

$$\mathcal{X}(t) = \begin{bmatrix} e^t & te^t \\ 0 & e^t \end{bmatrix}, \quad \mathcal{X}(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathcal{X}^{-1}(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

$$e^{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} t} = \mathcal{X}(t)\mathcal{X}^{-1}(0) = \begin{bmatrix} e^t & te^t \\ 0 & e^t \end{bmatrix}.$$

Particular solution:

$$X(t) = e^{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \int_0^t e^{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}(t-s)} \begin{bmatrix} e^s \\ e^s \end{bmatrix} ds$$

$$= \begin{bmatrix} e^t + te^t \\ e^t \end{bmatrix} + \begin{bmatrix} e^t & te^t \\ 0 & e^t \end{bmatrix} \int_0^t \begin{bmatrix} e^{-s} & -se^{-s} \\ 0 & e^{-s} \end{bmatrix} \begin{bmatrix} e^s \\ e^s \end{bmatrix} ds$$

sufficient to get to here.

$$= \begin{bmatrix} e^t + te^t \\ e^t \end{bmatrix} + \begin{bmatrix} e^t & te^t \\ 0 & e^t \end{bmatrix} \int_0^t \begin{bmatrix} 1 - s \\ 1 \end{bmatrix} ds$$

$$= \begin{bmatrix} e^t + te^t \\ e^t \end{bmatrix} + \begin{bmatrix} e^t & te^t \\ 0 & e^t \end{bmatrix} \begin{bmatrix} t - \frac{t^2}{2} \\ t \end{bmatrix}$$

$$= \begin{bmatrix} e^t + te^t \\ e^t \end{bmatrix} + \begin{bmatrix} te^t + \frac{1}{2}t^2 e^t \\ te^t \end{bmatrix} = \begin{bmatrix} e^t + 2te^t + \frac{1}{2}t^2 e^t \\ e^t + te^t \end{bmatrix}.$$

Solution: $c_1 e^t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^t \begin{bmatrix} t \\ 1 \end{bmatrix} + \begin{bmatrix} e^t + 2te^t + \frac{1}{2}t^2 e^t \\ e^t + te^t \end{bmatrix}.$