UNIVERSITY OF COLORADO AT BOULDER
MATH 3430 002 - MIDTERM 3

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Name: Solutions

Read all of the following information before starting the exam:

- **Show all work**, clearly and in order, if you want to get full credit. The graders reserve the right to take off points if they cannot see how you arrived at your answer (even if your final answer is correct). Points will also be removed for answers that are not clear.
- **Calculators, notes, phones, and other aids including all electronic devices, are not permitted for this test.**
- This test has 4 problems, 9 pages, and is worth 40 points. It is your responsibility to make sure that you have all of the pages!
- Good luck!

Problem	Grade
1. (/~~5~~)10	
2. (/10)	
3. (/10)	
4. (/~~15~~)10	
Total: (/40)	

Problem 1 (10 points). Solve the following IVP.

$$t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt} + 16y = 0, \quad y(1) = 1,\ y'(1) = 0. \qquad (t>0)$$

Let $y(t) = t^r$. Then,

$0 = t^2(r-1)r\,t^{r-2} + t\,r\,t^{r-1} + 16t^r$

$= (r^2+16)t^r = 0 \Rightarrow r = \pm 4i$

The general solution is $y(t) = c_1\cos(4\ln t) + c_2\sin(4\ln t)$.

$1 = y(1) = c_1$, $\quad y'(t) = -\frac{4}{t}\sin(4\ln t) + \frac{4c_2}{t}\cos(4\ln t)$

$\Rightarrow y'(1) = 0 = 4c_2 \Rightarrow c_2 = 0.$

Get: $y(t) = \cos(4\ln t)$.

Problem 2 (10 points).

/5 (a) Find the Laplace transform of $-t\sin(2t)+\delta(t-3)$, where $\delta(t-t_0)$ is the Dirac delta.

$$\mathcal{L}\{-t\sin(2t)\}+\mathcal{L}\{\delta(t-3)\}$$

$$=\frac{d}{ds}\frac{2}{s^2+4}+e^{-3s}$$

$$=\frac{-4s}{(s^2+4)^2}+e^{-3s}$$

/5 (b) Find the inverse Laplace transform of $\cancel{\frac{1}{(s-1)^2}}$. $\frac{1}{(s^2+1)^2}$

Hint: $\sin a\sin b = \frac{1}{2}(\cos(a-b)-\cos(a+b))$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)}\right\}=\sin t$$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)^2}\right\}=\sin t*\sin t=\int_0^t\sin(t-u)\sin u\,du$$

$$=\frac{1}{2}\int_0^t\cos(t-2u)-\cos(t)\,du$$

$$=\frac{1}{2}\left(\frac{\sin(t-2u)}{-2}-u\cos t\right)\Bigg|_0^t$$

$$=\frac{1}{2}\left(\frac{\sin(t)}{2}+\frac{\sin(t)}{2}-t\cos t\right)$$

$$=\frac{1}{2}(\sin t-t\cos t)$$

Check: $\mathcal{L}\{\frac{1}{2}(\sin t-t\cos t)\}=\frac{1}{2}\left(\frac{1}{s^2+1}+\frac{d}{ds}\frac{s}{s^2+1}\right)$

$$=\frac{1}{2}\left(\frac{s^2+1}{(s^2+1)^2}+\frac{s^2+1-2s^2}{(s^2+1)^2}\right)$$

$$=\frac{1}{(s^2+1)^2}$$ ✓

Problem 3 (10 points). Use Laplace transforms to solve the following IVP.

$$\frac{d^2y}{dt^2} + 4y = f(t), \quad y(0) = 0,\ y'(0) = 0,$$

where

$$f(t) = \begin{cases} t & \text{if } t \in (0,1), \\ 0 & \text{if } t > 1. \end{cases}$$

$\mathcal{L}\{LHS\} = s^2 Y(s) - s\underbrace{y(0)}_{0} - \underbrace{y'(0)}_{0} + 4Y(s)$

$\mathcal{L}\{RHS\} = \mathcal{L}\{(1-H_1(t))t\} = \mathcal{L}\{t\} - \mathcal{L}\{H_1(t)((t-1)+1)\}$

Since $\mathcal{L}\{t+1\} = \frac{1}{s^2} + \frac{1}{s}$, get

$\mathcal{L}\{H_1(t)((t-1)+1)\} = e^{-s}\left(\frac{1}{s^2} + \frac{1}{s}\right)$

So, $(s^2+4)Y(s) = -e^{-s}\left(\frac{1}{s^2} + \frac{1}{s}\right) + \frac{1}{s^2}$

$\Rightarrow Y(s) = \underbrace{\frac{-e^{-s}}{s^2(s^2+4)}}_{1} - \underbrace{\frac{e^{-s}}{s(s^2+4)}}_{2} + \underbrace{\frac{1}{s^2(s^2+4)}}_{3}$

1 : $\frac{1}{s^2(s^2+4)} = \frac{A}{s^2} + \frac{Bs+C}{s^2+4} \Rightarrow As^2 + 4A + Bs^3 + Cs^2 = 1$

$\Rightarrow \begin{cases} A+C=0 & \Rightarrow C = \frac{-1}{4} \\ B=0 \\ 4A=1 & \Rightarrow A=\frac{1}{4} \end{cases}$

2 : $\frac{1}{s(s^2+4)} = \frac{A}{s} + \frac{Bs+C}{s^2+4} \Rightarrow As^2 + 4A + Bs^2 + Cs = 1$

$\Rightarrow \begin{cases} A+B=0 & \Rightarrow B = \frac{-1}{4} \\ C=0 \\ 4A=1 & \Rightarrow A=\frac{1}{4} \end{cases}$

3 is the same as 1

(More room for #3.)

$$\text{So,}\quad Y(s) = -e^{-s}\left(\frac{1/4}{s^2} + \frac{-1/4}{s^2+4} + \frac{1/4}{s} + \frac{-1/4\, s}{s^2+4}\right) + \left(\frac{1/4}{s^2} - \frac{1/4}{s^2+4}\right)$$

$$\Rightarrow y(t) = \frac{-1}{4}H_1(t)\left((t-1) - \frac{1}{2}\sin(2(t-1)) + 1 - \cos(2(t-1))\right)$$
$$+ \frac{1}{4}\left(t - \frac{1}{2}\sin(2t)\right)$$

Problem 4 (10 points). Solve the following system of ODEs.

$$\dot{X} = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix} X.$$

<u>Eigenvalues</u>:

$$0 = \det\begin{bmatrix} 1-\lambda & 3 \\ 4 & 2-\lambda \end{bmatrix} = (1-\lambda)(2-\lambda) - 12 = \lambda^2 - 3\lambda - 10$$

$$= (\lambda - 5)(\lambda + 2)$$

$$\Rightarrow \lambda = -2, 5$$

<u>Eigenvectors</u>:

<u>$\lambda_1 = -2$</u>:

$$\begin{bmatrix} 3 & 3 \\ 4 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow x_1 + x_2 = 0$$

$$X_1 = e^{-2t}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

<u>$\lambda_2 = 5$</u>

$$\begin{bmatrix} -4 & 3 \\ 4 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow -4x_1 + 3x_2 = 0$$

$$\Rightarrow x_1 = \frac{3}{4}x_2$$

$$X_2 = e^{5t}\begin{bmatrix} 3/4 \\ 1 \end{bmatrix}$$

General solution: $X(t) = c_1 e^{-2t}\begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 e^{5t}\begin{bmatrix} 3/4 \\ 1 \end{bmatrix}$.

(More room for #4.)

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$$\sin^2(x) + \cos^2(x) = 1$$

$$\begin{aligned}
\sin(a \pm b) &= \sin(a)\cos(b) \pm \sin(b)\cos(a) \\
\cos(a \pm b) &= \cos(a)\cos(b) \mp \sin(a)\sin(b) \\
\sin(2x) &= 2\sin(x)\cos(x) \\
\cos(2x) &= \cos^2(x) - \sin^2(x) \\
&= 1 - 2\sin^2(x) \\
&= 2\cos^2(x) - 1
\end{aligned}$$

$$\int u\,dv = uv - \int v\,du$$

Table of Laplace Transforms

$f(t)$	$F(s)$	
$t^n \ (n \geq 0)$	$\frac{n!}{s^{n+1}}$	$s > 0$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$	$s > 0$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$	$s > 0$
e^{kt}	$\frac{1}{s-k}$	$s > k$
$f'(t)$	$sF(s) - f(0)$	
$f''(t)$	$s^2F(s) - sf(0) - f'(0)$	
$(-1)^n t^n f(t)$	$F^{(n)}(s)$	
$e^{kt} f(t)$	$F(s-k)$	
$H_k(t) f(t-k)$	$e^{-sk}F(s)$	
$\delta(t - t_0)$	$\begin{cases} e^{-st_0} & t_0 \geq 0 \\ 0 & t_0 < 0 \end{cases}$	
$(f * g)(t)$	$F(s)G(s)$	