UNIVERSITY OF COLORADO AT BOULDER
MATH 3430 002 - MIDTERM 1

September 18, 2015
Instructor: Jordan Watts

Name: Solutions

Read all of the following information before starting the exam:

- **Show all work**, clearly and in order, if you want to get full credit. The graders reserve the right to take off points if they cannot see how you arrived at your answer (even if your final answer is correct). Points will also be removed for answers that are not clear.
- **Calculators, notes, phones, and other aids including all electronic devices, are not permitted for this test.**
- This test has 4 problems, ~~8~~ 7 pages, and is worth 40 points. It is your responsibility to make sure that you have all of the pages!
- Good luck!

Problem	Grade
1. (/10)	
2. (/10)	
3. (/10)	
4. (/10)	
Total: (/40)	

Problem 1. Find the general solution to the following differential equation.

$$\frac{dy}{dt} = \frac{x}{y\sqrt{1-x^2}}.$$

Separable!

$$\int y\,dy = \int \frac{x}{\sqrt{1-x^2}}\,dx + C$$

$u = 1 - x^2$

$du = -2x\,dx$

$$\Rightarrow \quad \frac{1}{2}y^2 = -\frac{1}{2}\int \frac{du}{\sqrt{u}} + C$$

$$\Rightarrow \quad \frac{1}{2}y^2 = -\frac{1}{2}\cdot 2\sqrt{u} + C = -\sqrt{1-x^2} + C$$

$$\frac{1}{2}y^2 = -\sqrt{1-x^2} + C$$

Problem 2. (a) Solve the following initial value problem.

$$\frac{dy}{dt} + ty = t, \quad y(0) = 2.$$

Non-homog. FOLODE

$$\mu(t) = e^{\int t\,dt} = e^{t^2/2}, \quad \mu(0) = 1$$

$$y(t) = e^{-t^2/2}\left(\int_0^t e^{\tilde{t}^2/2}\tilde{t}\,d\tilde{t} + 2\right)$$

$u = \tilde{t}^2/2$

$du = \tilde{t}\,d\tilde{t}$

$\tilde{t} = 0 \Rightarrow u = 0$

$\tilde{t} = t \Rightarrow u = t^2/2$

$$= e^{-t^2/2}\left(\int_0^{t^2/2} e^u\,du + 2\right)$$

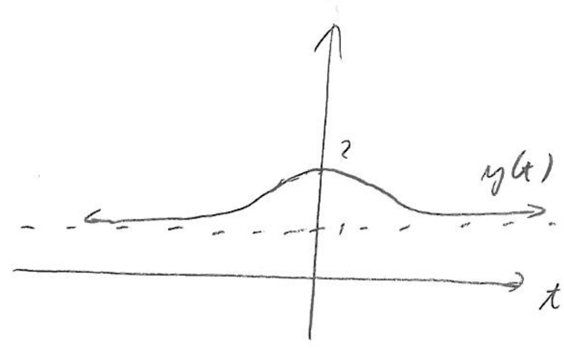
$$= e^{-t^2/2}\left(e^{t^2/2} - 1 + 2\right) = 1 + e^{-t^2/2}$$

$$y(t) = 1 + e^{-t^2/2}$$

(b) What is the domain of the solution, and what is the behaviour of the solution as t approaches the endpoints of the domain?

The domain is all real numbers, $(-\infty, \infty)$.

$\lim_{t \to -\infty} y(t) = 1 = \lim_{t \to \infty} y(t)$: the solution has a horizontal asymptote $y = 1$.



Problem 3.

(a) Is the following differential equation exact? Prove your answer.

$$2ty^4 + \sin(y) + (4t^2y^3 + t\cos(y))\frac{dy}{dt} = 0.$$

$M(y,t) = 2ty^4 + \sin y$

$N(y,t) = 4t^2y^3 + t\cos(y)$

$\frac{\partial M}{\partial y} = 8ty^3 + \cos(y)$, $\frac{\partial N}{\partial t} = 8ty^3 + \cos(y)$.

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t} \Rightarrow$ exact!

(b) Find the general solution of the above differential equation.

$$\phi(y,t) = \int M(y,t)\,dt + h(y)$$
$$= \int (2ty^4 + \sin y)\,dt + h(y)$$
$$= t^2y^4 + t\sin y + h(y)$$

$$N(y,t) = \frac{\partial \phi}{\partial y} = 4t^2y^3 + t\cos y + h'(y)$$
$$= 4t^2y^3 + t\cos y$$

$\Rightarrow h'(y) = 0 \Rightarrow h(y) = C$

So, the general solution is $\phi(y,t) = t^2y^4 + t\sin y = \text{const}$

Problem 4.

(a) Show that the following differential equation is not exact.

$$2y + t\frac{dy}{dt} = 0.$$

$M(y,t) = 2y$, $N(y,t) = t$

$$\frac{\partial M}{\partial y} = 2 \neq \frac{\partial N}{\partial t} = 1$$

Since these partials are not equal, the differential equation is not exact.

(b) Show for a general differential equation $M(y,t) + N(y,t)\frac{dy}{dt} = 0$ that if an integrating factor $\mu(y,t)$ can be chosen so that $\mu(y,t) = \mu(y)$, then $\mu(y) = e^{\int R(y)}$ where $R(y) = \frac{1}{M(y,t)}\left(\frac{\partial N}{\partial t} - \frac{\partial M}{\partial y}\right)$.

Assume $\mu(y,t) = \mu(y)$. Then,

$$\frac{\partial}{\partial y}\left(\mu(y) M(y,t)\right) = \frac{\partial}{\partial t}\left(\mu(y) N(y,t)\right)$$

$$\Rightarrow \frac{\partial \mu}{\partial y} M + \mu \frac{\partial M}{\partial y} = \mu(y)\frac{\partial N}{\partial t}$$

$$\Rightarrow \frac{\partial \mu}{\partial y} = \mu\left(\frac{\partial N/\partial t - \partial M/\partial y}{M}\right).$$

This is a homogeneous FOLODE, so it has a solution $\mu(t) = e^{\int R(y)}$ where $R(y) = \dfrac{\frac{\partial N}{\partial t} - \frac{\partial M}{\partial y}}{M}$

(c) Use the integrating factor derived in part (b) to solve the differential equation in part (a). (Assume $y \geq 0$.)

$$\mu(y) = e^{\int \frac{1-2}{2y} dy} = e^{\int \frac{-1}{2y} dy} = e^{-\frac{1}{2}\ln|y|} = |y|^{-1/2} = y^{-1/2}$$

$\mu M + \mu N \frac{dy}{dt} = 0$ is an exact differential eq'n.

$$\phi(y,t) = \int y^{-1/2} \cdot 2y \, dt + h(y) = 2\sqrt{y}\, t + h(y)$$

$$\mu(y)N(y,t) = \frac{\partial \phi}{\partial y} = \frac{t}{\sqrt{y}} + h'(y) = \frac{t}{\sqrt{y}} \Rightarrow h'(y) = 0 \Rightarrow h(y) = C$$

So, $\phi(y,t) = 2t\sqrt{y} = \text{const}$ is the general sol'n.