1. (25) For each of the following statements, either prove it is true, or provide a counterexample to show that it is false.

(i) If S is a non-empty bounded subset of $\mathbb{R}$ and $c \in \mathbb{R}$, let $c\,S = \{cs : \ s \in S\}$. Then if $c < 0$,

$$\sup c\,S \;=\; c \inf S.$$

(ii) If $f : I \to \mathbb{R}$ is continuous on I, where $I \subset \mathbb{R}$ is an interval, then f attains a maximum value on I.

(iii) If c is a real number in the open interval I, and if $f : I \to \mathbb{R}$ is differentiable at c, then f is continuous at c.

(iv) If $f : [a, b] \to \mathbb{R}$ is bounded on $[a, b]$, then f is Riemann integrable on $[a, b]$.

2. (25)

(i) Let (a_n) be a bounded sequence of real numbers, and suppose that (s_n) is a convergent sequence of real numbers with $\lim s_n = 0$. Prove using the definition of convergence that

$$\lim_{n\to\infty} a_n s_n = 0.$$

(ii) Define a sequence (s_n) by $s_1 = \sqrt{6}$ and $s_{n+1} = \sqrt{6 + s_n}$, $n \geq 1$. By using an appropriate convergence theorem or otherwise, prove that (s_n) is convergent, and calculate $\lim s_n$.

3. (25)

(i) Prove that the equation $x^3 = [\cos(x)]^2$ has a solution in the interval $[0, \frac{\pi}{2}]$.

(ii) Is the function $f(x) = x \sin(\frac{1}{x})$ uniformly continuous on $(0, \frac{1}{\pi})$? Justify your reasoning.

4. (25)

(i) Let f defined on $\mathbb{R}$ by

$$f(x) = \begin{cases} 0 & x \text{ rational}, \\ 3x^2 & x \text{ irrational}. \end{cases}$$

Prove that f is continuous at one point only and differentiable at one point only, and compute the value of the derivative of f at that point.

(ii) State, without proof, the Mean Value Theorem, and use it to show that if $x > 0$,

$$1 + x \; < \; e^x.$$

5. (25)

(i) Suppose $f : [0, 2] \to \mathbb{R}$ and that f is twice differentiable on $[0, 2]$, i.e. suppose f' and f'' both exist on $[0, 2]$, with $f(0) = f(1) = f(2) = 0$. Prove that there exists $c \in (0, 2)$ with $f''(c) = 0$.

(ii) Let $f : [a, b] \to \mathbb{R}$ be bounded and suppose $f(x) \geq 0$ for all $x \in [a, b]$. Prove that $L(f) \geq 0$.

(iii) As in part (ii) above, let $f : [a, b] \to \mathbb{R}$ be bounded and satisfy $f(x) \geq 0$ for all $x \in [a, b]$, and suppose in addition that f is contininous on $[a, b]$. Prove that if $L(f) = 0$, then $f(x) = 0$ for all $x \in [a, b]$.

[Hint: you can use the method of contradiction].

6. (25)

(i) Let $f : [a,b] \to \mathbb{R}$ be a bounded function. Suppose there exist a sequence of partitions $\{\mathcal{P}_n\}_{n=1}^{\infty}$ of $[a,b]$ such that

$$\lim_{n\to\infty} [U(f,\mathcal{P}_n) - L(f,\mathcal{P}_n)] \;=\; 0.$$

Prove that f is Riemann integrable on $[a,b]$, with

$$\lim_{n\to\infty} U(f,\mathcal{P}_n) = \int_a^b f(x)dx.$$

(ii) Define $F : \mathbb{R} \to \mathbb{R}$ by

$$F(x) = \int_x^{x^2} \sqrt{1 + e^{-t^2}}\, dt.$$

Prove by citing the appropriate theorem(s) that F is differentiable on $\mathbb{R}$, and calculate $F'(x)$. Be sure to justify your reasoning at every stage.

Name: ______________________________

University of Colorado

Mathematics 3001: Final Exam

May 9, 2012

Problem	Points	Score
1	25	
2	25	
3	25	
4	25	
5	25	
6	25	
Total	150	