UNIVERSITY OF COLORADO AT BOULDER
MATH 2300 880 - MIDTERM 3

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Instructor: Jordan Watts

Name: Solutions

Read all of the following information before starting the exam:

- **Show all work**, clearly and in order, if you want to get full credit. The graders reserve the right to take off points if they cannot see how you arrived at your answer (even if your final answer is correct). Points will also be removed for answers that are not clear.
- **Calculators, notes, phones, and other aids including all electronic devices, are not permitted for this test.**
- This test has 5 problems, 11 pages, and is worth 75 points. It is your responsibility to make sure that you have all of the pages!
- Good luck!

Problem	Grade
1. (/10)	
2. (/10)	
3. (/25)	
4. (/20)	
5. (/10)	
Total: (/75)	

$\frac{1}{4}$ 10 pts Find the 5^{th} degree Taylor polynomial of $\tan x$ about $x=0$.

Problem 3 (~~25 points~~).

/10 ~~(a) Show that the Taylor series of $\tan(x)$ about $x = 0$ is ??????.~~

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\tan x$	0
1	$\sec^2 x$	1
2	$2\sec^2 x \tan x$	0
3	$4\sec^2 x \tan^2 x + 2\sec^4 x$	2
4	$8\sec^2 x \tan^3 x + 8\sec^4 x \tan x + 8\sec^4 x \tan x$	0
5	$16\sec^2 x \tan^4 x + 24\sec^4 x \tan^2 x + 64\sec^4 x \tan^2 x + 16\sec^6 x$	16

$$P_n(x) = x + \frac{x^3}{3} + \frac{2}{15}x^5$$

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Problem 3 (25 points).

Hint: $\ln \frac{a}{b} = \ln a - \ln b$

/10 (a) Show that the Taylor series of $\ln\left(\frac{1+x}{1-x}\right)$ about $x = 0$ is $2\sum_{k=0}^{\infty} \frac{x^{2k+1}}{2k+1}$.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\ln\left(\frac{1+x}{1-x}\right)$	0
1	$\frac{1}{1+x} + \frac{1}{1-x}$	$2 = 2\cdot 0!$
2	$\frac{-1}{(1+x)^2} - \frac{-1}{(1-x)^2}$	0
3	$\frac{(-1)(-2)}{(1+x)^3} + \frac{(-1)(-2)}{(1-x)^3}$	$4 = 2\cdot 2!$
4	$\frac{(-1)(-2)(-3)}{(1+x)^4} - \frac{(-1)(-2)(-3)}{(1-x)^4}$	0
5	$\frac{(-1)(-2)(-3)(-4)}{(1+x)^5} + \frac{(-1)(-2)(-3)(-4)}{(1-x)^5}$	$2\cdot 4!$
$\vdots$		
k	$\frac{(-1)^{k-1}(k-1)!}{(1+x)^k} + (-1)^{k-1}\frac{(-1)^{k-1}(k-1)!}{(1-x)^k}$	$\begin{cases} 0 & k \text{ even} \\ 2(k-1)! & k \text{ odd} \end{cases}$
$2k+1$	———	$2(2k)!$

$$\sum_{k=0}^{\infty} \frac{2\cancel{(2k)!}}{\cancel{(2k+1)!}} x^{2k+1} = 2\sum_{k=0}^{\infty} \frac{x^{2k+1}}{2k+1}$$

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/5 (b) Find the radius of convergence of the Taylor series you found in part (a).

Ratio test:
$$\lim_{k\to\infty} \left| \frac{\frac{x^{2k+3}}{2k+3}}{\frac{x^{2k+1}}{2k+1}} \right| = \lim_{k\to\infty} \frac{x^2(2k+1)}{2k+3} = x^2 < 1$$

$\Rightarrow$ $-1 < x < 1$, and so the radius of convergence is 1.

/5 (c) Find the interval of convergence (including endpoints) of the Taylor series you found in part (a).

$\underline{x=-1}$: $$2\sum_{k=0}^{\infty} \frac{(-1)^{2k+1}}{2k+1} = -\sum_{k=0}^{\infty} \frac{1}{k+\frac{1}{2}}.$$

$\underline{x=1}$: $$2\sum_{k=0}^{\infty} \frac{(+1)^{2k+1}}{2k+1} = \sum_{k=0}^{\infty} \frac{1}{k+\frac{1}{2}}.$$

$\frac{1}{k+\frac{1}{2}} < \frac{1}{k}$, and so by the basic comparison test, these series diverge.

Interval of convergence: $(-1, 1)$

Is $\ln\left(\frac{1+x}{1-x}\right)$ equal to its Taylor series about $x=0$ on $\left(\frac{-1}{2}, \frac{1}{2}\right)$?

/5 (d) ~~Is tan(x) analytic at x = 0 (i.e. equal to its Taylor series about x = 0 on the interval of convergence)?~~ Justify your answer and show your work.

$$|R_k(x)| \leq \left(\frac{2 \cdot k!}{\left(\frac{1}{2}\right)^{k+1}}\right) \frac{|x|^{k+1}}{(k+1)!} = \frac{2\,(2|x|)^{k+1}}{k+1}$$

Since $x \in \left(\frac{-1}{2}, \frac{1}{2}\right)$, $2|x| < 1$, and so as $k \to \infty$, $(2|x|)^{k+1} \to 0$. So, $R_k(x) \to 0$ as $k \to \infty$,

and we conclude that $\ln\left(\frac{1+x}{1-x}\right)$ is equal to its Taylor series about $x=0$ on $\left(\frac{-1}{2}, \frac{1}{2}\right)$

Problem ~~4~~ 3 (20 points). Find power series representations about $x = 0$ for each of the following functions (the formulas at the back of the test may be helpful).

/5 (a) $\dfrac{x}{1+x^2}$ $\rightsquigarrow$ $x \displaystyle\sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n+1}$

/5 (b) $x\sin(x^2)$ $\rightsquigarrow$ $x \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+3}}{(2n+1)!}$

/5 (c) xe^{x^2} $\rightsquigarrow$ $x \sum_{n=0}^{\infty} \frac{(x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{n!}$

/5 (d) $\cosh(x)$ $= \frac{1}{2}\left(e^{x} + e^{-x}\right) \rightsquigarrow \frac{1}{2}\sum_{n=0}^{\infty} \frac{x^n + (-x)^n}{n!}$

$$= \sum_{n=0}^{\infty} \frac{(1+(-1)^n)x^n}{2n!}$$

$$= \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$$

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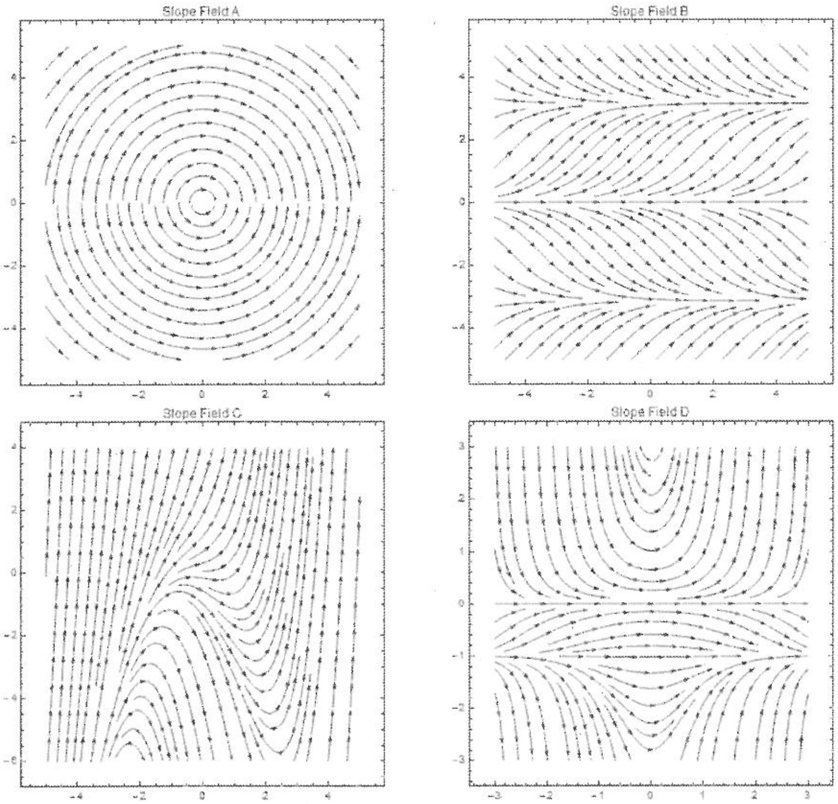
Problem 5 (10 points). Match the following differential equations with their slope fields by filling out the table below; put the letter corresponding to the correct slope field beside the number corresponding to the differential equation in the bottom-right table.



1) $y' = \sin(y)$	2) $y' = x(y^2 + y)$
3) $y' = x^2 + y$	4) $y' = -\frac{x}{y}$

1) B	2) D
3) C	4) A

more to last

Problem 5 (10 points). Find what $\sum_{n=0}^{\infty} \frac{n}{5^n}$ is equal to. Show your work!

- $\frac{n}{5^n} = n\left(\frac{1}{5}\right)^n$ Consider $\sum_{n=0}^{\infty} n\left(\frac{x}{5}\right)^n$.

- This is equal to $\frac{x}{5}\sum_{n=1}^{\infty} n\left(\frac{x}{5}\right)^{n-1}$. Integrating $\sum_{n=1}^{\infty} n\left(\frac{x}{5}\right)^{n-1}$, we get $C+\sum_{n=1}^{\infty} \cancel{n}\left(\frac{x}{5}\right)^n \cdot \frac{1}{\cancel{n}} \cdot 5 = 5\sum_{n=1}^{\infty}\left(\frac{x}{5}\right)^n + C$.

- Choose C to be whatever you want; how about $C=5$ so we get $5\sum_{n=0}^{\infty}\left(\frac{x}{5}\right)^n$ as our antiderivative.

- This is equal to the geometric series where $a=5$, $r=\frac{x}{5}$. So, we have that this antiderivative is equal to $\frac{5}{1-\frac{x}{5}}$ $= \frac{25}{5-x}$.

- Differentiating, we get $\sum_{n=1}^{\infty} n\left(\frac{x}{5}\right)^{n-1} = \frac{25}{(5-x)^2}$.

- Finally, $\sum_{n=0}^{\infty} n\left(\frac{x}{5}\right)^n = \frac{x}{5}\sum_{n=1}^{\infty} n\left(\frac{x}{5}\right)^{n-1} = \frac{5x}{(5-x)^2}$.

Plug in 1 for x: $\frac{5}{4^2} = \frac{5}{16}$