MATH 2300
FINAL REVIEW

Final Exam Review

2015/12/09

#1. (a) $\int \frac{\ln x}{x}\,dx = I$

SOL'N: Let $u = \ln x$

$du = \frac{1}{x}\,dx$

So, $I = \int u\,du = \frac{1}{2}u^2 + C = \frac{1}{2}(\ln x)^2 + C$

(b) $\int_0^1 e^x \cos(2x)\,dx = I$

SOL'N: $u_1 = e^x \quad dv_1 = \cos(2x)\,dx$

$du_1 = e^x\,dx \quad v_1 = \frac{1}{2}\sin(2x)$

$u_2 = e^x \quad dv_2 = \sin(2x)\,dx$

$du_2 = e^x\,dx \quad v_2 = \frac{-1}{2}\cos(2x)$

$I = \frac{1}{2}e^x \sin(2x)\Big|_0^1 - \frac{1}{2}\int_0^1 e^x \sin(2x)\,dx$

$= \frac{1}{2}e\sin 2 - 0 - \frac{1}{2}\left(\frac{-1}{2}e^x\cos(2x)\Big|_0^1 + \frac{1}{2}\underbrace{\int_0^1 \cos(2x)e^x\,dx}_{I}\right)$

$= \frac{1}{2}e\sin 2 + \frac{1}{4}e\cos 2 - \frac{1}{4} - \frac{1}{4}I$

$\Rightarrow I = \frac{2}{5}\left(e\sin 2 + \frac{1}{2}e\cos 2 - \frac{1}{2}\right)$

(c) $\int \sinh x \sin x\,dx = I$

SOL'N: $u_1 = \sinh x \quad dv_1 = \sin x\,dx$

$du_1 = \cosh x\,dx \quad v_1 = -\cos x$

$u_2 = \cosh x \quad dv_2 = \cos x\,dx$

$du_2 = \sinh x \quad v_2 = \sin x$

$I = -\cos x \sinh x + \int \cos x \cosh x\,dx$

$= -\cos x \sinh x + \cosh x \sin x - \underbrace{\int \sin x \sinh x\,dx}_{I}$

$\Rightarrow I = \frac{1}{2}(-\cos x \sinh x + \cosh x \sin x) + C$

d) $\int_0^{\pi/2} \sin^4(x)\,dx = I$

<u>SOL'N:</u>

$$I = \int_0^{\pi/2} (\sin^2 x - \sin^2 x \cos^2 x)\,dx$$

$$= \int_0^{\pi/2} \left(\frac{1}{2} - \frac{1}{2}\cos(2x)\right)dx - \int_0^{\pi/2} \frac{1}{4}\sin^2(2x)\,dx$$

$$= \frac{1}{2}\left(x - \frac{1}{2}\sin(2x)\right)\Big|_0^{\pi/2} - \frac{1}{8}\int_0^{\pi/2} (1 - \cos(4x))\,dx$$

$$= \frac{1}{2}\left(\frac{\pi}{2} - 0\right) - \frac{1}{8}\left(x - \frac{1}{4}\sin(4x)\right)\Big|_0^{\pi/2}$$

$$= \frac{\pi}{4} - \frac{1}{8}\left(\frac{\pi}{2}\right) = \frac{3\pi}{16}$$

e) $\int \sin^3 x \cos^3 x = I$

<u>SOL'N:</u>

$$I = \int (\sin^3 x \cos x - \sin^5 x \cos x)\,dx$$

$u = \sin x$

$du = \cos x\,dx$

$$I = \int (u^3 - u^5)\,du$$

$$= \frac{1}{4}u^4 - \frac{1}{6}u^6 + C$$

$$= \frac{1}{4}\sin^4 x - \frac{1}{6}\sin^6 x + C$$

f) $\int_0^2 \frac{x^2\,dx}{\sqrt{x^2+4}} = I$

<u>SOL'N:</u>

Let $x = 2\tan\theta$

$dx = 2\sec^2\theta$

$x = 0 \Rightarrow \theta = 0$

$x = 2 \Rightarrow \theta = \frac{\pi}{4}$

$$I = 2\int_0^{\pi/4} \frac{4\tan^2\theta\sec^2\theta\,d\theta}{2\sec\theta}$$

$$= 4\int_0^{\pi/4} \tan^2\theta\sec\theta\,d\theta$$

$$= 4\int_0^{\pi/4} (\sec^3\theta - \sec\theta)\,d\theta$$

$$\int_0^{\pi/4} \sec^3\theta\, d\theta \qquad u = \sec\theta \qquad dv = \sec^2\theta\, d\theta$$

$$du = \sec\theta\tan\theta \quad v = \tan\theta$$

$$= \sec\theta\tan\theta \Big|_0^{\pi/4} - \underbrace{\int_0^{\pi/4} \sec\theta\tan^2\theta}_{= \frac{1}{4} I}$$

$$\text{So, } I = 4\sec\theta\tan\theta \Big|_0^{\pi/4} - I - 4\int_0^{\pi/4} \sec\theta\, d\theta$$

$$= 4\sqrt{2} - I - 4\ln|\sec\theta + \tan\theta|\Big|_0^{\pi/4}$$

$$\Rightarrow I = 2\sqrt{2} - 2\ln(\sqrt{2}+1)$$

9) $I = \int \frac{x^4+1}{x^3+9x}\, dx$

$$\begin{array}{r|l} & x \\ \hline x^3+9x & x^4+1 \\ & -(x^4+9x^2) \\ \hline & -9x^2+1 \end{array}$$

$$\text{So, } \frac{x^4+1}{x^3+9x} = x + \frac{1-9x^2}{x^3+9x}$$

$$\frac{1-9x^2}{x^3+9x} = \frac{A}{x} + \frac{Bx+C}{x^2+9}$$

$$\Rightarrow 1-9x^2 = Ax^2 + 9A + Bx^2 + Cx$$

$$\Rightarrow \begin{cases} A+B = -9 \Rightarrow B = -\frac{82}{9} \\ C = 0 \\ 9A = 1 \Rightarrow A = \frac{1}{9} \end{cases}$$

$$\frac{1-9x^2}{x^3+9x} = \frac{1}{9x} - \frac{\frac{82}{9}x}{x^2+9}$$

$$\text{So, } I = \int\left(x + \frac{1}{9x} - \frac{82}{9}\frac{x}{x^2+9}\right)dx \qquad u = x^2+9$$

$$= \frac{1}{2}x^2 + \frac{1}{9}\ln|x| - \frac{82}{9}\int \frac{1}{2u}\, du \qquad du = 2x\, dx$$

$$= \frac{1}{2}x^2 + \frac{1}{9}\ln|x| - \frac{41}{9}\ln|x^2+9| + C$$

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(b) $\int_{-\pi/2}^{\pi/2} \sin(2x)\cos(3x)\,dx = I$

SOL'N: Recall: $\sin(a \pm b) = \sin a \cos b \pm \sin b \cos a$

$\Rightarrow \sin(a)\cos(b) = \frac{1}{2}(\sin(a+b) + \sin(a-b))$

So, $I = \frac{1}{2}\int_{-\pi/2}^{\pi/2} (\sin(5x) - \sin(x))\,dx$

$= \frac{1}{2}\left(-\frac{1}{5}\cos(5x) + \cos x\right)\Big|_{-\pi/2}^{\pi/2}$

$= 0$ (or, just use the fact that the original integrand is odd.)

#2 (a). $\int_{-\infty}^{\infty} \frac{dx}{x^2+1} = \lim_{a\to-\infty} \lim_{b\to\infty} \int_a^b \frac{dx}{x^2+1}$

$= \lim_{a\to-\infty} \lim_{b\to\infty} (\arctan b - \arctan a)$

$= \frac{\pi}{2} + \frac{\pi}{2} = \pi$

(b) $x + x^3 \geq x$ on $[0,1]$.

$\Rightarrow \sqrt{x+x^3} \geq \sqrt{x}$

$\Rightarrow \frac{1}{\sqrt{x+x^3}} \leq \frac{1}{\sqrt{x}}$

$\Rightarrow \int_0^1 \frac{dx}{\sqrt{x+x^3}} \leq \int_0^1 \frac{1}{\sqrt{x}}$ ← converges, and so on $[0,1)$ the original integral converges

Similarly, on $[1,\infty)$, $x + x^3 \geq x^3$.

So, $\frac{1}{\sqrt{x+x^3}} \leq x^{-3/2}$ on $[0,\infty)$.

$\Rightarrow \int_1^\infty \frac{dx}{\sqrt{x+x^3}} \leq \int_1^\infty x^{-3/2}\,dx$ ← converges (p-test)

We conclude that $\int_0^\infty \frac{dx}{\sqrt{x+x^3}}$ converges.

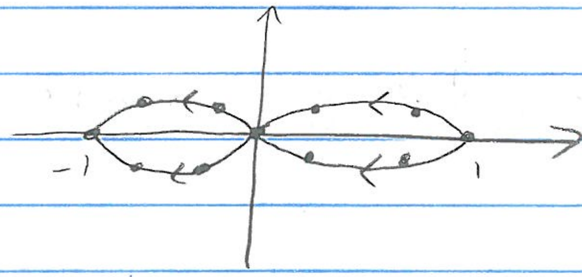
#2. ⓒ. On $[2,\infty)$, $\ln x < x$

$\Rightarrow \frac{1}{\ln x} > \frac{1}{x}$

$\Rightarrow \int_2^\infty \frac{dx}{\ln x} \geq \int_2^\infty \frac{dx}{x} \longleftarrow$ diverges by p-test

So, $\int_2^\infty \frac{dx}{\ln x}$ diverges.

#3 ⓑ

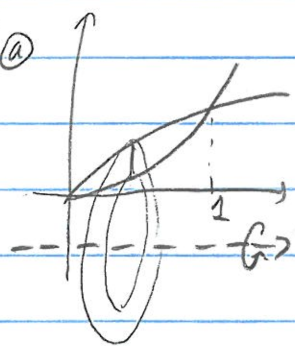


Warning, only need to let $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$

ⓒ The right leaf is obtained by letting θ be between $-\frac{\pi}{4}$ and $\frac{\pi}{4}$.

$$\int_{-\pi/4}^{\pi/4} \frac{1}{2}\left(\sqrt{\cos(2\theta)}\right)^2 d\theta = \frac{1}{2}\cdot\frac{1}{2}\sin(2\theta)\Big|_{-\pi/4}^{\pi/4} = \frac{1}{4}\cdot 2 = \frac{1}{2}$$

#4 ⓐ



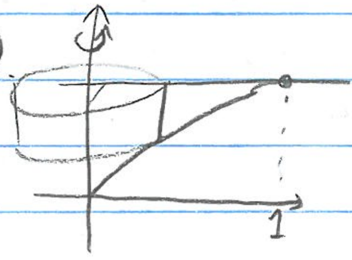
$$\text{Volume} = \int_0^1 \pi\left((\sqrt{x}+1)^2 - (x^2+1)^2\right)dx$$

$$= \pi\int_0^1 (x + 2\sqrt{x} + 1) - (x^4 + 2x^2 + 1)\,dx$$

$$= \pi\left(-\frac{1}{5}x^5 - \frac{2}{3}x^3 + \frac{1}{2}x^2 + \frac{4}{3}x^{3/2}\right)\Big|_0^1$$

$$= \pi\left(\frac{7}{6} - \frac{1}{5}\right) = \frac{29\pi}{30}$$

ⓑ.



$$\text{Volume} = \int_0^1 2\pi x(1-\sqrt{x})\,dx$$

$$= 2\pi\left(\frac{1}{2}x^2 - \frac{2}{5}x^{5/2}\right)\Big|_0^1$$

$$= \frac{\pi}{5}$$

#5. (a) $f(x) = 2e^{x} + \frac{1}{8}e^{-x}$

$f'(x) = 2e^{x} - \frac{1}{8}e^{-x}$

$(f'(x))^2 = 4e^{2x} - \frac{1}{2} + \frac{1}{64}e^{-2x}$

$$s = \int_0^{\ln 2} \sqrt{1 + f'(x)^2}\, dx$$ (using parametrisation $c(t) = (t, f(t))$.)

$$= \int_0^{\ln 2} \sqrt{4e^{2x} + \frac{1}{2} + \frac{1}{64}e^{-2x}}\, dx$$

$$= \int_0^{\ln 2} \sqrt{(2e^{x} + \frac{1}{8}e^{-x})^2}\, dx$$

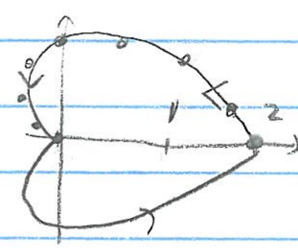
$$= \left(2e^{x} - \frac{1}{8}e^{-x}\right)\Big|_0^{\ln 2} = 4 - \frac{1}{16} - 2 + \frac{1}{8} = \frac{33}{16}$$

(b) $r(\theta) = 1 + \cos\theta$

$r(\theta)^2 = 1 + 2\cos\theta + \cos^2\theta$

$r'(\theta) = -\sin\theta$

$(r'(\theta))^2 = \sin^2\theta$



$$s = 2\int_0^{\pi} \sqrt{1 + 2\cos\theta + \cos^2\theta + \sin^2\theta}\, d\theta$$

$$= \int_0^{\pi} \sqrt{2 + 2\cos\theta}\, d\theta \qquad \left(\cdot \frac{\sqrt{2 - 2\cos\theta}}{\sqrt{2 - 2\cos\theta}}\right)$$

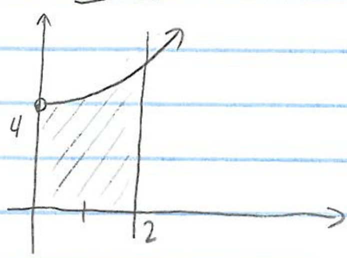
$$= \int_0^{\pi} \frac{2\sin\theta}{\sqrt{2 - 2\cos\theta}}\, d\theta$$

$u = 2 - 2\cos\theta \quad \Big| \quad \theta = 0 \Rightarrow u = 0$

$du = +2\sin\theta \quad \Big| \quad \theta = \pi \Rightarrow u = 4$

$$= \int_0^{4} \frac{du}{\sqrt{u}} = 2\sqrt{u}\Big|_0^4 = 4$$

#7 Sol'n:



$$M_{x=0} = \int_0^2 x(x^2+4)\,dx$$
$$= \left(\tfrac{1}{4}x^4 + 2x^2\right)\Big|_0^2$$
$$= 4+8 = 12$$

$$m = \int_0^2 (x^2+4)\,dx = \left(\tfrac{1}{3}x^3+4x\right)\Big|_0^2 = \tfrac{8}{3} + 8 = \tfrac{32}{3}$$

$$M_{y=0} = \int_0^2 \tfrac{1}{2}(x^2+4)^2\,dx = \int_0^2 \tfrac{1}{2}(x^4+8x^2+16)\,dx$$
$$= \left(\tfrac{1}{10}x^5 + \tfrac{4}{3}x^3 + 8x\right)\Big|_0^2 = \tfrac{32}{10} + \tfrac{32}{3} + 16$$

$$\bar{x} = \frac{M_{x=0}}{m} = \frac{12\cdot 3}{32} = \frac{9}{8}$$
$$\bar{y} = \frac{M_{y=0}}{m} = \frac{3}{32}\left(\frac{32}{10} + \frac{32}{3} + \frac{32}{2}\right) = \frac{168}{32} = \frac{21}{4}$$

$$(\bar{x}, \bar{y}) = (9/8,\ 21/4)$$

#8 (a) $\left\{\frac{1}{n}\sin\left(\frac{1}{n}\right)\right\}$

Sol'n: Since $\left|\sin\left(\frac{1}{n}\right)\right| \le 1$ for all n, and $\lim_{n\to\infty} \frac{1}{n} = 0$, the sequence converges

(b) $\sum_{n=0}^{\infty} n^2$

Sol'n: this series diverges by the divergence test since $\lim_{n\to\infty} n^2 = \infty$.

(c) $\sum_{n=0}^{\infty} \frac{1}{n^2+2}$

Sol'n:

$$n^2+2 \ge n^2 \Rightarrow \frac{1}{n^2+2} \le \frac{1}{n^2}$$

Since $\sum \frac{1}{n^2}$ converges by the p-series test, $\sum_{n=0}^{\infty} \frac{1}{n^2+2}$ converges by the basic comparison test.

(8)

(d) $\sum_{n=0}^{\infty} \frac{1}{n^2-2}$

Sol'n: $\lim_{n\to\infty} \frac{1/n^2}{1/(n^2-2)} = \lim_{n\to\infty}\left(1-\frac{2}{n^2}\right) = 1 \in (0,\infty)$

By the limit comparison test, the series converges.

(e). $\sum_{n=0}^{\infty} \frac{\ln n}{n}$

Sol'n: Eventually, $\ln n > 1$, and so $\frac{\ln n}{n} > \frac{1}{n}$.

Since $\sum \frac{1}{n}$ diverges, by the basic comparison test, $\sum_{n=0}^{\infty}\frac{\ln n}{n}$ diverges.

(f) $\sum_{n=0}^{\infty} \frac{\cos(\pi n)\cos(n)}{n}$

Sol'n: Since $\cos(\pi n) = (-1)^n$, this is an alternating series. Since $|\cos(n)| \le 1$ for all n, we have that $\lim_{n\to\infty} \frac{\cos(n)}{n} = 0$. So, by the alternating series test, the series diverges.

#9. Sol'n: To find the radius of convergence of $\sum_{n=1}^{\infty} \frac{x^n}{n^n}$, we apply the root test:

$\lim_{n\to\infty} \left|\frac{x^n}{n^n}\right|^{1/n} = \lim_{n\to\infty} \frac{|x|}{n} = 0$. So, the radius of convergence is ∞, and the interval of convergence is $(-\infty,\infty)$.

#10 (a) $\sum_{n=}^{\infty} \frac{2}{4^n}$

Sol'n:
$= 2 \frac{1}{1-1/4} = 8/3$ since this series is geometric and $|\frac{1}{4}|<1$.

(b) $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$

Sol'n: Recall that the Taylor series of $\arctan x$ about $x=0$ is
$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$
and that $\arctan x$ is analytic at $x=0$.

The radius of convergence is 1, and so by Abel's theorem, $\arctan 1 = \sum_{n=0}^{\infty} (-1)^n/(2n+1)$.
But $\arctan 1 = \pi/4$, and so the series equals $\pi/4$.

(c) $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}$

Sol'n: Recall $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$. So, $\cos 1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}$

(d) $\sum_{n=0}^{\infty} n/3^n$

Sol'n:
From the geometric series, $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$.

So, $\frac{d}{dx}\left(\sum_{n=0}^{\infty} x^n\right) = \sum_{n=0}^{\infty} n x^{n-1} = \frac{+1}{(1-x)^2}$

So, $\sum_{n=0}^{\infty} n x^n = \frac{x}{(1-x)^2}$. Since $|\frac{1}{3}|<1$, we have
$$\sum_{n=0}^{\infty} n\left(\tfrac{1}{3}\right)^n = 1/3\left(\frac{1}{(1-1/3)^2}\right) = 3/4$$

#11 sol'n:

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\tan(2x)$	0
1	$2\sec^2(2x)$	2
2	$8\sec^2(2x)\tan(2x)$	0
3	$32\sec^2(2x)\tan^2(2x)+16\sec^4(2x)$	16
4	$128\sec^2(2x)\tan^3(2x)+256\sec^4(2x)\tan(2x)$	0
5	$512\sec^2x\tan^4x+2048\sec^4x\tan^2x+512\sec^6x$	512

$$P_5(x) = 2x + \frac{8}{3}x^3 + \frac{64}{15}x^5$$

#12. sol'n:

$$\sinh x = \frac{1}{2}(e^x - e^{-x}) = \frac{1}{2}\left(\sum_{n=0}^{\infty}\frac{x^n}{n!} - \sum_{n=0}^{\infty}\frac{(-1)^n x^n}{n!}\right)$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\frac{(1-(-1)^n)x^n}{n!}$$ ← when n is even, get 0; when n is odd, get $\frac{2x^n}{n!}$

$$= \sum_{k=0}^{\infty}\frac{x^{2k+1}}{(2k+1)!}$$

Replace n with $2k+1$.

#13. sol'n:

$$\frac{\sin x}{x} = \frac{1}{x}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n+1)!}$$

#14. $y' = x - y$, $y(0) = 1$.

sol'n:

$y_0 = 1$, $x_0 = 0$

$y_1 = y_0 + h(x_0 - y_0) = 1 + \frac{1}{2}(0-1) = +\frac{1}{2}$, $x_1 = \frac{1}{2}$

$y_2 = y_1 + h(x_1 - y_1) = +\frac{1}{2} + \frac{1}{2}\left(\frac{1}{2} - \frac{1}{2}\right) = \frac{1}{2}$, $x_2 = 1$

$y_3 = y_2 + h(x_2 - y_2) = \frac{1}{2} + \frac{1}{2}\left(1 - \frac{1}{2}\right) = \frac{3}{4}$, $x_3 = \frac{3}{2}$

So, $y(3/2) \approx \frac{3}{4}$.

#15. $y' = \sec(y)\cos(t)$

Sol'n: Separable!

$$\int \cos y \, dy = \int \cos t \, dt$$

$\Rightarrow \sin y = \sin t + C$

$\Rightarrow y = \arcsin(\sin t + C)$

#16. $y(t) = Ce^{kt}$, $y(0) = 100g \Rightarrow C = 100$

$50 = 100e^{1200k}$

$\Rightarrow \frac{1}{2} = e^{1200k}$

$\Rightarrow -\ln 2 = 1200k \Rightarrow k = \frac{-1}{1200}\ln 2$

$10 = 100e^{-t\ln 2/1200}$

$\Rightarrow \ln\frac{1}{10} = -t\ln 2/1200$

$\Rightarrow t = 1200\ln 10/\ln 2$

#17. $P'(t) = kP\left(1 - \frac{P}{M}\right)$, $M = 1000$, $P(0) = 100$, $P(1) = 150$

$P(10) = ?$

Sol'n: $$\int \frac{dP}{P\left(1 - \frac{P}{M}\right)} = kt + C$$

$$\boxed{\frac{M}{P(M-P)} = \frac{1}{P} + \frac{1}{M-P}}$$

$$= \int\left(\frac{1}{P} + \frac{1}{M-P}\right)dP$$

$$= (\ln|P| - \ln|M-P|)$$

$\Rightarrow \frac{P}{M-P} = Ce^{kt} \Rightarrow P(1 + Ce^{kt}) = MCe^{kt}$

$\Rightarrow P(t) = \frac{MCe^{kt}}{1 + Ce^{kt}}$

$P(0) = \frac{MC}{1+C} \Rightarrow P(0) + CP(0) = MC$

$\Rightarrow C = \frac{P(0)}{M - P(0)} = \frac{100}{900} = \frac{1}{9}$

So, $P(t) = \dfrac{\frac{1000}{9}}{\frac{1}{9} + e^{-kt}} = \dfrac{1000}{1+9e^{-kt}}$

$P(1) = 150 = \dfrac{1000}{1+9e^{-k}}$

$\Rightarrow 1+9e^{-k} = \frac{20}{3}$

$\Rightarrow e^{-k} = \frac{17}{27}$

$\Rightarrow k = \ln\frac{27}{17}$

So, $P(10) = \dfrac{1000}{1+9e^{-10\ln(27/17)}}$.

#18. $S(0) = 20g$, $C(0) = \frac{20g}{150L} = \frac{2}{15}\ g/L$

$\binom{\text{rate}}{\text{in}} = \frac{3g}{L}\cdot\frac{2L}{min} = \frac{6g}{min}$

$\binom{\text{rate}}{\text{out}} = \frac{S(t)}{150L}\cdot\frac{2L}{min} = \frac{S(t)}{75}\frac{g}{min}$

$S'(t) = \binom{\text{rate}}{\text{in}} - \binom{\text{rate}}{\text{out}} = 6 - \frac{1}{75}S(t)$

$\Rightarrow \displaystyle\int \frac{dS}{6-\frac{1}{75}S} = t + C$

$\Rightarrow -75\ln\left|6-\frac{1}{75}S\right| = t + C$

$\Rightarrow \ln\left|6-\frac{S}{75}\right| = -1/75\,t + C$

$\Rightarrow 6 - \frac{S}{75} = Ce^{-t/75}$

$\Rightarrow S(t) = 450 - 75Ce^{-t/75}$

$S(0) = 20 = 450 - 75C \Rightarrow C = \frac{430}{75} = \frac{86}{15}$

So, $S(t) = 450 - 430e^{-t/75}$

Need $C(t) = 1 = \frac{S(t)}{150L} \Rightarrow S(t) = 150g$

$\Rightarrow 150 = 450 - 430e^{-t/75}$

$\Rightarrow \frac{30}{43} = e^{-t/75} \Rightarrow t = 75\ln\frac{43}{30}$ min.

#19. $y'' + xy = x$

Sol'n:

Set $y(x) = \sum_{n=0}^{\infty} a_n x^n$

$$y'(x) = \sum_{n=0}^{\infty} a_n n x^{n-1}, \quad y''(x) = \sum_{n=0}^{\infty} a_n (n-1) n x^{n-2}$$

$$x = \underbrace{\sum_{n=0}^{\infty} a_n (n-1) n x^{n-2}}_{n \to n+2} + \underbrace{x \sum_{n=0}^{\infty} a_n x^n}_{n \to n-1}$$

$$= \sum_{\cancel{n=-2}\ n=0}^{\infty} a_{n+2} (n+1)(n+2) x^n + \sum_{n=1}^{\infty} a_{n-1} x^n$$

$$= 2a_2 + \sum_{n=1}^{\infty} \left[a_{n+2} (n+1)(n+2) + a_{n-1} \right] x^n$$

$$\Rightarrow 0 = \underbrace{2a_2}_{=0} + \underbrace{(6a_3 + a_0 - 1)}_{=0} x + \sum_{n=2}^{\infty} \underbrace{\left[a_{n+2} (n+1)(n+2) + a_{n-1} \right]}_{=0 \text{ for each } n \geq 2} x^n$$

$$\Rightarrow a_2 = 0, \quad a_3 = \frac{1}{6}(1 - a_0), \quad \boxed{a_{n+2} = \frac{-a_{n-1}}{(n+1)(n+2)} \quad (n \geq 2)}$$

$$a_4 = \frac{-a_1}{3 \cdot 4}$$

$$a_5 = \frac{-a_2}{4 \cdot 5} = 0$$

$$a_6 = \frac{-a_3}{5 \cdot 6} = \frac{-\frac{1}{6}(1 - a_0)}{5 \cdot 6}$$

$$a_7 = \frac{-a_4}{6 \cdot 7} = \frac{a_1}{3 \cdot 4 \cdot 6 \cdot 7}$$

$$a_8 = \frac{-a_5}{7 \cdot 8} = 0$$

$$a_9 = \frac{-a_6}{8 \cdot 9} = \frac{\frac{1}{6}(1 - a_0)}{5 \cdot 6 \cdot 8 \cdot 9}$$

$$\begin{cases} a_{3k+1} = \dfrac{(-1)^k a_1}{(3 \cdot 6 \cdot 9 \cdot \ldots)(4 \cdot 7 \cdot 10 \cdot \ldots)} \\ a_{3k+2} = 0 \\ a_{3k+3} = \dfrac{(-1)^k \frac{1}{6}(1 - a_0)}{(5 \cdot 8 \cdot 11 \cdot \ldots)(6 \cdot 9 \cdot 12 \cdot \ldots)} \end{cases}$$

for $k \geq 1$.

(14)

So, $y(t) = a_0 + a_1 x + a_2 x^2 + \dots$

$$= a_0 + a_1 x + \frac{1}{6}(1-a_0)x^3 + a_1 \sum_{k=1}^{\infty} \frac{(-1)^k x^{3k+1}}{(3\cdot 6\cdot 9 \cdots)(4\cdot 7\cdot 10 \cdots)}$$

$$+ \frac{1}{6}(1-a_0) \sum_{k=1}^{\infty} \frac{(-1)^k x^{3k+3}}{(5\cdot 8\cdot 11 \cdots)(6\cdot 9\cdot 12 \cdots)}$$

#20. ⓐ. $(r, \theta) = (1, \frac{\pi}{3})$

$\Rightarrow \begin{cases} x = 1\cdot \cos\frac{\pi}{3} = \frac{1}{2} \\ y = 1 \cdot \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2} \end{cases}$ $(x, y) = (\frac{1}{2}, \frac{\sqrt{3}}{2})$

ⓑ $(r, \theta) = (3, \frac{-5\pi}{6})$

$\Rightarrow \begin{cases} x = 3\cos(\frac{-5\pi}{6}) = \frac{-3\sqrt{3}}{2} \\ y = 3\sin(\frac{-5\pi}{6}) = \frac{-3}{2} \end{cases}$ $(x, y) = (\frac{-3\sqrt{3}}{2}, \frac{-3}{2})$

ⓒ $(r, \theta) = (-1, \pi)$

$\Rightarrow \begin{cases} x = -1\cdot \cos\pi = 1 \\ y = -1 \cdot \sin\pi = 0 \end{cases}$ $(x, y) = (1, 0)$

ⓓ $(r, \theta) = (-2, \frac{-4\pi}{3})$

$\Rightarrow \begin{cases} x = -2 \cdot \cos(\frac{-4\pi}{3}) = 1 \\ y = -2 \cdot \sin(\frac{-4\pi}{3}) = -\sqrt{3} \end{cases}$ $(x, y) = (1, -\sqrt{3})$

#21 ⓐ $(x, y) = (1, 2)$

$r^2 = x^2 + y^2 = 1^2 + 2^2 = 5$

$\Rightarrow r = \sqrt{5}$, $\tan\theta = \frac{2}{1}$

$\Rightarrow \theta = \arctan 2$ (1^{st} quadrant)

So, $(r, \theta) = (\sqrt{5}, \arctan 2)$

(b). $(x,y)=(-3,4)$

$\Rightarrow r^2=(-3)^2+4^2=25$

$\Rightarrow r=5$, $\tan\theta=\frac{4}{-3} \Rightarrow \theta=\arctan\left(\frac{-4}{3}\right)+\pi$ 2nd quadrant

So, $(r,\theta)=\left(5, \arctan\left(\frac{-4}{3}\right)+\pi\right)$

(c) $(x,y)=(4,-3)$

$\Rightarrow r^2=4^2+(-3)^2=25$

$\Rightarrow r=5$, $\tan\theta=\frac{-3}{4} \Rightarrow \theta=\arctan\left(\frac{-3}{4}\right)$ (4th quadrant)

So, $(r,\theta)=\left(5, \arctan\left(\frac{-3}{4}\right)\right)$

#22. $c(t)=(\sin t, t^2, \cos(3t))$

$c'(t)=(\cos t, 2t, -3\sin(3t))$

$c(0)=(0,0,1)$, $c'(0)=(-1,0,0)$

So, the tangent line at $c(0)$ is

$$L(t)=(0,0,1)+t(-1,0,0)=(-t,0,1).$$

#23. $r=1+\cos^2(\theta)$

Sol'n: Horizontal: $\frac{dr}{d\theta}\sin\theta+r\cos\theta=0$

$\Rightarrow -2\cos\theta\sin^2\theta+\cos\theta+\cos^3\theta=0$

$\Rightarrow \cos\theta(\cos^2\theta+1-2\sin^2\theta)=0$

$\Rightarrow \cos\theta(3\cos^2\theta-1)=0$

$\Rightarrow \cos\theta=0$ or $\cos^2\theta=\frac{1}{3}$

$\Rightarrow \boxed{\theta=\frac{\pi}{2}, \frac{3\pi}{2}, \arccos\frac{1}{\sqrt{3}}, \arccos\frac{-1}{\sqrt{3}}}$

Vertical: $\frac{dr}{d\theta}\cos\theta-r\sin\theta=0$

$\Rightarrow -2\cos^2\theta\sin\theta-\sin\theta-\sin\theta\cos^2\theta=0$

$\Rightarrow \sin\theta\underbrace{(-3\cos^2\theta-1)}_{\neq 0}=0 \Rightarrow \boxed{\theta=0,\pi}$