General comments. Your proof write-up does not have to meet “textbook” or “journal” standards but it should be coherent and consistent. To ensure that, always re-read your proof and ask yourself:

- Is each claim (statement) you wrote supported with a clear justification?
- Are there any leaps in reasoning? (Assumptions or statements you are considering without mentioning them).
- Does your conclusion matches exactly what we wanted to prove?

Introduction. Euclid and his postulates.

1. Observe Figure .
    a. Explain the historical event that it refers to.
    b. Calculate the Earth’s circumference.
2. Observe Figure 1.
    a. What is the relationship among angles α_1, α_2, α_3, α_4?
    b. Use the previous result to justify a relationship formula for 5 angles.
    c. What is the formula for n angles? How can we prove it? (You don’t have to reproduce the complete proof, just make sure you can tell what kind of proof it is and describe its components.)
3. Discuss Euclid’s contribution to geometry.
    a. Discuss postulates 1-3.
    b. Discuss the fifth postulate and its historical role.(You don’t have to memorize the postulates, if requested, they will be provided)
4. Prove elementary theorems (we may have not discussed these but you should be able to prove them as they are elementary):
    a. The vertical angles theorem.
    b. Interior angle sum in a triangle theorem.
    c. Interior angle sum in a quadrilateral theorem.

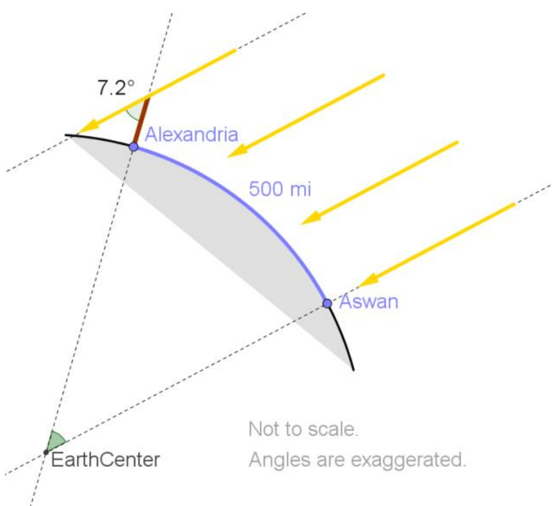


Figure1

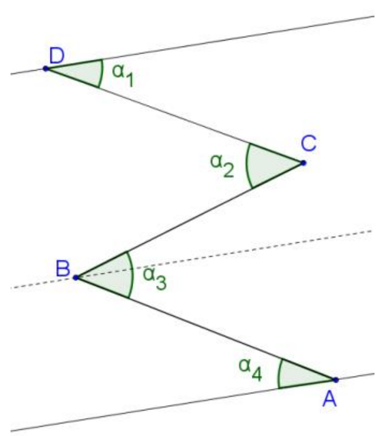


Figure 1

    d. Which of the theorems above are not part of the absolute (neutral) geometry? Explain.

Axiomatic systems and Finite Geometries.

1. Explain three properties of axiomatic systems.
2. Consider the following system of axioms:

   Axiom 1. Every ant has at least two paths.
   Axiom 2. Every path has at least two ants.
   Axiom 3. There exists at least one ant.

   a. What are the undefined terms? (Don't forget the undefined "relation").
   b. Prove the theorem: *There exists at least one path*.
   c. What is the minimum number of paths? Prove your answers from axioms.
   d. Find three different (=non-isomorphic) models of the system. Represent them with a drawing.
   e. Is this system categorical? Explain.
   f. Discuss the properties of this system.

      i. Does the consistency hold? Explain.
      ii. Does the independence hold? Explain. (Use only one of three possible cases to illustrate your point.)
      iii. Is the system complete? Explain.

   g. Suggest a change or changes to the axioms that would:

      iv. Make the system inconsistent
      v. Make the system complete.

3. Consider the following system of axioms:

   Axiom 1: There exist five points.
   Axiom 2: There exist two lines.
   Axiom 3: Each line contains at least two points.

   a. Is this system categorical?
   b. Is it complete? If you think it is so, explain why. If you think it is not, give an example of a theorem that cannot be proven or disproven based on the axioms.
   c. Are all three axioms independent? Explain.

4. Three point geometry:

   Axiom 1. There exist exactly three distinct points in the geometry.
   Axiom 2. Each two distinct points are on exactly one line.
   Axiom 3. Not all the points of the geometry are on the same line.
   Axiom 4. Each two distinct lines are on at least one point.

   a. Draw its model.
   b. Is the system categorical? Explain.

c. Is it complete? Explain.

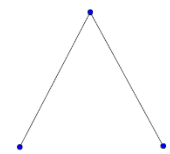
d. Prove:

i. Theorem 1: Each two distinct lines are on exactly one point.

ii. Theorem 2. The geometry has exactly three lines.

5. A student is showing the independence of the Axiom 3 in the following way:

Looking at the model on the right, we can see that Ax1 is not satisfied, Ax2 is not satisfied, but Ax 3 is satisfied. Therefore the Axiom 3 must be independent.



Discuss the student's solution. What would you do so that the student understands why his solution is incorrect?

For further practice, see for example Tim Peil's website http://web.mnstate.edu/peil/geometry/C1AxiomSystem/AxSysWorksheet.htm .

Triangle Congruence

6. Explain the intuitive understanding of congruence. What is it based on – how can we tell if two shapes (not necessarily triangles) are congruent? What middle school students can do to decide if two shapes are congruent?
7. Give a triangle congruence definition and explain how it differs from congruence theorems and postulates.
8. Show that it is possible to use compass and ruler to:

e. Construct an angle bisector of a given angle.

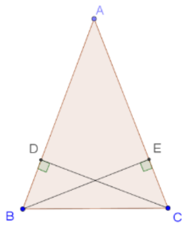
f. Copy (Transfer) a given angle onto a given ray.

9. Formulate and prove the Isosceles triangle theorem. Can you do it without a reference to an angle bisector?
10. Discuss SAS, SSS and ASA conditions. Are they theorems or postulates? Explain. Prove the ones you classified as theorems.
11. Discuss AAS condition. How is it different from ASA? Prove it. (You are allowed to use the parallel postulate).
12. Discuss an "SSA" condition. Is there a valid theorem (or theorems) that refer to two pairs of congruent sides and one pair of congruent angles? If so, formulate it (them).

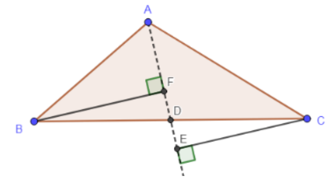
Applications of triangle congruence.

13. Typical school application problems for triangle congruence: https://www.geogebra.org/m/s53r7rus
14. Prove the converse of the Isosceles triangles theorem: If two angles in a triangle are congruent, then the sides opposite these angles are congruent.
15. Prove: If a triangle is equiangular, then it is also equilateral.
16. Prove: If a triangle is isosceles, then the median falling on the base of the triangle is a perpendicular bisector of that base.

17. Prove: If a triangle is isosceles, then the median falling on the base of the triangle is an angle bisector of the apex angle. (Apex is the vertex opposite to the base.)



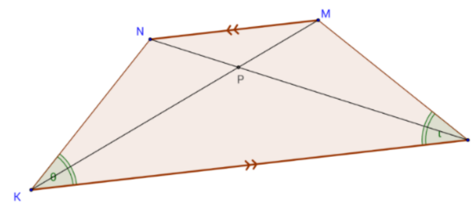
18. Prove: If ΔABC is isosceles, then $\Delta CBE \cong \Delta BCD$ (see the picture).
19. D is the midpoint of BC. Prove that BF is congruent to CE. (see the picture).



20. Prove the properties of quadrilaterals (see the [Geogebra Tube book](), chapter Applications). Selected examples (make sure to check out the link for more problems):

g. Diagonals of a kite are perpendicular to each other.

h. Isosceles trapezoid is a trapezoid with congruent base angles. Prove that in an isosceles trapezoid KLMN:



iii. $KN \cong LM$

iv. Using the previous result, show that the diagonals are congruent.

v. Triangles KPN and LPM in an isosceles trapezoid KLMN are congruent.

i. The diagonals in a parallelogram bisect each other.

Note: The above quadrilateral problems are focused on the application of triangle congruence, not on your knowledge of quadrilaterals. You don't have to memorize the definitions of quadrilaterals (they will be provided). Images depicting the situations (such as the one above) can also be provided if needed.