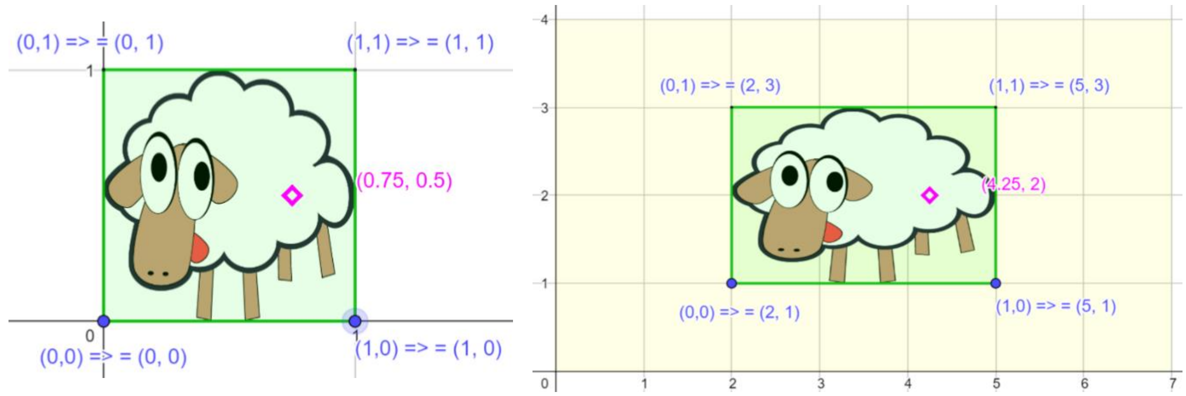
1. Coordinate systems. Operations with vectors and points.

Problem 1. Local and global coordinates

A picture of a sheep is created in local coordinates (local frame has corners (0,0), (1,0), (1,1), (0,1). The picture is then placed on a screen as shown (global frame has corners (0,0), (7,0), (7,4), and (0,4). Find conversion equations that will convert coordinates of a point given in a local frame into coordinates in the global frame. ([Applet we used in class]())



Problem 2. Aspect Ratio

TV sizes are specified by a diagonal. Calculate the dimensions (width and height) of a 40''

a) standard TV (Aspect ratio of width to height is 4:3)
b) HDTV (Aspect ratio is 16:9)

You can leave answers in a fraction form.

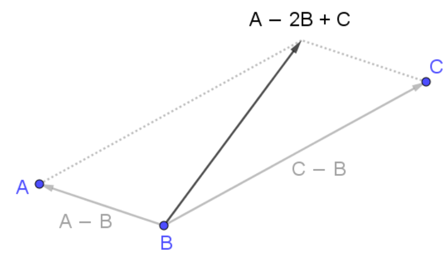
Problem 3. Geometrically meaningful operations

Decide which of the following operations are geometrically meaningful (i.e. independent of the coordinate system). Provide a brief explanation for your answer (typically, if an operation is independent of the coordinate system, you don't need coordinates to explain what the result is and you can do it graphically).

a) Adding two vectors ($\vec{v} + \vec{u}$)
b) Subtracting two vectors ($\vec{v} - \vec{u}$)
c) Adding a vector and a point (D + $\vec{u}$)
d) Adding two points (D + I)
e) Subtracting two points (D - I)
f) Multiplying a vector by a scalar ($3 \cdot \vec{u}$)
g) Multiplying a point by a scalar ($3 \cdot B$)
h) Operation on two points X(x_1, x_2) and Y(y_1, y_2) that results in a point $\left(\frac{x_1+y_1}{2}, \frac{x_2+y_2}{2}\right)$. How would you describe (geometrically) the result of this operation?

Problem 4. Geometrically meaningful operations

Coordinate independent operations can be extended to include more than two objects. If it is possible to manipulate the objects algebraically to represent coordinate-independent combinations, then such operation will itself be coordinate independent.



Example: A – 2B + C is coordinate independent as it can be expressed as the sum of two vectors: (A – B) + (C – B). Also, it is possible to solve it graphically without knowing the coordinates of the points A, B and C (see the picture).

Decide if the following operations are geometrically meaningful. If they are, draw their representation:

a) A + 2B – C
b) L – M + K
c) $\vec{u} - C$
d) $C - \vec{u}$
e) A + 2B – 2C
f) $\vec{u} - 2\vec{v} + \vec{w}$

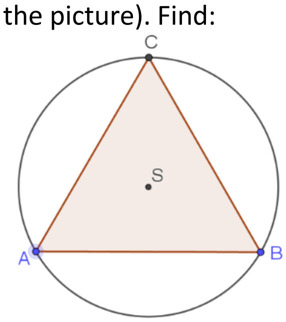
Note: Notation $\overrightarrow{SA} = A - S$ will be used in the following problems.

Problem 5.

Given two points A, B, what is the geometric meaning of $A + \frac{1}{2}(B - A)$?

Problem 6. Vector addition

ABC is an equilateral triangle and S is the center of its circumscribed circle (see the picture). Find:



a) $\overrightarrow{SA} + \overrightarrow{SB} + \overrightarrow{SC}$
b) $\overrightarrow{SA} + \overrightarrow{SB} - \overrightarrow{SC}$
c) $2\overrightarrow{SA} + 2\overrightarrow{SB} + \overrightarrow{SC}$
d) $\overrightarrow{AS} + \overrightarrow{BS} + \overrightarrow{SC}$
e) $2\overrightarrow{AS} + 2\overrightarrow{BS} - 2\overrightarrow{SC}$

Problem 7. Vector addition

ABCD is a square (A and C are opposite vertices) and the intersection of its diagonals is S. Find:

a) $\overrightarrow{SA} + \overrightarrow{SC}$
b) $\overrightarrow{SA} - \overrightarrow{SC}$
c) $\overrightarrow{SB} + 2\overrightarrow{SD}$
d) $\overrightarrow{SA} - 3\overrightarrow{SC}$
e) $\overrightarrow{SA} + \overrightarrow{SB} + \overrightarrow{SC} + \overrightarrow{SD}$
f) $\overrightarrow{SA} - \overrightarrow{SB} + \overrightarrow{SC} - \overrightarrow{SD}$
g) $\overrightarrow{SA} + \overrightarrow{SB} + \overrightarrow{SC} - \overrightarrow{SD}$
h) $\overrightarrow{SA} - \overrightarrow{SB} + \overrightarrow{SC} - 2\overrightarrow{SD}$

Problem 8. Vector addition

Force F_1 has the magnitude of 5N. Force F_2 has the magnitude of 8N. They act at the same point and their vectors form an angle α = 120° . Graphically find the vector of a third force $\overrightarrow{F_3}$ so that all three forces $\overrightarrow{F_1}, \overrightarrow{F_2}, \overrightarrow{F_3}$ are in equilibrium. You may use Geogebra, but hand drawn solution is also acceptable.

2. Vector magnitude. Dot product.

Problem 9. **Magnitudes and angles**
Find the lengths of sides and interior angles of a triangle ABC if

a) A=(1,1), B=(2, -1), C=(3,2)
b) A=(1,2,-3), B=(-3,3,-2), C=(-1, 1, -1)
c) A=(2,-1,-2), B=(2,-4,-5), C=(-1, -4, -2)

Problem 10.
Given two vectors $\vec{a}$ = (3,-2) and $\vec{b}$ = (-1,5), find a vector $\vec{c}$ so that $\vec{a} \cdot \vec{c} = 17$ and $\vec{b} \cdot \vec{c} = 3$.

Problem 11. **Angle of two vectors**
Find angles formed by these vectors:

a) (-4, -2) and (-1,-3)
b) (-2,0) and (-2, $-2\sqrt{3}$)
c) (-2, 1, -9) and $(-\frac{1}{2}, 2, \frac{1}{3})$

Problem 12. **Angle of two vectors**

a) From a vertex of a square, two lines are drawn such that they split both sides opposite to the vertex in halves. Find the angle of the two lines.
b) Triangle OAB has vertices O=(0,0), A=(2a,0), B=(a, a). The median OM connects O and the midpoint of the line segment AB. Find the angle between the side OB and median OM.
c) Three consecutive vertices of a parallelogram are A=(-3,-2,0), B=(3,-3,1), C=(5,0,2). Find the fourth vertex D of the parallelogram ABCD. Find the angle between vectors $\overrightarrow{AC}$ and $\overrightarrow{BD}$.

Problem 13. **Perpendicular (orthogonal) vectors**
Find a (non-zero) perpendicular vector to given vectors and explain why they are perpendicular:

a) (-3, 4)
b) (1, -2, -3)
c) How many such vectors can you find in a)? How many such vectors can you find in b)?

Problem 14.
Unit vectors $\vec{m}$ and $\vec{n}$ form a 30°angle. Calculate $(\vec{m} + \vec{n})^2$

3. Parametric and standard equation of a line.

Problem 15. **Parametric equation of a line. Direction and normal vectors.**

Line p is given by its parametric equation: X = (1, 3) + t (2, 1), t∈R. Find:

a) Its direction vector
b) Its normal vector
c) Direction vector of a line perpendicular to p
d) Normal vector of a line perpendicular to p
e) Parametric equation of a line perpendicular to p and going through A=(4,2)
f) Intersection point of the given line p and perpendicular line you got in e)
g) Equation of the line p in the standard form.

h) Draw both lines in Geogebra and verify your answers.

Problem 16. Parametric equation of a line, line segment, ray.
Find a parametric equation of

a) a line going through point A=(3,-7) in the direction of the vector **u** = (-2,5)
b) a line going through point A=(5,0) in the direction of the vector **u** = (0,2)
c) a line going through point A=(0,0) in the direction of the vector **u** = (4,0)
d) a line AB contains points A=(4,0) and B=(2,3)
e) a ray AB with endpoint A=(-3,2) and going through B=(1,2)
f) a line segment AB with endpoints A=(0,-5) and B=(3,-3)
g) a line going through A in the direction of the vector C-B if A=(2,-5), B=(2,-4), C=(3,-1)
h) a line going through A in the direction of the vector C-B if A=(-3,0), B=(-2,-7), C=(-2,-5)

Problem 17.
Find a parametric equation of a line going through A=(-2,3) and parallel to

a) x axis
b) y axis
c) line y = x
d) Express these lines in standard form.
e) Which of these lines is not a function? Explain.

Problem 18. Problem solving with parametric equations.
Given three points A(3,-2), B(1,4), C(-1,-3), find a point D so that the line CD cuts the line segment AB in its midpoint S and |CD| = 3|CS| (the length of CD is 3x the length of CS). You may use help of Geogebra but make sure you can do all the calculations.

Problem 19. Problem solving with parametric equations.
A point moves along a line at a constant speed. At the time t=0 sec, it is positioned at A=(1,2), at the time =1 sec it is positioned at B=(3,5).

a) Find a parametric equation of the trajectory.
b) Calculate the coordinates of the point at t=15 sec.
c) When will the point be positioned at (7,11) ?

Problem 20. Parametric equation of a line in space
In space, the parametric equation of a line is composed of three separate equations: for x, y and z. Given three points A(5, 3, 6), B(-1,7,-2), C(-9,-5, 4),

a) Find parametric equation of the line AB
b) Check numerically if the point C belongs to the line AB
c) Find the parametric equation of a line going through the point A and the midpoint of BC.

Problem 21. Standard equation of a line

Find a standard equation of a line that

a) goes through A(3, -4), B(-7, 1)
b) goes through A(2, -7), B(-3, 5)
c) goes through A(6,2), B(6, -7)
d) goes through A and is perpendicular to vector $\overrightarrow{BC}$ if A=(4,-7), B=(-9,5), C=(-6,1)
e) goes through A and is perpendicular to vector $\overrightarrow{BC}$ if A=(-5,-3), B=(11,-2), C=(2,-2)

Problem 22. Problem solving with standard equations

Find a standard equation of a tangent to a circle if the point of tangency is T=(6,2) and the center of the circle is (3,-4). (Note: Tangent is always perpendicular to the radius at the point of tangency).

Problem 23. Problem solving with standard equations

Find a standard equation of a line parallel to 7x-3y+2=0 and going through point M=(3,5).

Problem 24. Intersections – given standard form

Find the coordinates of the vertices of a triangle if its three sides have equations:

7x-4y-1=0, x-2y+7=0, 2x+y+4=0

Problem 25. Intersections - given parametric form

Find the intersection of the given lines p and q with parametric equations:

a) line p: x = 3+5t, y=2-7t, $t \in \mathrm{R}$
   line q: x=1-s, y=2s, $s \in \mathrm{R}$
b) line p: x =4-2t , y=1+3t, z=-5-3t, $t \in \mathrm{R}$
   line q: x=7-7s, y=2+5s, z=-8-3s, $s \in \mathrm{R}$

Problem 26. Intersections – given parametric and standard form

a) Find the intersection of a line 4x-5y+16=0 with another line given parametrically: x=2-3t, y=-4+2t, $t \in \mathrm{R}$.
b) Do these two objects intersect? A line 5x + 8y +17 =0 and a line segment given parametrically x=2-4t, y=-3 + 2t, $t \in [0,1]$. (Note that you don't have to find the coordinates of the intersection, just decide if they intersect).

Problem 27. Distance of a point from a line

Find the distance of the point M from the line p if

a) M=(3,-7), p: 4x – 3y + 7 = 0
b) M=(1,3), p: (1, -2) + (-3,4)t
c) M=(4,3), p: 3x + 4y – 10 = 0
d) M=(2,1), p: 3x + 4y – 10 = 0
e) M=(1,0), p: 3x + 4y – 10 = 0

Problem 28. Distance of a point from a line

Explain why the lines 2x – 3y – 6 = 0 and 4x – 6y -25 = 0 are parallel and find their distance.

Problem 29. Distance of a point from a line
Find the height BD of a triangle ABC with vertices A=(-3,0), B=(2,5), C=(3,2)

Problem 30. Linear combinations
Express the given vector $\vec{d}$ as a linear combination of the vectors $\vec{a}$ and $\vec{b}$ (and $\vec{c}$, if given). If it is not possible, explain why.

a) $Express\ \vec{d} = \begin{bmatrix} 12 \\ 7 \end{bmatrix}\ as\ a\ linear\ combination\ of\ \vec{a} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}\ and\ \vec{b} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$
b) $Express\ \vec{d} = \begin{bmatrix} -2 \\ 7 \end{bmatrix}\ as\ a\ linear\ combination\ of\ \vec{a} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}\ and\ \vec{b} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$
c) $Express\ \vec{d} = \begin{bmatrix} -2 \\ 7 \end{bmatrix}\ as\ a\ linear\ combination\ of\ \vec{a} = \begin{bmatrix} -2 \\ -3 \end{bmatrix}\ and\ \vec{b} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$
d) $Express\ \vec{d} = \begin{bmatrix} -4 \\ 12 \end{bmatrix}\ as\ a\ linear\ combination\ of\ \vec{a} = \begin{bmatrix} -2 \\ -3 \end{bmatrix}\ and\ \vec{b} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$
e) $Express\ \vec{d} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}\ as\ a\ linear\ combination\ of\ \vec{a} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\ and\ \vec{b} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\ and\ \vec{c} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Problem 31. Normalized vectors
a) Explain what a normalized vector is.
b) Normalize vector $\begin{bmatrix} 8 \\ 6 \end{bmatrix}$
c) Normalize vector $\begin{bmatrix} 8 \\ 4 \end{bmatrix}$

Problem 32. Normalized vectors
Modify the standard equations of the following lines so that the left side of the equation outputs the distance of a point given by its coordinates from the line.

a) $3x + 3y - \frac{1}{3} = 0$
b) $14x - 7y + 28 = 0$

Problem 33. Linear transformations*[1]
Exercise 2.2.11 (pg. 63) Let T : $R^2 \rightarrow R^2$ be a transformation. In each case find the matrix.

a) T is a reflection in the y axis.
b) T is a reflection in the line y = x.
c) T is a reflection in the line y = −x.
d) T is a counter clockwise rotation through π/2 .
e) T is a counter clockwise rotation through angle 30°
f) T is a <u>clockwise</u> rotation through angle 135°

[1] Problems marked with an asterisk and starting with "Exercise" are taken from the 2019 edition of open textbook "Linear Algebra with Applications" by W. Keith Nicholson. The textbook and student solution manual are on my website.

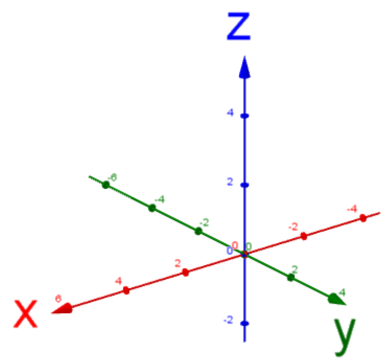
Problem 34. Linear transformations*

Exercise 2.2.12 (pg.63) The transformation T : $R^3 \to R^2$ is defined by $T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$. for all $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ in R^3 . Find the matrix of T.

Problem 35. Linear transformations*

Exercise 2.2.13 (pg 63) Let T : $R^3 \to R^3$ be a transformation. In each case find its matrix.

a) T is a reflection in the x−y plane.
b) T is a reflection in the y−z plane
c) T is a rotation about z-axis 90° counterclockwise, . when looking from “above”, i.e. from the positive half of the z axis. Axes are oriented as in this picture. This problem is not from the book.



Problem 36. Orthogonal projection

Find Orthogonal projection of a vector $\begin{bmatrix} 2 \\ 8 \end{bmatrix}$ onto line -3x + 5y = 0.

Problem 37. Orthogonal projection - applications *

A railroad car left on an east-west track without its brake is pushed by a wind blowing toward the northeast at fifteen miles per hour; what speed will the car reach? ()

Problem 38. Orthogonal Projection *

Exercise 4.2.10 on page240

Problem 39. Orthogonal Decomposition *

Exercise 4.2.11 on page 240

Problem 40. Matrix operations*

Exercise 2.1.1 through Exercise 2.1.4 on pages 44, 45.

Problem 41. Matrix multiplication*

Exercise 2.3.1 through Exercise 2.3.5 on pages 76, 77.

a) Find a 2x2 matrix A such that $A^2 = 0$. Does A have to be a zero matrix?
b) Find 2×2 matrices A and B such that AB = 0 but BA $\neq$ 0

Problem 42. Matrix multiplication – applications*

Exercise 2.3.25 on page 78

Problem 43. Composition of Linear transformations as matrix multiplication

In Exercise 2.6.12 on page 116, you are asked to replace two consecutive transformations with a single one (rotation or reflection). Think about why this can be solved simply by multiplying two transformation matrices (the order matters!). Then answer the questions.

Problem 44. Determinants

Exercises 3.1.1 through 3.1.3. For 3.1.3 on page 155. Keep in mind that only square matrices have determinants. Also you can always check your calculations in Geogebra (command Determinant).

Problem 45. Determinant - properties

Use your knowledge of cofactor expansion to reason about the following situations. Justify your answer:

a) Explain what will happen to the value of determinant if one row (or column) is multiplied by a scalar.

Problem 46. Determinants - applications

Find the area (volume) of region (parallelogram or parallelepiped) defined by the vectors:

a) $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ *and* $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$

b) $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$

Problem 47. Matrix Rank

What is the rank of these matrices? Are they full rank or rank deficient?

a) $\begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$ b) $\begin{bmatrix} 1 & 4 \\ 3 & 12 \end{bmatrix}$ c) $\begin{bmatrix} 1 & 3 & 5 \\ 3 & 1 & 7 \end{bmatrix}$

Problem 48. Cramer's rule

a) Derive the Cramer's rule (showing the solution for one variable, for example $y = \frac{|A_2|}{|A|}$ will do).
b) What can the determinant of the coefficient matrix A tell us? (specifically, what does |A| = 0 indicates?)

Solve using the Cramer's rule:

c)

3x + 2y = 7

4x – 5y = 40

d)

5x + 2y = 4

7x + 4y = 8

e) For what values of a has the system a unique solution? What is the solution?

ax – 3y = 1

ax – 2y = 2

Problem 49. Gaussian Elimination

a) $2x - 3y + z - 2 = 0$
$x + 5y - 4z + 5 = 0$
$4x + y - 3z + 4 = 0$

b) $2x - 4y + 3z = 1$
$x - 2y + 4z = 3$
$3x - y + 5z = 2$

c) $3x + 2y - z = 0$
$2x - y + 3z = 0$
$x + 3y - 4z = 0$

d) $x + 2y + 3z = 4$
$2x + y - z = 3$
$3x + 3y + 2z = 7$

e) $2x - y + 3z = 0$
$x + 2y - 5z = 0$
$3x + y - 2z = 0$

f) $x - 2y + z = 4$
$2x + 3y - z = 3$
$4x - y + z = 11$

g) $x + 2y + 3z = 4$
$2x + 4y + 6z = 3$
$3x + y - z = 1$

h) $x + y - z = 2$
$y + 4z = 7$
$x + 2y + 3z = 5$

i) $2x + y = 8$
$y + 2z = 16$
$2x + u = 20$
$-u = 0$

j) $x + y + z + u - v = 26$
$x + y + z - u - v = 16$
$x + y - z - u - v = 6$
$x - y - z - u - v = -10$
$3x + 5y - 6z - 5u - 7v = 0$

k) $2x - 4y - z = 2$
$8y - 4x + 2z = -4$
$x - 2y - 0.5z = 1$

More exercises for solving the linear systems are in Linear Algebra: A First course by K.Kuttler (see the link on my website), problems 1.2.27-1.2.40 on pages 45-47

Problem 50. Linear systems – Elimination **[2]

(Page 42 – 43). A linear system is called consistent, if the system has at least one solution. Consider the following augmented matrices in which $*$ denotes any number (including zero) and $\blacksquare$ denotes a nonzero number. Determine whether the given linear systems are consistent. If consistent, is the solution unique?

a.
$$\left[\begin{array}{ccccc|c} \blacksquare & * & * & * & * & * \\ 0 & \blacksquare & * & * & 0 & * \\ 0 & 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * \end{array}\right]$$

b.
$$\left[\begin{array}{ccc|c} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & \blacksquare & * \end{array}\right]$$

c.
$$\left[\begin{array}{ccccc|c} \blacksquare & * & * & * & * & * \\ 0 & \blacksquare & 0 & * & 0 & * \\ 0 & 0 & 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * \end{array}\right]$$

d.
$$\left[\begin{array}{ccccc|c} \blacksquare & * & * & * & * & * \\ 0 & \blacksquare & * & * & 0 & * \\ 0 & 0 & 0 & 0 & \blacksquare & 0 \\ 0 & 0 & 0 & 0 & * & \blacksquare \end{array}\right]$$

Problem 51. Linear systems

a) Suppose a system of equations has fewer equations than variables. Will such a system necessarily be consistent? If so, explain why and if not, give an example which is not consistent.

b) If a system of equations has more equations than variables, can it have a solution? If so, give an example and if not, tell why not.

Problem 52. Linear systems

(Page 43) Find h such that the following augmented matrices represent linear system that are

a) inconsistent $\left[\begin{array}{cc|c} 2 & h & 4 \\ 3 & 6 & 7 \end{array}\right]$ b) consistent $\left[\begin{array}{cc|c} 1 & h & 3 \\ 2 & 4 & 6 \end{array}\right]$ c) consistent $\left[\begin{array}{cc|c} 1 & 1 & 4 \\ 3 & h & 12 \end{array}\right]$

---

[2] Problems marked with two asterisks are taken from the Open Textbook: Linear Algebra: A First course by K.Kuttler (see the link on my website).

Problem 53. Linear systems

(Page 43-44) Choose h and k such that the augmented matrix shown has each of the following:

(i) one solution

(ii) no solution

(iii) infinitely many solutions

a)

$$\left[\begin{array}{cc|c} 1 & h & 2 \\ 2 & 4 & k \end{array}\right]$$

b)

$$\left[\begin{array}{cc|c} 1 & 2 & 2 \\ 2 & h & k \end{array}\right]$$

Problem 54. Linear systems – Applications

(Page 42) Four times the weight of Gaston is 150 pounds more than the weight of Ichabod. Four times the weight of Ichabod is 660 pounds less than seventeen times the weight of Gaston. Four times the weight of Gaston plus the weight of Siegfried equals 290 pounds. Brunhilde would balance all three of the others. Find the weights of the four people.

Problem 55. Standard and parametric equation of a plane

a) Find a parametric equation of the plane $3x – 6y + z – 12 = 0$

b) Find a standard equation of the plane given parametrically: $X = (2,1,2) + (1,2,3)t + (-2, 1, 0)s$

c) Find a parametric equation of the plane given by three points: A(1,3,-1), B(2,3,3), C(-2,-5,-7)

d) Find a parametric equation of the plane given by three points: A(-1,-1,0), B(1,1,2), C(2,2,3)

e) Given points M(0, -1, 3) and N(1,3,5), find a standard equation of the plane going through the point M and perpendicular to the vector N – M.

f) Check if the point L(0,1,2) belongs to the plane given by the points A(1,0,3), B(-2,3,0), and C(-3,-2,4).

g) Find an equation of the plane (any form) going through the axis z and point P(2,-4,3)

Problem 56. Inverse matrix **

(Page 95) Find A^{-1} if possible. If A^{-1} does not exist, explain why.

a)
b)
c)
d)

$$A = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix} \quad A = \begin{bmatrix} 0 & 1 \\ 5 & 3 \end{bmatrix} \quad A = \begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix} \quad A = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$$

(Page 95-96)

e)
f)
g)

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 1 & 0 & 2 \end{bmatrix} \quad A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 3 & 4 \\ 1 & 0 & 2 \end{bmatrix} \quad A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 4 & 5 & 10 \end{bmatrix}$$

Problem 57. Inverse Matrix - Solving linear systems **

(Page 96) Using the inverse of the matrix, find the solution to the systems:

(a)

$$\begin{bmatrix} 1 & 0 & 3 \\ 2 & 3 & 4 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

(b)

$$\begin{bmatrix} 1 & 0 & 3 \\ 2 & 3 & 4 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix}$$

Now give the solution in terms of a, b, and c to the following:

$$\begin{bmatrix} 1 & 0 & 3 \\ 2 & 3 & 4 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

Problem 58. Matrix rank**

(Page 47) Find the rank of these matrices.

a)

$$\begin{bmatrix} 4 & -16 & -1 & -5 \\ 1 & -4 & 0 & -1 \\ 1 & -4 & -1 & -2 \end{bmatrix}$$

b)

$$\begin{bmatrix} 3 & 6 & 5 & 12 \\ 1 & 2 & 2 & 5 \\ 1 & 2 & 1 & 2 \end{bmatrix}$$

c)

$$\begin{bmatrix} 0 & 0 & -1 & 0 & 3 \\ 1 & 4 & 1 & 0 & -8 \\ 1 & 4 & 0 & 1 & 2 \\ -1 & -4 & 0 & -1 & -2 \end{bmatrix}$$

More exercises are in the book** , 1.2.49-1.2.56, page 48-49.

Problem 59. Linear dependence

Determine if the following sets of vectors is linearly independent. What is the rank of these matrices and what does it say about linear independence?

a) $\begin{bmatrix} 1 & 2 & 2 \\ 2 & 4 & 1 \\ 4 & 2 & 1 \end{bmatrix}$

b) $\begin{bmatrix} 2 & 4 & 2 \\ 2 & 3 & 1 \\ 3 & 1 & -2 \end{bmatrix}$

c) $\begin{bmatrix} 2 & 4 & 2 & 1 \\ 2 & 3 & 1 & 5 \\ 3 & 1 & -2 & 2 \\ 1 & 2 & 6 & 9 \end{bmatrix}$

d) $\begin{bmatrix} 2 & 4 & 2 & 1 \\ 2 & 3 & 1 & 5 \\ 5 & 5 & 0 & 3 \\ 0 & 1 & 1 & -4 \end{bmatrix}$

e) $\begin{bmatrix} 4 & 7 & 3 & 6 \\ 2 & 3 & 1 & 5 \\ 2 & 4 & 2 & 1 \\ 0 & 1 & 1 & -4 \end{bmatrix}$

f) What can you say about linear independence of row vectors in Problem 58?

Problem 60. Matrix rank **

Suppose A is an m×n matrix. Explain why the rank of A is always no larger than min(m,n) .

Problem 61. Matrix rank and solutions to linear systems**

(Page 49) Consider the matrix equation AX = B. If the following situations are possible, describe the solution set. That is, tell whether there exists a unique solution, no solution or infinitely many solutions. Here, [A|B] denotes the augmented matrix.

a) A is a 5×6 matrix, rank(A) = 4 and rank[A|B] = 4.
b) A is a 3×4 matrix, rank(A) = 3 and rank[A|B] = 2.

c) A is a 4×2 matrix, rank(A) = 4 and rank[A|B] = 4.
d) A is a 5×5 matrix, rank(A) = 4 and rank[A|B] = 5.
e) A is a 4×2 matrix, rank(A) = 2 and rank[A|B] = 2.

Problem 62. Basis of a vector space

Determine if the following set of vector is a basis for the given space. Explain.

a) $\begin{bmatrix}1\\-2\end{bmatrix},\begin{bmatrix}-2\\4\end{bmatrix}$ in R^2

b) $\begin{bmatrix}1\\-2\end{bmatrix},\begin{bmatrix}-2\\-4\end{bmatrix}$ in R^2

c) $\begin{bmatrix}1\\-2\end{bmatrix},\begin{bmatrix}-2\\4\end{bmatrix},\begin{bmatrix}3\\5\end{bmatrix}$ in R^2

d) $\begin{bmatrix}1\\2\\4\end{bmatrix},\begin{bmatrix}2\\4\\2\end{bmatrix},\begin{bmatrix}2\\1\\1\end{bmatrix}$ in R^3

e) $\begin{bmatrix}2\\2\\3\end{bmatrix},\begin{bmatrix}4\\3\\1\end{bmatrix},\begin{bmatrix}2\\1\\-2\end{bmatrix}$ in R^3

f) $\begin{bmatrix}1\\2\\4\end{bmatrix},\begin{bmatrix}2\\4\\2\end{bmatrix}$ in R^2

Problem 63. Linear Transformation given by arbitrary vectors

Find the matrix of a linear transformation T such that

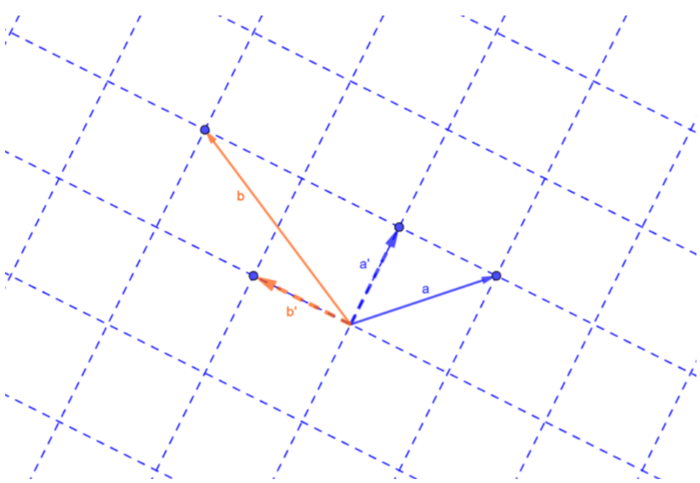
a) $T\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}-1\\0\end{bmatrix}$ and $T\begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}-2\\3\end{bmatrix}$

b) $$T\begin{bmatrix}1\\2\\-6\end{bmatrix} = \begin{bmatrix}5\\1\\3\end{bmatrix}$$
$$T\begin{bmatrix}-1\\-1\\5\end{bmatrix} = \begin{bmatrix}1\\1\\5\end{bmatrix}$$
$$T\begin{bmatrix}0\\-1\\2\end{bmatrix} = \begin{bmatrix}5\\3\\-2\end{bmatrix}$$

More exercises are in the book** , 9.9.4-9.9.7, page 552-553.

Problem 64. Change of basis

This image shows a space with basis a' and b' (dashed vectors, system 1). Vectors a and b (solid line) are another basis of this space (system 2).



a) Find the change of basis matrix that converts coordinates of a vector in system 2 into system 1.
b) Vector **v** has the coordinates $\begin{bmatrix}2\\-1\end{bmatrix}$ in system 2. What will be its coordinates in system 1?
c) Find the change of basis matrix that converts coordinates of a vector in system 1 into system 2.

d) Vector **v** has the coordinates $\begin{bmatrix} -3 \\ 3 \end{bmatrix}$ in system 1. What will be its coordinates in system 2?

Problem 65. Eigen vectors and eigen values

(Page 360) Find the eigenvalues and eigenvectors :

a) $\begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}$

b) $\begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$

c) $\begin{bmatrix} 2 & -2 \\ 0 & 2 \end{bmatrix}$

d) $\begin{bmatrix} 1 & 0 & 2 \\ 2 & 2 & 2 \\ 3 & 0 & 2 \end{bmatrix}$