# Lab 7: Linking Rates of Change: The Chain Rule

## Laboratory Experience

So far we have noticed certain patterns for finding derivatives of basic functions such as power functions, trigonometric functions, logarithmic functions, etc. However, what happens when we combine these basic functions through composition? As we have seen, the derivative of a function such as $f(x) = x^3 - x^2 + 3x - 5$ can be obtained using the power rule to get $f'(x) = 3x^2 - 2x + 3$. However, most functions we use can be thought of as compositions of more simple functions. For example, $f(x) = e^{x^2}$ could be thought of as the composition of two more basic functions $f(x) = g(h(x))$ where $g(x) = e^x$ and $h(x) = x^2$. We will now investigate what happens when we have two composed functions and wish to find the derivative.

When we talk about taking derivatives, we frequently use the phrase, "with respect to". Why do we bother appending this phrase to our language? We will explore this phrase during this investigation and try to answer this question.

1. To begin, what does a derivative represent? What does rate of change mean? Consider the function $f(x) = 3x$. Make a table of values for *f* below.

| $x$ | $f(x)$ |
|---|---|
| 1 | |
| 2 | |
| 3 | |
| 4 | |
| 5 | |
| 6 | |
| 7 | |
| 8 | |
| 9 | |

   a. Describe how the output of *f* changes as *x* changes by 1 each time.

b. Now suppose that we attach a rate of change to the values of $x$. Suppose that the values of $x$ in the table are set to change by 1 unit every second. How fast would $f$ change every second?

c. Suppose we program the calculator to input the given values for $x$ into $f(x)$ at half of the rate as before (2 seconds between entries). How fast would $f$ change every 2 seconds? Every 1 second?

d. What if we programmed the calculator to input the given values for $x$ into $f(x)$ twice as fast as the original rate (every half second)? How fast would $f$ change every second?

e. Describe any pattern you see in the relationship between the rate of change of $f$ and the rate of change of $x$.

2. Suppose you have a pump designed to pump water. The cylinder for the pump holds 2 liters of water and as you pull the handle up and down, the cylinder first fills with water and then the water is forced out (see diagram below).


a. Describe the rate of water that is pumped with each up and down motion of the handle. Give units per pump.

b. Now imagine the pump is being used so that and up and down motion of the handle occurs every second. What is the rate of water flow out of the pump per second?

c. Now imagine the pump is being used so that an up and down motion of the handle occurs every half second. What is the pump rate (i.e. rate of up and down motions per second)? What is the rate of water flow out of the pump per second?

d. Now imagine the pump is being used so that an up and down motion of the handle occurs every three seconds. What is the pump rate (i.e. rate of up and down motions per second)? What is the rate of water flow out of the pump per second?

e. How do the pump rate (i.e. rate of up and down motions per second) and the per pump water flow rate combine to give the total rate of water flow per second?

In the last question, we looked at what happens when the values of $x$ are changing at a constant rate. This prompts the question of how would $f$ change if $x$ were changing at a variable rate? To investigate this question, we will hold $f$ fixed as in the last question and replace $x$ with a function of time, call it $x(t)$. Now we have created a new function that is a composition of two functions, $f(x(t))$. Consider $x(t)=t^2$ and $f(x)=3x$. If we use the same analogy of a time-dependent $x$, notice that the rate at which $x$ changes is also changes.

3. Consider the function $h(x)=(x^3-3x^2+x+2)^3$. This function can be thought of as the composition of two functions $f(x)=x^3$ and $g(x)=x^3-3x^2+x+2$ where $h(x)=f(g(x))$. Upon first inspection, the obvious solution for finding $h'$ is to simply apply the power rule to obtain $h'(x)=3(x^3-3x^2+x+2)^2$.
   a. Use your numerical derivative command to investigate the slope of the function $h$ at various values of $x$. Record your data in the table provided (See Figure 1).

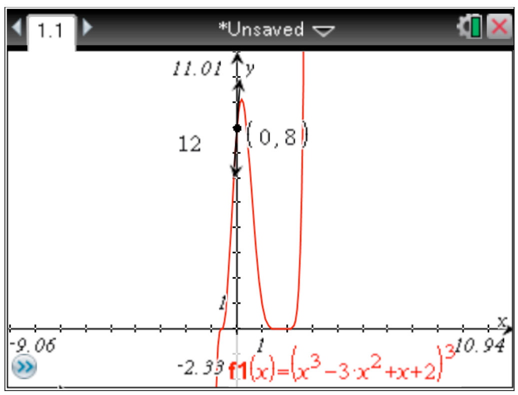


Figure 1

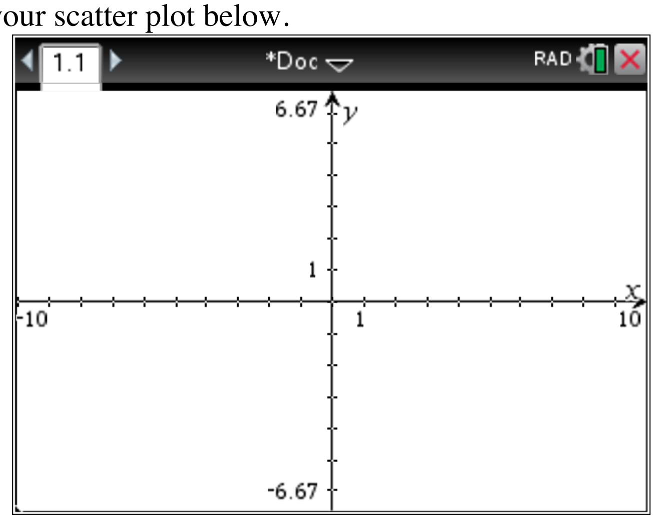
| $x$ | Approximation of $h'(x)$ from dy/dx | Value of $3(x^3-3x^2+x+2)^2$ | Correction Factor |
|---|---|---|---|
| 0 | 12 | | |
| 0.25 | | | |
| 0.5 | | | |
| 0.75 | | | |
| 1 | | | |
| 1.25 | | | |
| 1.5 | | | |
| 1.75 | | | |
| 2.25 | | | |
| 2.5 | | | |

b. Now record the values for your assumed derivative function $h'(x) = 3(x^3 - 3x^2 + x + 2)^2$ for the same *x* values. Does the value you obtain from the numerical derivative command match the value you obtain from the assumption that $h'(x) = 3(x^3 - 3x^2 + x + 2)^2$? Explain.

c. In the fourth column record the factor you could multiply the third column (proposed derivative) by to get the value appearing in the second column (actual derivative). In other words, for a given *x*, what do you need to multiply $3(x^3 - 3x^2 + x + 2)^2$ by to get the numerical approximation for the slope of *h* found by using the dy/dx command? Record the values of these "correction factors" in the fourth column of the table. Are there any values that might cause you concern?

4. Now that we have calculated the values we need to "fix" the assumed derivative function, in other words, try to find a relationship between *x* and the correction factors. Use the data to construct a scatter plot of *x* and the *correction factors*.
On a TI-Nspire CX CAS, after you have collected your data, enter the data into a **Lists & Spreadsheets** page and perform a scatter plot. Place your *x* data in the first column and your *correction factor* data in the second column. Label the columns at the very top so that you can refer to them in the **Graphs** page when doing the scatter plot.

a. Graph your scatter plot below.



b. What do you notice about the data pattern? Do you recognize a function that fits the data? Try graphing several functions until you have one that appears to fit the data. What is your function? Compare it with other groups in the class. Record your observations/answers to these questions.

c. Using your data from the **List & Spreadsheets** page, use the curve-fitting capabilities of your graphing calculator to find a function that fits your scatter plot. For example, if you thought a quadratic would fit your data, you would choose the quadratic regression command and store the regression equation in $f2(x)$ on the TI-Nspire. Describe any relationship between your function for the "correction factor" and the original function $h$.

5. To test your conjecture about the relationship between the function for the correction factor and the original function, repeat the above experiment for the function $f(x) = (\sin x)^3$. Here the assumed derivative might be $f'(x) = 3(\sin x)^2$ since $\frac{d}{dx}(x^3) = 3x^2$.

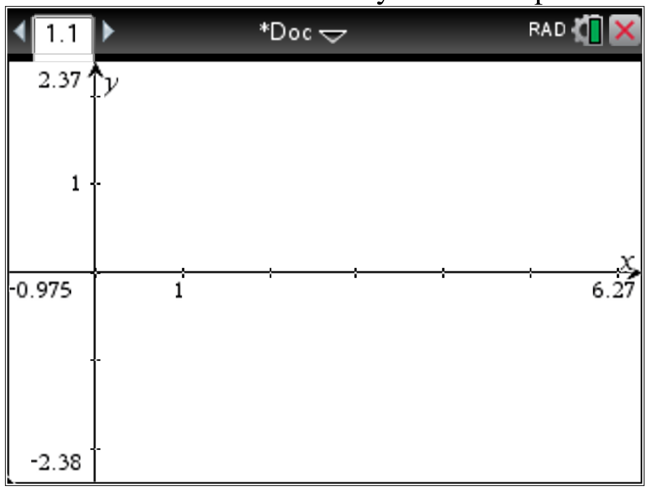
   a. Use your numerical derivative command to investigate the slope of the function *h* at various values of *x*. Record your data in the table provided.

   b. Now record the values for your assumed derivative function $f'(x) = 3(\sin x)^2$ for the same *x* values. Does the value you obtain from the numerical derivative command match the value you obtain from the assumption that $f'(x) = 3(\sin x)^2$? Explain.

   c. Find a "correction factor" as you did earlier in this lab and record values in the fourth column.

| $x$ | Approximation of $f'(x)$ from dy/dx | Value of $3(\sin x)^2$ | Correction Factor |
|---|---|---|---|
| 0 | | | |
| 0.5 | | | |
| 1 | | | |
| 1.5 | | | |
| 2 | | | |
| 2.5 | | | |
| 3 | | | |
| 3.5 | | | |
| 4 | | | |
| 4.5 | | | |
| 5 | | | |

d. Sketch your scatter plot for the correction factors as a function of $x$. Label your axes and then find a function that fits your scatter plot and sketch it.




e. Did your conjecture from the first part of the lab hold for the function $f$? Explain.

6. Based on the data you have obtained thus far, write a general statement for finding the derivative of a composed function $h$ where $h(x) = f(g(x))$.