Lab 6: Derivatives of Exponential Functions

Laboratory Experience

1. In this lab we will explore the derivative of exponential functions. We will start with probably the most familiar exponential function to most students, $f(x) = 2^x$.
   a. It is common to see students take the derivative of this using the power rule and get that $f'(x) = x2^{x-1}$. Have your computer plot both $f(x) = 2^x$ and $f'(x) = x2^{x-1}$. Consider what the graph of $f(x) = 2^x$ would look like if its derivative were negative. Explain why $f'(x) = x2^{x-1}$ **cannot** be the derivative of $f(x) = 2^x$.

   b. To get a handle on what the actual derivative of this function might be, let's go back to the definition of the derivative. Consider that $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$. Write and expression for $\frac{d}{dx}(2^x)$ based on the definition of derivative and show the intermediate steps that lead to the fact that $\frac{d}{dx}(2^x) = 2^x \cdot \lim_{h \to 0} \frac{2^h - 1}{h}$.

c. Use your computer to compute $\lim_{h \to 0} \frac{2^h - 1}{h}$ for various small values of h (here use the decimal approximation by pressing [ctrl] [enter] rather than just [enter]). Does this limit appear to exist? Notice that the derivative of this function appears to be the original function itself simply multiplied by this constant.

d. Repeating the process used in parts (b) and (c), find an expression for the derivative of $f(x) = 3^x$. Notice again that the derivative of this function appears to be the original function itself simply multiplied by a slightly different constant.

e. Algebraically show that in general, $\frac{d}{dx}\left(a^x\right) = a^x \cdot \lim_{h \to 0} \frac{a^h - 1}{h}$.

2. Wouldn't it be nice if we could find a function that was its own derivative? Since the derivatives of exponential functions are simply the original exponential function multiplied by the "fudge factor" $\lim_{h \to 0} \frac{a^h - 1}{h}$, we could accomplish this by finding a base, *a*, such that $\lim_{h \to 0} \frac{a^h - 1}{h} = 1$. This would give us $\frac{d}{dx}\left(a^x\right) = a^x \cdot \lim_{h \to 0} \frac{a^h - 1}{h} = a^x \cdot 1 = a^x$ for this special *a*. Notice that from our previous experiments, for $a = 2$ the value of $\lim_{h \to 0} \frac{a^h - 1}{h}$ was less than 1 while for $a = 3$ the value was greater than 1. Experiment with other values for *a* to approximate a value for, *a*, such that $\lim_{h \to 0} \frac{a^h - 1}{h} = 1$. What is your value for *a*? Have you ever seen this number before? Is so, explain it significance.

3. Now that we have determined a base for an exponential function that yields a function who is its own derivative, the next natural question might be, "Is there a pattern to the 'fudge factors' we have observed with various values of *a*?" We will now explore this question.

   a. Using various values for *a* between 1 and 10, record your results for $\lim_{h\to 0}\frac{a^h-1}{h}$ in the following table. (*Here use the decimal approximation by pressing* ctrl enter *rather than just* enter.)

a	$\lim_{h\to 0}\frac{a^h-1}{h}$
2	
3	
4	
5	
6	
7	
8	
9	

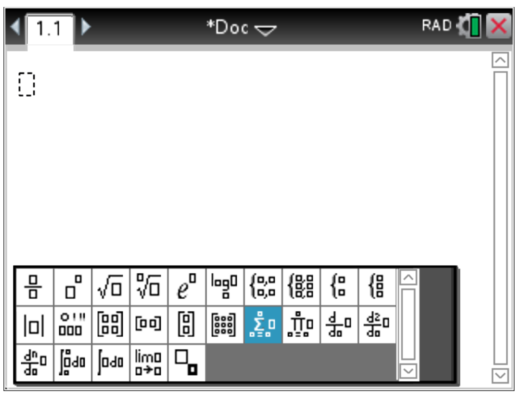
   b. Produce a scatter plot of your data from part (a).

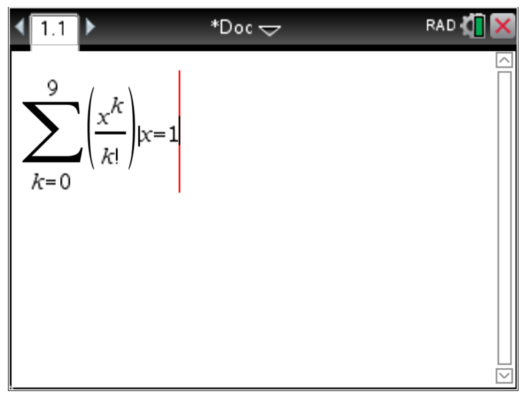
   c. Using the curve-fitting capabilities of your CAS, find a function that describes the relationship between *a* and $\lim_{h\to 0}\frac{a^h-1}{h}$, and sketch it on top of your scatter plot.

d. Using your observations thus far, give an algebraic expression for the general form of $\frac{d}{dx}\left(a^x\right)$.

4. We can also express an exponential function using series. Consider the infinite series $1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots$.

   a. Using your sum command and setting $x=1$, sum the first 10 terms of this series. What number do you get? (Note that the factorial symbol (!) is located under the **Probability** menu or by pressing [?!▸])





   b. Repeat this using $x=2$. Try to discover where this number comes from. How is it related to the value of *a* that you found earlier such that $\lim_{h\to 0}\frac{a^h-1}{h}=1$.

c. In general, what function does the series $1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots$ appear to be describing?

d. Compute the derivative of the infinite series $1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots$ term-by-term using the power rule. What does it equal? Based on your answer to 4(c), is this surprising? Explain.

5. For what value of a are the graphs of $y=a^x$ and $y=\log_a x$ tangent to each other? You may wish to note that your CAS has a $\log_a x$ key by pressing [ctrl] [log]. Use your computer algebra system to approximate a solution before trying to solve it analytically.