

## Lab 2: Introduction to the Limit of a Function

### Pre-Lab

To begin, we need to make sure you are familiar with some of the basic functionalities of your Computer Algebra System (CAS) that you will use throughout the exploration. To achieve this please perform the following tasks.

Make sure you are able to evaluate a function using your CAS from a numerical and graphical environment.

- Define a function  $f(x) = x^2$  on your CAS. Now evaluate your function on the home screen for  $x = 1, 2.4$ , and  $3$ .
- Use the **Table** feature of your CAS to evaluate  $f$  beginning at  $x = 3$  and continuing to  $5$  stepping by  $0.2$ .
- Evaluate  $f$  using the “such that” key, “|”, so that it can be edited easily. Consider the following command line:  $f(x)|x = 3$ .
- Sketch a graph of  $f$  and trace directly to the point  $(6.231, f(6.231))$ .

Make sure you can use your CAS to perform algebraic manipulations.

- Expand the expression  $(2x - 3)^7$ .
- Factor the expression  $2x^5 - x^4 - x^3 - x^2 - x - 3$

### Laboratory Experience

In this exploration we will examine the behavior of functions near a specific point. We cannot always evaluate a function for a specific value (e.g.  $f(x) = \frac{\sin(x)}{x}$  at  $x = 0$ ) and so we need to rely on other methods to understand a function’s behavior as we get nearer to that point. This lab will explore the concept of *limit* as it is related to function behavior.

- Consider the function,  $f$ , defined by the rule  $f(x) = \frac{x^5 - 1}{x - 1}$ .
  - As you evaluate  $f$  at values close to  $2$ , but slightly smaller than  $2$ , what do you think the resulting value of  $f(x)$  is doing as you get close to  $2$ , but not equal to  $2$ ?
  - Perform the same procedure as in part (a) for values of  $x$  slightly larger than  $2$ . Discuss your observations.

c. Evaluate  $f$  at  $x = 2$ . What do you notice?

d. Graph  $f$  and trace directly to  $x = 2$ . What do you get? Make a sketch of your graph stating the viewing window.



The value you have been exploring is called the *limit* of the function and in this case would be denoted as  $\lim_{x \rightarrow 2} f(x)$ . The basic idea is that the limit is the value the function "settles" on as you get closer and closer to an input,  $a$ , without necessarily reaching  $a$ .

2. Use the same function as in problem 1. This time consider what happens near  $x = 1$ .
- Compute the numerator and denominator of  $f$  separately and then the quotient. You may want to use your **Table** feature or a spreadsheet here defining three separate functions for the numerator, denominator, and quotient respectively (this is provided in the *.tms* file). What are your conclusions for  $\lim_{x \rightarrow 1} f(x)$ ? State any observations you have made.

$x$	$x^5 - 1$	$x - 1$	$\frac{x^5 - 1}{x - 1}$
0.9			
0.99			
0.999			
0.9999			
1.1			
1.01			
1.001			
1.0001			

b. On the **Calculator** page, now try to directly evaluate  $f$  as you did in part (c) of question 1, this time for  $x = 1$ . Discuss your result.

c. Graph  $f$  and trace directly to  $x = 1$ . What do you get? Make a sketch of your graph stating the viewing window.



d. Factor the numerator of the function. Discuss your observations.

The result of this attempt raises the question, “When can you find the limit of a function by direct evaluation of the function?” The answer to this question is connected to the concept of *continuity*. Think about what the word *continuous* means to you.

e. How is your understanding of the word *continuous* related to the graph from part (c)?

- f. If a function does not have a value at a specified point, can it have a limit at that point? Explain your reasoning as it is related to your attempts to evaluate the function at points near the desired value in questions 1 and 2 as well as your earlier conclusions for  $\lim_{x \rightarrow 2} f(x)$  and  $\lim_{x \rightarrow 1} f(x)$ .
3. Using experimentation on your CAS, attempt to evaluate the following limits. Use either your graphing utility or direct evaluation of the function at points near the desired point.
- a.  $\lim_{x \rightarrow 0} (1+x)^{\frac{1}{x}}$  [You may recognize this number as a familiar constant]

b.  $\lim_{x \rightarrow 0} \frac{\sin(5x)}{x}$

c.  $\lim_{x \rightarrow 3} \frac{x^2 - 9}{x - 3}$

d.  $\lim_{x \rightarrow 2} f(x)$  where  $f$  is defined by the following:  $f(x) = \begin{cases} 2x & \text{for } x < 2 \\ 6 & \text{for } x = 2 \\ \frac{x^3 - 2x^2}{x - 2} & \text{for } x > 2 \end{cases}$

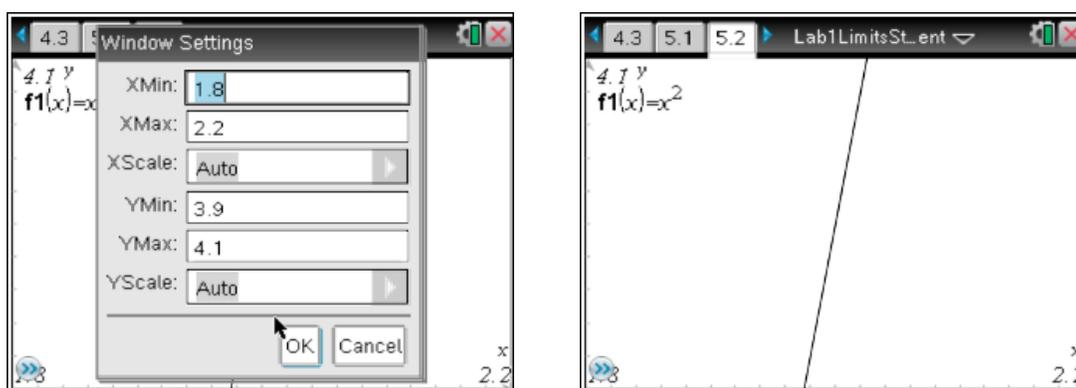
4. In mathematics there are times that limits do not exist. For the following limits, try to determine if they do indeed exist. If they exist give the limit as well as your justification for it. If it does not exist, give your evidence and reasoning for your conclusion.

a.  $\lim_{x \rightarrow 5} \frac{|x - 5|}{x - 5}$

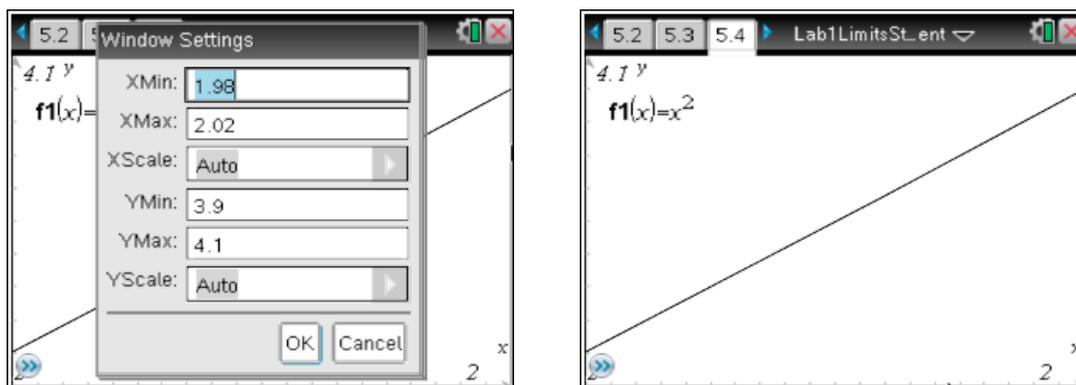
b.  $\lim_{x \rightarrow 2} \frac{x^2 - 9}{x - 2}$

c.  $\lim_{x \rightarrow 0} \frac{\sin(x)}{x}$

In this section you will explore the formal definition of limit. The formal definition of limit is sort of like a game played between two people. First make a conjecture for the limit of the function as  $x \rightarrow a$  and call it  $L$ . Once you have an idea as to what the limit of the function is, the first player will select a value called  $\epsilon > 0$ . Once s/he has done this, the challenge is for the second player to select a value called  $\delta > 0$  so that the symmetric window about the point  $(a, L)$  defined by the viewing window  $[a - \delta, a + \delta]$  by  $[L - \epsilon, L + \epsilon]$  captures the graph of  $f$  and does not allow it to escape out of the top or bottom of the viewing window when the y-min setting is  $L - \epsilon$  and the y-max setting is  $L + \epsilon$ . For example, suppose the function is  $f(x) = x^2$  and  $a = 2$ . If we assume  $\lim_{x \rightarrow 2} x^2 = 4$  and the first player chooses  $\epsilon = 0.1$ , a choice of  $\delta = 0.2$  does not satisfy the needed conditions as can be seen by the screen images below. The graph escapes out of the top and bottom of the screen.

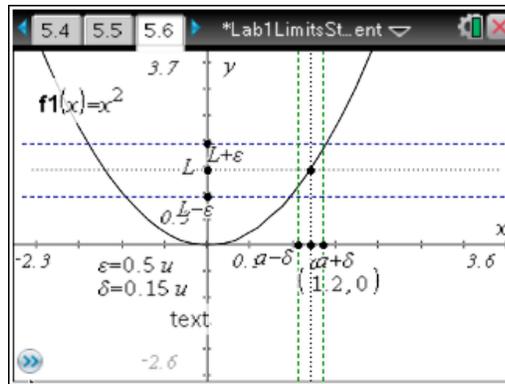
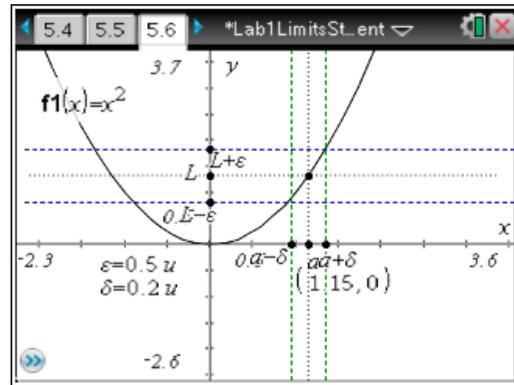
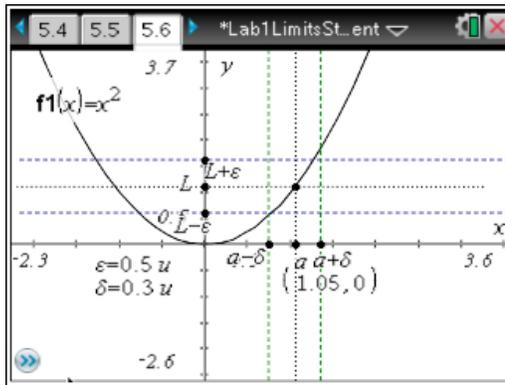


However, a choice of  $\delta = 0.02$  does not allow the graph to escape.



We can also look at this idea from the standpoint of a target. If we wish to get a certain “closeness”, call it  $\epsilon$ , to the bulls eye,  $L$ , from what closeness to  $a$ , call it  $\delta$ , do we have to shot to guarantee we come within our desired closeness to  $L$ ? The images below show this idea by marking the desired intervals around  $L$  and  $a$  with dashed lines. Here we are looking at the limit given by  $\lim_{x \rightarrow 1} f(x) = L$  where  $a = 1$ . Using a TI-Nspire file, you can manipulate  $\epsilon$  and  $\delta$  by dragging the points  $L + \epsilon$  or  $a + \delta$  or by simply changing the values of  $\epsilon$  and  $\delta$  given on the screen. In this case, the desire is to keep the graph from escaping out of the

rectangle centered around the point  $(a, L)$ . This rectangle can be thought of as the viewing window you used earlier in this same question. In this example, we are choosing  $\epsilon = 0.5$ .



Notice that when  $\delta = 0.2$ , we cannot be certain visually if we are safe from  $f$  escaping out the top of the interval. If we use  $\delta = 0.15$ , we seem to be safe. The main idea here is that as long as we choose an input value within  $0.15$  of  $x = 1$ ,  $0.85 < x < 1.15$ , we are guaranteed to have outputs that fall within  $0.5$  of our proposed limit,  $L = 1$ . In other words, an input of  $0.85 < x < 1.15$ , will insure that our output is  $0.5 < y < 1.5$ . We will address our uncertainty about  $\delta = 0.2$  later in a more analytic way.

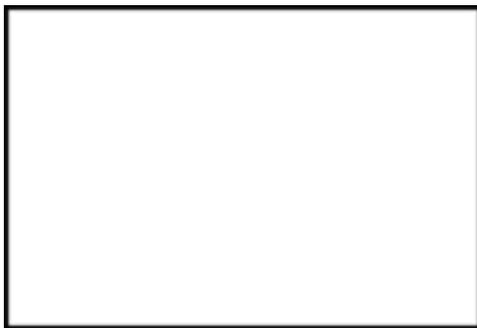
5. For this question, you will use functions you explored earlier in this lab.
  - a. For the function from question 1, have one person choose the value of  $\epsilon$  and a second person try to find an appropriate value of  $\delta$  by setting the viewing window or using the file for the TI-Nspire and manipulating  $\epsilon$  and  $\delta$  accordingly by first using  $a = 2$  and then  $a = 1$ . In the table below record your guess for the limit,  $L$ . Record your results for at least three different choices of  $\epsilon$  and  $\delta$  *sketching your graphs as well*.

$f(x) = \frac{x^5 - 1}{x - 1}$ $a = 2$ $L =$			$f(x) = \frac{x^5 - 1}{x - 1}$ $a = 1$ $L =$		
Choice of $\epsilon$	Choice of $\delta$	Viewing Window	Choice of $\epsilon$	Choice of $\delta$	Viewing Window



- b. Repeat the activity from part (a) using the functions from question 3 parts (b) and (c).

$f(x) = \frac{\sin(5x)}{x}$ $a=0$ $L=$			$f(x) = \frac{x^2 - 9}{x - 3}$ $a=3$ $L=$		
Choice of $\varepsilon$	Choice of $\delta$	Viewing Window	Choice of $\varepsilon$	Choice of $\delta$	Viewing Window

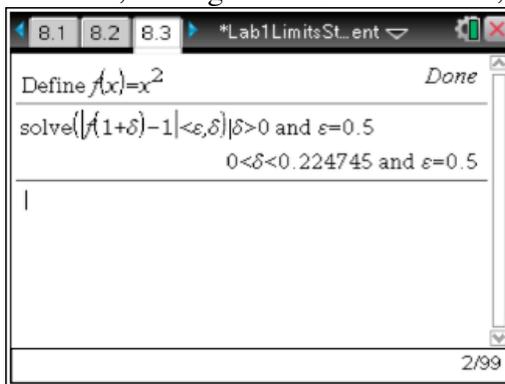


6. Consider the following definition of limit. We define  $\lim_{x \rightarrow a} f(x)$  to be the number  $L$  (if it exists) such that for **any**  $\varepsilon > 0$  (as small as we want), there is a  $\delta > 0$  (sufficiently small) such that if  $|x - a| < \delta$  and  $x \neq a$ , then  $|f(x) - L| < \varepsilon$ . Explain in your own words and/or symbols how the activities you did in question 6 are related to this definition.

7. Now that we have spent some time on the graphical and numerical representations of limit, how would we address our earlier concern about  $\lim_{x \rightarrow a} f(x) = L$  where  $a = 1$  for  $f(x) = x^2$ ?

In our earlier example we could not be visually certain that when we chose  $\varepsilon = 0.5$ , a  $\delta = 0.2$  would work to guarantee any input into  $f$  coming from the interval  $[a - \delta, a + \delta]$  would generate an output falling in the interval  $[L - \varepsilon, L + \varepsilon]$ . We can use our computer algebra system (CAS) to try and determine analytically if this value for delta would work. In fact, we can try to find a relationship between  $\varepsilon$  and  $\delta$  that will guarantee that for **any**  $\varepsilon > 0$  (as small as we want), there is a  $\delta > 0$  (sufficiently small) such that if  $|x - a| < \delta$  and  $x \neq a$ , then  $|f(x) - L| < \varepsilon$ . To do this, we can assign  $x - a = \delta$  assuming  $\delta > 0$  and this implies that  $x = a + \delta$ .

- a. Now to find a relationship between  $\varepsilon$  and  $\delta$ , we now solve  $|f(x) - L| < \varepsilon$  where  $x = a + \delta$ . To do this use your CAS solve command to solve for  $\delta$  as shown below using  $a = 1$  and  $\varepsilon = 0.5$  as in the example. We can see that the  $\delta$  of 0.2 would have worked even though we could not visually see this from the graphical representation. In fact, as long as  $0 < \delta < 0.224745$ , we would be safe.



- b. Now use the process from part (8a) to find a general relationship between  $\varepsilon$  and  $\delta$  for this case where  $a = 1$  and  $L = 1$ .
- c. Now use the process from part (8a) to find an acceptable  $\delta$  given that  $\varepsilon = 0.1$  for evaluating the limit,  $\lim_{x \rightarrow 1} x^2$  as you did in questions 5 and 6 using the adjustment of the viewing window or the edited graph. How does your analytic result compare to the result shown in question 5?
- d. Now use the process from part (8a) to show that  $\lim_{x \rightarrow 2} x^2 = 4$  by finding the relationship between  $\varepsilon$  and  $\delta$  giving a way to generate a  $\delta$  from any specific  $\varepsilon$ . Explain how this process is related to the formal definition of limit given in question 7.