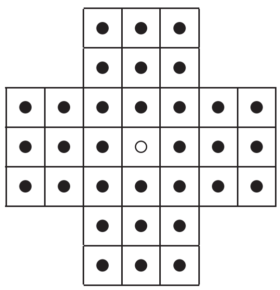
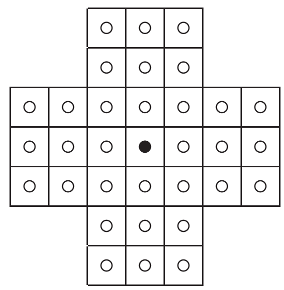
Peg Solitaire

Peg Solitaire is a game that consists of a board with 33 holes arranged in the pattern given in the pictures below. At the start, every hole except the center is filled with a peg. The player then starts jumping pegs. Any peg that is jumped over is removed, just as in checkers. Vertical and horizontal jumps are allowed, but diagonal jumps are forbidden. The goal is to reach a position where only one peg remains, and that peg is in the center hole.



Start



Finish

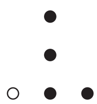
You can vary the problem by choosing some hole other than the center hole. In fact, you may pick any of the 33 holes, leave it empty at the start, and finish with one peg in that hole. We shall use the notation below to describe the spaces in the board.

		c1	*d1*	*e1*		
		c2	*d2*	*e2*		
a3	*b3*	*c3*	*d3*	*e3*	*f3*	*g3*
a4	*b4*	*c4*	*d4*	*e4*	*f4*	*g4*
a5	*b5*	*c5*	*d5*	*e5*	*f5*	*g5*
		c6	*d6*	*e6*		
		c7	*d7*	*e7*		

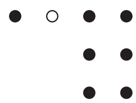
Coordinates

Block Moves

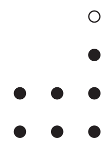
Before tackling a big problem, a good strategy is to start with several smaller problems. Some examples of smaller problems are given below. In each case, you need to reduce the given configuration to one peg. These configurations are called **block moves** or **packages**.



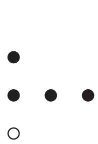
3–removal 1



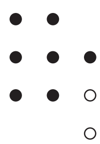
6–removal 1



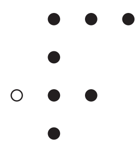
6–removal 2



3–removal 2



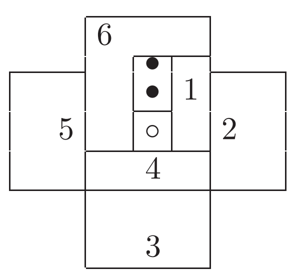
6–removal 3



L–removal

In the 3–removals, you should try to remove the line of 3 pegs and retain the other peg. Note that the second 3–removal is very similar to the first. In the 6–removals, try to remove the rectangular block of 6 pegs and retain the other peg. In the L–removal, remove the L–shaped collection of pegs.

The problem described on the first page is the *d*4–reversal. This means we start with pegs in every hole except *d*4, and we end with one peg in *d*4 and holes elsewhere. The diagram below shows one solution of this problem. The first block is 3–removal 1, the second block is 6–removal 1, etc.



Instead of the $d4$–reversal, you can pick any other space on the board and do the corresponding reversal problem. At first glance, you might think that this means that there are 33 different one–peg reversal problems. But by rotation and reflection, these can be reduced to 7 essentially distinct problems. Besides the $d4$–reversal, the problems are the $c1, c2, c3, d1, d2$, and $d3$–reversals. Most of these problems can be solved using block moves, and I leave them to you as exercises. The one exception is the $d1$-reversal.

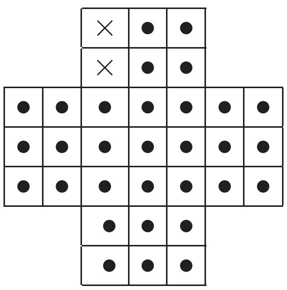
Homework

1. Do the $c1$–$c2$–reversal problem. In other words, start with holes only at $c1$ and $c2$, and end with pegs only at $c1$ and $c2$.

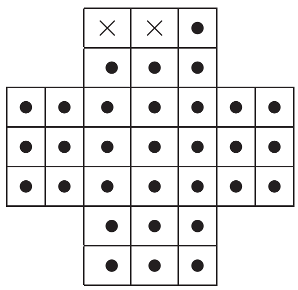
2. Do the $c1$–$d1$–reversal problem.

3. Do the $c1$–$c4$–reversal problem.

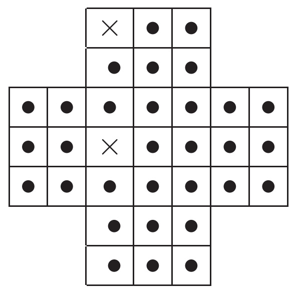
4. Do the $c4$–$e4$–reversal problem.



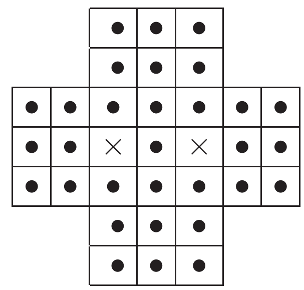
Problem 1



Problem 2

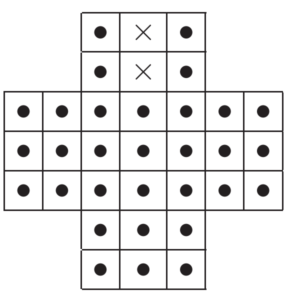


Problem 3

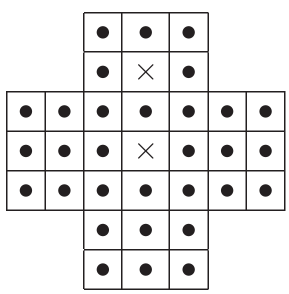


Problem 4

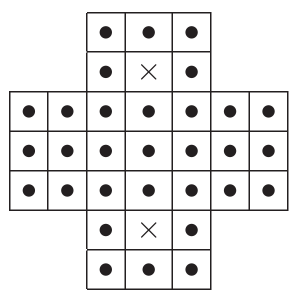
The four problems mentioned above are examples of two–peg reversal problems. There are 81 such problems, and all but 4 of them are possible. The impossible ones are $d1$–$d2$, $d2$–$d4$, $d2$–$d6$, and $d2$–$b4$.



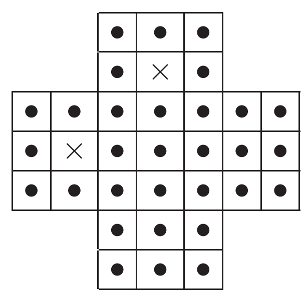
$d1$–$d2$



$d2$–$d4$



$d2$–$d6$

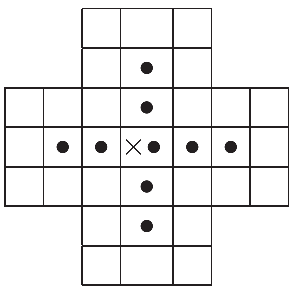


$d2$–$b4$

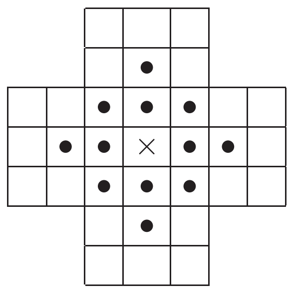
Impossible Problems

Other problems

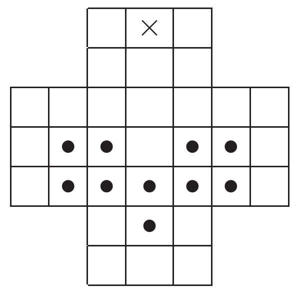
Here are some other simple problems that can be done on a Peg Solitaire board. The symbol • represents a peg at the beginning of the problem; the symbol × represents a peg present at the end of the problem.



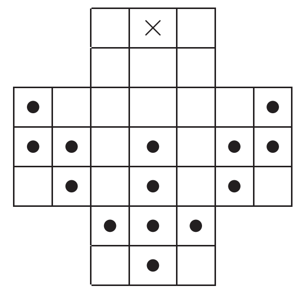
Cross



Diamond

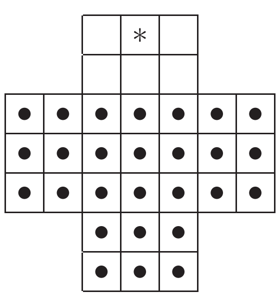


Crossbow

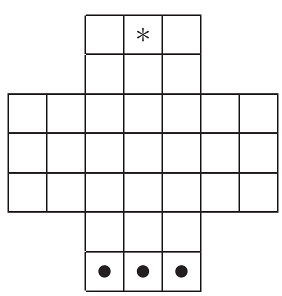


Longbow

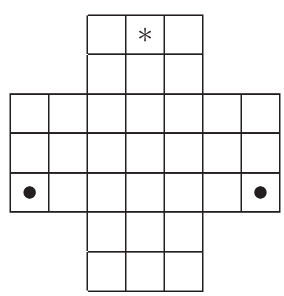
Here are three problems known collectively as "the lecturer and her audience." All three problems start from the position in the first picture below. In the first problem, the lecturer remains a motionless gongoozler while everyone departs except for three sleeping students in the back row $c7/d7/e7$. The second problem ends with two sleeping students in the corners $a5/g5$. In the third problem, the lecturer remains motionless until the end, when, disgusted with the course of events, she collects the three remaining students and marches out the back door.



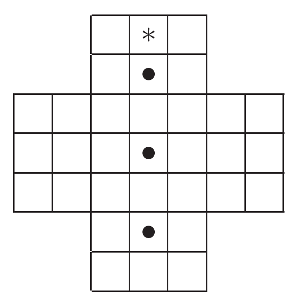
Start



Problem 1



Problem 2



Problem 3

Homework

5. Do the Cross problem.
6. Do the Diamond problem.
7. Do the Crossbow problem.
8. Do the Longbow problem.
9. Do the first Lecturer problem.

Peg Algebra

Let r, s, t and u denote four spaces on a peg board. Suppose we start with pegs in s and t. We can jump from t to r. Let us think of st as equal to r. Alternatively, we can jump from s to u, so $st = u$

○ ● ● ○ ⟶ ● ○ ○ ○ ○ ● ● ○ ⟶ ○ ○ ○ ●
r s t u r s t u r s t u r s t u

Since $st = r$ and $st = u$, this means that in our algebra, $r = u$. In other words,

places three apart in a line are equal.

Starting from the equations $st = u$ and $tu = s$, we multiply to get $st^2u = su$. Cancelling out the factors of su, we get $t^2 = 1$. In other words,

two pegs in the same place cancel.

Combining these first two rules, we see that $ru = 1$; i.e.

two pegs three apart cancel.

Since $r = st$ and $r^2 = 1$, we get $rst = 1$. Thus

any three pegs in a line cancel.

With the last rule, it is easy to see that the value of the whole board is 1. Thus any position is equal to its complement.

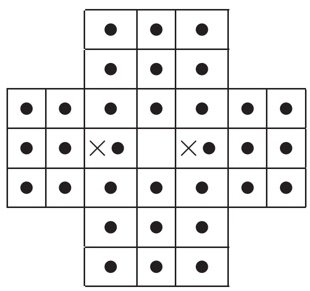
What do these strange algebraic rules tell us about Peg Solitaire? They give us an easy way to show that certain problems are impossible. For example, is it possible to start with a hole in the center ($d4$) and end with a peg at $d3$? No, because the two positions do not have the same algebraic values. On the other hand, it is possible to end with one peg in $d1$. (Try it. If you need help, go back and look at the solution we gave earlier to the $d4$–reversal problem.) It is also possible to end with one peg in any of the positions $a4$, $g4$, or $d7$; just take the $d1$ solution and use rotation.

Problems

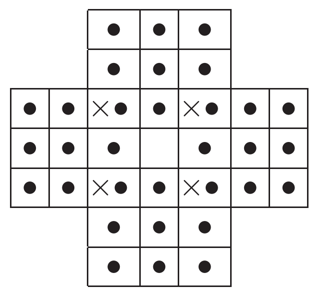
10. Suppose we start with a peg in every hole except $c4$, and we jump pegs until only one peg is on the board. According to our algebraic rules, there are four different possible locations for the last peg. Find these four locations, and solve each of the corresponding problems.

11. Start with pegs in every hole except $d4$. End with pegs in $c4$ and $e4$.

12. Start with pegs in every hole except $d4$. End with pegs in $c3$, $e3$, $c5$, and $e5$.



Problem 11



Problem 12

Further Reading

Much more information about Peg Solitaire can be found in the following two books.

1. *The Ins and Outs of Peg Solitaire* by John Beasley. This is the definitive reference for Peg Solitaire. It is published by Oxford University Press. A paperback edition is due in August 1992; the cost is about $9.

2. *Winning Ways for Your Mathematical Plays* by E.R. Berlekamp, J. H. Conway, and R.K. Guy. This two–volume work, published by Academic Press, covers a number of topics in recreational mathematics. Chapter 23 of the second volume is devoted to Peg Solitaire.