AN ESTIMATE FOR THE NUMBER OF REDUCIBLE BESSEL POLYNOMIALS OF BOUNDED DEGREE

M. Filaseta and S.W. Graham

1. Introduction

The nth Bessel polynomial is

$$y_n(x) = \sum_{j=0}^{n} \frac{(n+j)!}{2^j (n-j)! j!} x^j.$$

In [3], E. Grosswald conjectured that $y_n(x)$ is irreducible over the rationals for every positive integer n. In [1], the first author proved that almost all $y_n(x)$ are irreducible and later [2] sharpened this by showing that the number of $n \le t$ for which $y_n(x)$ is reducible is $\ll t/\log\log\log t$. The object of this paper is to give a further sharpening.

Theorem. *The number of $n \le t$ for which $y_n(x)$ is reducible is $\ll t^{2/3}$.*

The first author's earlier work used the Tchebotarev Density Theorem, but the proof given here uses only elementary estimates. Our starting point is the Corollary to Lemma 2 in [1], which states that if

$$(1) \qquad \Bigg(\prod_{p \mid n(n+1)} p\Bigg)^2 \Bigg(\prod_{\substack{p \mid (n-1) \\ p \text{ odd}}} p\Bigg) \Bigg(\prod_{\substack{p \mid (n+2) \\ p > 3}} p\Bigg) > n^2(n+1)^2,$$

then $y_n(x)$ is irreducible. We shall show that (1) holds for most n by showing that the non-squarefree part of $(n-1)n(n+1)(n+2)$ is typically very small.

2. Preliminaries

For every positive integer n, we define

$$a_n = \prod_{\substack{p^\alpha \| n \\ \alpha \text{ odd}}} p \text{ and } b_n = \prod_{p^\alpha \| n} p^{[\alpha/2]},$$

where $p^\alpha \parallel n$ denotes, as usual, that p^α is the highest power of p dividing n. We then have that $n = a_n b_n^2$ and that

$$(2) \qquad a_n \le \prod_{p \mid n} p.$$

In the next lemma, we use (2) to state (1) in a more usable form.

The second author was supported in part by a grant from the National Security Agency

Lemma 1. *If $y_n(x)$ is reducible and $t < n \le 2t$ then*

$$b_{n-1} b_n^2 b_{n+1}^2 b_{n+2} > \tfrac{1}{3} t.$$

Proof. From (1) and (2), we see that if $y_n(x)$ is reducible, then

$$\frac{n-1}{b_{n-1}^2} \cdot \frac{n^2}{b_n^4} \cdot \frac{(n+1)^2}{b_{n+1}^4} \cdot \frac{n+2}{b_{n+2}^2} \le 6n^2(n+1)^2.$$

The result now follows.

Lemma 2. *If y is a positive real number, then*

$$\#\{n \in (t, 2t] : b_n > y\} \ll \frac{t}{y} + t^{1/2}.$$

Proof. The left–hand side is at most

$$\sum_{t<n\le 2t} \sum_{\substack{b^2 | n \\ b>y}} 1 \ll \sum_{y<b\le\sqrt{2t}} \left(\frac{t}{b^2} + 1 \right) \ll \frac{t}{y} + t^{1/2}.$$

Lemma 3. *If $z \ge 2$ and y are real numbers, then*

$$\#\{n \in (t, 2t] : b_n b_{n+1} > z, b_n \le y, \text{ and } b_{n+1} \le y\} \ll \frac{t \log z}{z} + y^2.$$

Proof. The left–hand side is

$$\begin{aligned} &\le \sum_{t<n\le 2t} \sum_{\substack{b^2 | n, c^2 | (n+1) \\ bc > z, b \le y, c \le y}} 1 \ll \sum_{\substack{bc>z \\ b\le y, c\le y}} \left(\frac{t}{b^2 c^2} + 1 \right) \\ &\ll y^2 + \sum_{bc \ge z} \frac{t}{b^2 c^2}. \end{aligned} \tag{3}$$

Now the last sum in (3) is at most

$$t \sum_{r \ge z} d(r) r^{-2}, \tag{4}$$

where $d(r)$ denotes the number of divisors of r. Using the elementary estimate $\sum_{r\le x} d(r) \ll x \log x$ and partial summation, we find that (4) is

$$\ll \frac{t \log z}{z}.$$

This completes the proof.

3. Proof of the theorem

We will bound

$$\#\{n \in (t, 2t] : b_{n-1}b_n^2b_{n+1}^2b_{n+2} > \tfrac{1}{3}t\}. \tag{5}$$

By Lemma 2, those n with any of $b_{n-1}, b_n, b_{n+1}, b_{n+2}$ greater than $t^{1/3}$ contribute $\ll t^{2/3}$. The remaining n all have $b_{n+j} \le t^{1/3}$ for $-1 \le j \le 2$. By Lemma 3, those n with any of $b_{n-1}b_n, b_nb_{n+1}, b_{n+1}b_{n+2}$ greater than $t^{1/3}\log t$ contribute $\ll t^{2/3}$. The remaining n all have

$$b_{n-1}b_n, b_nb_{n+1}, b_{n+1}b_{n+2} \le t^{1/3}\log t.$$

Using the condition in (5), we see that

$$b_{n-1}b_n \cdot b_nb_{n+1} \cdot b_{n+1}b_{n+2} > \tfrac{1}{3}t,$$

so in fact the remaining n satisfy the stronger conditions

$$\tfrac{1}{3}t^{1/3}\log^{-2} t \le b_{n-1}b_n, b_nb_{n+1}, b_{n+1}b_{n+2} \le t^{1/3}\log t. \tag{6}$$

Now consider those n satisfying (6) with $b_n > t^{2/9}$. Then $b_{n-1}, b_{n+1} < t^{1/9}\log t$ and $b_{n+2} > \frac{1}{3}t^{2/9}\log^{-3} t$. In other words, these n have

$$b_n \le t^{1/3}, b_{n+2} \le t^{1/3} \text{ and } b_nb_{n+2} > \tfrac{1}{3}t^{4/9}\log^{-2} t.$$

By an easy variant of the argument giving Lemma 3, these n contribute

$$\ll t^{5/9}\log^3 t + t^{2/3} \ll t^{2/3}.$$

A similar argument can be used to get the same bound for those n with $b_{n+1} > t^{2/9}$.

The remaining n have $b_n, b_{n+1} \le t^{2/9}$. By (6), $b_{n-1} \ge \frac{1}{3}t^{1/9}\log^{-2} t$ and

$$\tfrac{1}{9}t^{4/9}\log^{-4} t \le b_{n-1}b_nb_{n+1} \le t^{5/9}\log t.$$

The number of such n is

$$\ll \sum_{\frac{1}{9}t^{4/9}\log^{-2} t \le m \le t^{5/9}\log t} \left(\frac{t}{m^2} + 1\right) d_3(m) \tag{7}$$

where $d_3(m)$ denotes the number of ways of writing m as a product of three factors. Using the trivial estimate $\sum_{m\le x} d_3(m) \ll x\log^2 x$ and partial summation, we see that (7) is

$$\ll t^{5/9}\log^4 t \ll t^{2/3}.$$

This concludes the proof.

Acknowledgements. Part of the work for this paper was done while the second author was on sabbatical leave at the University of Illinois. He thanks them for their hospitality.

References

1. M. Filaseta, *The irreducibility of almost all Bessel polynomials*, J. No. Theory **27** (1987), 22–32.
2. M. Filaseta, *On an irreducibility theorem of I. Schur*, Acta Arith. (to appear).
3. E. Grosswald, *Bessel Polynomials*, Lecture Notes in Math. Vol. 698, Springer-Verlag, Berlin, 1978.

Department of Mathematics
University of South Carolina
Columbia, SC 29208 USA
E-mail address: filaseta@milo.math.scarolina.edu

Department of Mathematics
Michigan Technological University
Houghton, MI 49931 USA
E-mail address: swgraham@math1.math.mtu.edu